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ON NEW GENERALIZED INEQUALITIES VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRATION

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Abstract. In this paper, we generalize Montogomery identities for Riemann-Liouville fractional integrals by using a new peano kernel. We also use this Montogomery identities to establish some new Ostrowski type integral inequalities. Some inequalities via convex function are also discussed.

1. Introduction

Fractional calculus has been known since the 17th century. Recently, the interest in fractional analysis has been growing continually due to its useful applications in many fields of sciences such as electromagnetic waves [9], visco-elastic systems [5], quantum evolution of complex systems [12], diffusion waves [20], physics [10], engineering [19], finance [14], social sciences [3],[22] mathematical biology [2],[21] and chaos theory [16]-[8]. Moreover, fractional calculus developed not only in pure theoretical field but also in diverse fields ranging from physical sciences and engineering to biological sciences and economics [4]-[24]. Further work in this direction has long been overdue.

In 1938, Ostrowski [18] established an interesting integral inequality associated with differentiable mappings. This Ostrowski inequality has powerful applications in a number of fields such as numerical integration, probability and optimization theory, stochastic, information and integral operator theory. In 2009, Anastassiou [1] gave the fractional version of Ostrowski inequality. Zeki [23] gave a generalization of Anastassiou [1]. There is not much work done in this direction and needs to be explored as fractional Ostrowski inequality is expected to have applications in many areas in the same way as its counterpart [18].

Ostrowski [18] proved the following interesting integral inequality:

Theorem 1. *Let* $f : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b), *whose derivative* $f' : (a,b) \to \mathbb{R}$ *is bounded on* (a,b), *i.e.* $\|f'\|_\infty =$

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$\sup_{t\in[a,b]} |f'(t)| < \infty$ *then*

$$|S(f;a,b)| \leq \left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\frac{M}{b-a} \tag{1.1}$$

for all $x \in [a,b]$.

For some generalizations of this classic theorem see the book by Mitrinović [17]. A simple proof of the above can be obtained by using the following identity:

Lemma 1. *If* $f : [a,b] \to \mathbb{R}$ *is differentiable on* $[a,b]$ *with the first derivative* f' *integrable on* $[a,b]$*, then the Montgomery's identity holds:*

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\int_a^b P_1(x,t)f\,'(t)dt, \tag{1.2}$$

where $P_1(x,t)$ *is the peano kerenel defined by*

$$P_1(x,t) = \begin{cases} t-a & \text{if} \quad a \leq t \leq x \leq b \\ t-b & \text{if} \quad a \leq x < t \leq b \end{cases}$$

Anastassiou used the following lemma to prove his inequality in [1].

Lemma 2. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be differentiable on* I^0 *with* $a, b \in I$ $(a < b)$ *and* $f\,' \in L_1[a,b]$*, then*

$$f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^{\alpha}(f(b)) - J_a^{\alpha-1}(P_2(x,b)f(b)) + J_a^{\alpha}(P_2(x,b)f\,'(b)), \quad \alpha \geq 1 \tag{1.3}$$

where $P_2(x,t)$ *is the fractional peano kernel defined by*

$$P_2(x,t) = \begin{cases} \frac{t-a}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha) & \text{if} \quad a \leq t \leq x \leq b \\ \frac{t-b}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha) & \text{if} \quad a \leq x < t \leq b \end{cases}$$

In this paper, we will use the Riemann-Liouville fractional integrals to establish some new integral inequalities of Ostrowski's type. Define $\Omega_1(x,t)$ as

$$\Omega_1(x,t) = \begin{cases} \frac{1}{b-a}\left[t - \left(a + h\frac{b-a}{2}\right)\right], & a \leq t \leq x \\ \frac{1}{b-a}\left[t - \left(b - h\frac{b-a}{2}\right)\right], & x < t \leq b \end{cases} \tag{1.4}$$

The kernel $\Omega_1(x,t)$ is an extension of $P_1(x,t)$ which will be adapted to define the fractional version of $P_1(x,t)$. From our results, some classical Ostrowski's inequalities can be deduced as special cases. Now we will give a definition and a result which is useful in understanding our derivations and results. They will also help in connecting our work with available literature.

Definition 1. *The Riemann-Liouville fractional integral operator of order* $\alpha \geq 0$ *is defined as*

$$J_a^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt,$$

where

$$J_a^0 f(x) = f(x).$$

2. Main Results

In this section, we will state and prove our main results. But before we do that we will prove a useful identity with the help of the following kernel. We specially wrote this kernel for the said purpose.

Lemma 3. *Let* $f : [a,b] \to \mathbb{R}$ *be an absolutely continuous mapping. Let* $\Omega_2(x,.) : [a,b] \to \mathbb{R}$, *be the fractional peano type kernel which is given by*

$$\Omega_2(x,t) = \begin{cases} \frac{1}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha)\left(t-\left(a+h\frac{b-a}{2}\right)\right), & a \le t \le x \\ \frac{1}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha)\left(t-\left(b-h\frac{b-a}{2}\right)\right), & x < t \le b \end{cases} \tag{2.1}$$

for all $x \in \left[a+h\frac{b-a}{2}, b-h\frac{b-a}{2}\right]$ *and* $h \in [0,1]$, *then the following identity holds:*

$$(1-h)f(x) = \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha}J_a^\alpha(f(b)) - \frac{h}{2}(b-x)^{1-\alpha}(b-a)^{\alpha-2}f(a) \tag{2.2}$$
$$-J_a^{\alpha-1}(\Omega_2(x,b)f(b)) + J_a^\alpha(\Omega_2(x,b)f'(b)), \alpha \ge 1.$$

Proof. From (2.1)

$$\begin{aligned} &\Gamma(\alpha)J_a^\alpha(\Omega_2(x,b)f'(b)) \\ =& \int_a^b (b-t)^{\alpha-1}\Omega_2(x,t)f'(t)dt \\ =& \frac{1}{b-a}\int_a^x (b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right)f'(t)dt \\ &+\frac{1}{b-a}\int_x^b (b-t)^{\alpha-1}\left(t-\left(b-h\frac{b-a}{2}\right)\right)f'(t)dt \\ =& \frac{1}{b-a}(J_1+J_2), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} J_1 &= \int_a^x (b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right)f'(t)dt \\ &= (b-x)^{\alpha-1}\left(x-\left(a+h\frac{b-a}{2}\right)\right)f(x) + (b-a)^{\alpha-1}\left(h\frac{b-a}{2}\right)f(a) \\ &\quad -\int_a^x (b-t)^{\alpha-1}f(t)dt + (\alpha-1)\int_a^x (b-t)^{\alpha-2}\left(t-\left(a+h\frac{b-a}{2}\right)\right)f(t)dt \end{aligned}$$

and

$$\begin{aligned} J_2 &= \int_x^b (b-t)^{\alpha-1}\left(t-\left(b-h\frac{b-a}{2}\right)\right)f'(t)dt \\ &= -(b-x)^{\alpha-1}\left(x-\left(b-h\frac{b-a}{2}\right)\right)f(x) - \int_x^b (b-t)^{\alpha-1}f(t)dt \\ &\quad +(\alpha-1)\int_x^b (b-t)^{\alpha-2}\left(t-\left(b-h\frac{b-a}{2}\right)\right)f(t)dt. \end{aligned}$$

Putting J_1 and J_2 in (2.3), we get

$$
\begin{aligned}
&\Gamma(\alpha) J_a^\alpha \left(\Omega_2(x,b) f'(b)\right) \\
=\; & \frac{1}{b-a}(b-x)^{\alpha-1}\left(x-\left(a+h\frac{b-a}{2}\right)\right) f(x) + (b-a)^{\alpha-2}\left(h\frac{b-a}{2}\right) f(a) \\
& -\frac{1}{b-a}\int_a^x (b-t)^{\alpha-1} f(t)dt + \frac{1}{b-a}(\alpha-1)\int_a^x (b-t)^{\alpha-2}\left(t-\left(a+h\frac{b-a}{2}\right)\right) f(t)dt \\
& -\frac{1}{b-a}(b-x)^{\alpha-1}\left(x-\left(b-h\frac{b-a}{2}\right)\right) f(x) - \frac{1}{b-a}\int_x^b (b-t)^{\alpha-1} f(t)dt \\
& +\frac{1}{b-a}(\alpha-1)\int_x^b (b-t)^{\alpha-2}\left(t-\left(b-h\frac{b-a}{2}\right)\right) f(t)dt.
\end{aligned}
$$

Thus, (2.3) becomes

$$
\begin{aligned}
&\Gamma(\alpha) J_a^\alpha \left(\Omega_2(x,b) f'(b)\right) \\
=\; & (1-h) f(x)(b-x)^{\alpha-1} - \frac{1}{b-a}\int_a^b (b-t)^{\alpha-1} f(t)dt + (b-a)^{\alpha-2}\left(h\frac{b-a}{2}\right) f(a) \\
& +\frac{\alpha-1}{b-a}\left(\begin{array}{c} \int_a^x (b-t)^{\alpha-2}\left(t-\left(a+h\frac{b-a}{2}\right)\right) f(t)dt \\ +\int_x^b (b-t)^{\alpha-2}\left(t-\left(b-h\frac{b-a}{2}\right)\right) f(t)dt \end{array}\right).
\end{aligned}
$$

Multiplying the above equation with $(b-x)^{1-\alpha}$, and after some subtle algebraic manipulation, we rewrite the above equation as:

$$
\begin{aligned}
&(b-x)^{1-\alpha}\Gamma(\alpha) J_a^\alpha \left(\Omega_2(x,b) f'(b)\right) \\
=\; & (1-h) f(x) + \frac{h}{2}(b-x)^{1-\alpha}(b-a)^{\alpha-2} f(a) \\
& -\frac{1}{b-a}(b-x)^{1-\alpha}\int_a^b (b-t)^{\alpha-1} f(t)dt \\
& +\frac{(\alpha-1)}{b-a}\left(\begin{array}{c} \int_a^x (b-x)^{1-\alpha}(b-t)^{\alpha-2}\left(t-\left(a+h\frac{b-a}{2}\right)\right) f(t)dt \\ +\int_x^b (b-x)^{1-\alpha}(b-t)^{\alpha-2}\left(t-\left(b-h\frac{b-a}{2}\right)\right) f(t)dt \end{array}\right),
\end{aligned}
$$

or

$$
\begin{aligned}
&J_a^\alpha \left(\Omega_2(x,b) f'(b)\right) \\
=\; & (1-h) f(x) + \frac{h}{2}(b-x)^{1-\alpha}(b-a)^{\alpha-2} f(a) \\
& -\frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^\alpha (f(b)) + J_a^{\alpha-1}\left(\Omega_2(x,b) f(b)\right).
\end{aligned}
$$

Hence required identity (2.2) is proved. □

Remark 1. *If we choose $h = 0$, the formula we obtained reduces to fractional Montgomery Identity given in (1.3).*

Remark 2. *If we choose $h = 0$ and $\alpha = 1$, the formula we get reduces to classical Montgomery Identity given in (1.2).*

Now we state and prove our main result which is Ostrowski fractional inequality.

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be differentiable on (a, b) such that $f'(b) \in L_1(a, b)$, where $a < b$. If $|f'(x)| \leq M$ for every $x \in [a, b]$ and $\alpha \geq 1$, then the following Ostrowski fractional inequality holds:*

$$\begin{array}{c} \left| (1-h) f(x) + \frac{h}{2} (b-x)^{1-\alpha} (b-a)^{\alpha-2} f(a) \right. \\ \left. - \frac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^{\alpha} (f(b)) + J_a^{\alpha-1} (\Omega_2(x,b) f(b)) \right| \end{array} \tag{2.4}$$
$$\leq \frac{M}{\alpha(\alpha+1)} \left(\begin{array}{c} \frac{2(b-x)(\alpha+1)}{b-a} \left(\frac{a+b}{2} - x \right) - \frac{2}{b-a} (b-x)^2 \\ + (b-a)^{\alpha} (b-x)^{1-\alpha} - (\alpha+1) \frac{h}{2} (b-a)^{\alpha} (b-x)^{1-\alpha} \end{array} \right).$$

Proof. From Lemma 3, we get

$$\begin{aligned} & \left| (1-h) f(x) + \tfrac{h}{2} (b-x)^{1-\alpha} (b-a)^{\alpha-2} f(a) - \tfrac{\Gamma(\alpha)}{b-a} (b-x)^{1-\alpha} J_a^{\alpha} (f(b)) + J_a^{\alpha-1} (\Omega_2(x,b) f(b)) \right| \\ \leq\ & \frac{1}{\Gamma(\alpha)} \left| \int_a^b (b-t)^{\alpha-1} \Omega_2(x,t) f'(t) dt \right| \\ \leq\ & \frac{M}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} |\Omega_2(x,t)| \, dt \\ =\ & M (b-x)^{1-\alpha} \left(\begin{array}{c} \int_a^x \frac{1}{b-a} (b-t)^{\alpha-1} \left(t - \left(a + h\frac{b-a}{2} \right) \right) dt \\ + \int_x^b \frac{1}{b-a} (b-t)^{\alpha-1} \left(\left(b - h\frac{b-a}{2} \right) - t \right) dt \end{array} \right) \\ =\ & M (b-x)^{1-\alpha} (J_3 + J_4), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} J_3 &= \frac{1}{b-a} \int_a^x (b-t)^{\alpha-1} \left(t - \left(a + h\frac{b-a}{2} \right) \right) dt \\ &= -\frac{1}{\alpha(b-a)} \left(x - \left(a + h\frac{b-a}{2} \right) \right) (b-x)^{\alpha} - \frac{h}{2\alpha} (b-a)^{\alpha} \\ &\quad - \frac{1}{\alpha(\alpha+1)} \frac{1}{b-a} (b-x)^{\alpha+1} + \frac{1}{\alpha(\alpha+1)} (b-a)^{\alpha}, \end{aligned}$$

and

$$\begin{aligned} J_4 &= \frac{1}{b-a} \int_x^b (b-t)^{\alpha-1} \left(\left(b - h\frac{b-a}{2} \right) - t \right) dt \\ &= \frac{1}{\alpha(b-a)} \left(\left(b - h\frac{b-a}{2} \right) - x \right) (b-x)^{\alpha} - \frac{1}{\alpha(\alpha+1)(b-a)} (b-x)^{\alpha+1}. \end{aligned}$$

Using J_3 and J_4 in (2.5), we get

$$\begin{aligned} & \left| (1-h) f(x) + \tfrac{h}{2}(b-x)^{1-\alpha}(b-a)^{\alpha-2} f(a) \right. \\ & \left. -\tfrac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^{\alpha}(f(b)) + J_a^{\alpha-1}(\Omega_2(x,b)f(b)) \right| \\ \leq\ & M \begin{pmatrix} -\frac{1}{\alpha(b-a)}\left[x-\left(a+h\frac{b-a}{2}\right)\right](b-x) - \frac{h}{2\alpha}(b-a)^{\alpha}(b-x)^{1-\alpha} \\ -\frac{1}{\alpha(\alpha+1)}\frac{1}{b-a}(b-x)^2 + \frac{1}{\alpha(\alpha+1)}(b-a)^{\alpha}(b-x)^{1-\alpha} \\ +\frac{1}{\alpha(b-a)}\left[\left(b-h\frac{b-a}{2}\right)-x\right](b-x) - \frac{1}{\alpha(\alpha+1)(b-a)}(b-x)^2 \end{pmatrix} \\ =\ & M \begin{pmatrix} \frac{2(b-x)}{\alpha(b-a)}\left(\frac{a+b}{2}-x\right) - \frac{2}{\alpha(\alpha+1)(b-a)}(b-x)^2 \\ +\frac{1}{\alpha(\alpha+1)}(b-a)^{\alpha}(b-x)^{1-\alpha} - \frac{h}{2\alpha}(b-a)^{\alpha}(b-x)^{1-\alpha} \end{pmatrix} \\ =\ & \frac{M}{\alpha(\alpha+1)} \begin{pmatrix} \frac{2(b-x)(\alpha+1)}{b-a}\left(\frac{a+b}{2}-x\right) - \frac{2}{b-a}(b-x)^2 \\ +(b-a)^{\alpha}(b-x)^{1-\alpha} - (\alpha+1)\frac{h}{2}(b-a)^{\alpha}(b-x)^{1-\alpha} \end{pmatrix}. \end{aligned}$$

Which is our required inequality. □

Remark 3. *If we take $h = 0$ in Theorem 2, it reduces to the result proved by Anastassiou et. al. in* [1]*. So, our results are generalizations of the corresponding results of Anastassiou et. al.* [1]*.*

3. An Ostrowski Type Fractional Inequality via convexity

Definition 2. *A function $f(x)$ is convex on an interval $[a,b]$ if for any two points x_1 and x_2 in $[a,b]$ and any $\lambda \in (0,1)$,*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $x \in (a,b)$, the following identity holds:*

$$\begin{aligned} & \frac{1}{\alpha(\alpha+1)} \begin{pmatrix} [-\frac{1}{b-a}(\alpha+1)(x-\left(a+h\frac{b-a}{2}\right))(b-x) - \frac{h(\alpha+1)}{2}(b-a)^{\alpha}(b-x)^{1-\alpha} \\ -\frac{1}{b-a}(b-x)^2 + (b-a)^{\alpha}(b-x)^{1-\alpha}]f_-'(x) \\ -[\frac{1}{b-a}(\alpha+1)\left(\left(b-h\frac{b-a}{2}\right)-x\right)(b-x) + \frac{1}{b-a}(b-x)^2]f_+'(x) \end{pmatrix} \\ \leq\ & \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^{\alpha}(f(b)) - J_a^{\alpha-1}(\Omega_2(x,b)f(b)) \\ & -\frac{h}{2}(b-x)^{1-\alpha}(b-a)^{\alpha-2} f(a) - (1-h) f(x). \end{aligned} \tag{3.1}$$

Proof. From Lemma 3, we have

$$\begin{aligned} & (1-h) f(x) - \frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^{\alpha}(f(b)) \qquad (3.2) \\ & +\frac{h}{2}(b-x)^{1-\alpha}(b-a)^{\alpha-2} f(a) + J_a^{\alpha-1}(\Omega_2(x,b)f(b)) \\ =\ & \frac{1}{b-a}(b-x)^{1-\alpha} \begin{pmatrix} \int_a^x (b-t)^{\alpha-1}\left[t-\left(a+h\frac{b-a}{2}\right)\right] f'(t)dt \\ -\int_x^b (b-t)^{\alpha-1}\left[\left(b-h\frac{b-a}{2}\right)-t\right] f'(t)dt \end{pmatrix}. \end{aligned}$$

Since f is convex, then for any $x \in (a,b)$ we have the following inequalities

$$f'(t) \leq f_-'(x) \qquad \text{for} \quad t \in [a,x] \tag{3.3}$$

$$f'(t) \geq f'_+(x) \qquad \text{for} \quad t \in [x,b] \tag{3.4}$$

If we multiply (3.3) by $(b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right) \geq 0,\ t \in [a,x]$, $\alpha \geq 1$ and integrate on $[a,x]$, we get

$$\begin{aligned}
&\int_a^x (b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right) f'(t)dt \\
\leq\ &\int_a^x (b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right) f'_-(x)\ dt \\
=\ &\frac{1}{\alpha(\alpha+1)}\left(\begin{array}{c}(b-a)^{\alpha+1}-(b-x)^{\alpha+1} \\ -(\alpha+1)\left(x-\left(a+h\frac{b-a}{2}\right)\right)(b-x)^{\alpha} \\ -\frac{h(\alpha+1)}{2}(b-a)^{\alpha+1}\end{array}\right) f'_-(x).
\end{aligned} \tag{3.5}$$

If we multiply (3.4) by $(b-t)^{\alpha-1}\left(\left(b-h\frac{b-a}{2}\right)-t\right)$, $\alpha \geq 0,\ t \in [x,b]$, $\alpha \geq 1$ and integrate on $[x,b]$, we also get

$$\begin{aligned}
&\int_x^b (b-t)^{\alpha-1}\left(\left(b-h\frac{b-a}{2}\right)-t\right) f'(t)dt \qquad (3.6)\\
\geq\ &\int_x^b (b-t)^{\alpha-1}\left(\left(b-h\frac{b-a}{2}\right)-t\right) f'_+(x)dt \\
=\ &\frac{1}{\alpha(\alpha+1)}((\alpha+1)\left(\left(b-h\frac{b-a}{2}\right)-x\right)(b-x)^{\alpha}-(b-x)^{\alpha+1})f'_+(x).
\end{aligned}$$

Finally, if we subtract (3.5) from (3.6), and use the representation (3.2), we obtain the desired inequality (3.1). □

Corollary 1. *Under the assumptions of theorem 3 and with* $\alpha = 1$, $h = 0$ *one has,*

$$\frac{1}{2(b-a)}\left((x-a)^2 f'_-(x)-(b-x)^2 f'_+(x)\right) \leq \frac{1}{b-a}\int_a^b f(t)dt - f(x)$$

which is Dragomir's result given in [6].

Remark 4. *If we take* $x = \frac{a+b}{2}$ *in above corollary, we get*

$$0 \leq \frac{b-a}{8}\left(f'_-(\frac{a+b}{2})-f'_+(\frac{a+b}{2})\right) \leq \frac{1}{b-a}\int_a^b f(t)dt - f(\frac{a+b}{2})$$

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then for any $x \in (a,b)$, the following identity holds:*

$$\frac{\Gamma(\alpha)}{b-a}(b-x)^{1-\alpha} J_a^{\alpha}(f(b)) - J_a^{\alpha-1}(\Omega_2(x,b)f(b)) \tag{3.7}$$

$$-\frac{h}{2}(b-x)^{1-\alpha}(b-a)^{\alpha-2} f(a) - (1-h) f(x)$$

$$\leq \frac{1}{\alpha(\alpha+1)} \begin{pmatrix} (\frac{1}{b-a}(\alpha+1)\left(\left(b-h\frac{b-a}{2}\right)-x\right)(b-x) \\ -\frac{1}{b-a}(b-x)^2) f_-'(b) \\ -(-((b-a)^{\alpha}(b-x)^{1-\alpha} - \frac{1}{b-a}(b-x)^2 \\ -\frac{1}{b-a}(\alpha+1)\left(x-\left(a+h\frac{b-a}{2}\right)\right)(b-x) \\ -\frac{h(\alpha+1)}{2}(b-a)^{\alpha}(b-x)^{1-\alpha})) f_+'(a) \end{pmatrix}.$$

Proof. Assume that $f_+'(a)$ and $f_-'(b)$ are finite. Since f is convex, then for any $x \in (a,b)$ we have the following inequalities

$$f'(t) \geq f_+'(a) \qquad \text{for} \quad t \in [a,x] \tag{3.8}$$

$$f'(t) \leq f_-'(b) \qquad \text{for} \quad t \in [x,b]. \tag{3.9}$$

If we multiply (3.7) by $(b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right) \geq 0,\ t \in [a,x]$, $\alpha \geq 1$ and integrate on $[a,x]$, we get

$$\int_a^x (b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right) f'(t)dt$$

$$\geq \int_a^x (b-t)^{\alpha-1}\left(t-\left(a+h\frac{b-a}{2}\right)\right) f_+'(a)\, dt$$

$$= \frac{1}{\alpha(\alpha+1)} \begin{pmatrix} (b-a)^{\alpha+1} - (b-x)^{\alpha+1} \\ -(\alpha+1)\left(x-\left(a+h\frac{b-a}{2}\right)\right)(b-x)^{\alpha} \\ -\frac{h(\alpha+1)}{2}(b-a)^{\alpha+1} \end{pmatrix} f_+'(a). \tag{3.10}$$

If we multiply (3.8) by $(b-t)^{\alpha-1}\left(\left(b-h\frac{b-a}{2}\right)-t\right)$, $\alpha \geq 0,\ t \in [x,b]$,$\alpha \geq 1$ and integrate on $[x,b]$, we also get

$$\int_x^b (b-t)^{\alpha-1}\left(\left(b-h\frac{b-a}{2}\right)-t\right) f'(t)dt \tag{3.11}$$

$$\leq \int_x^b (b-t)^{\alpha-1}\left(t-\left(b-h\frac{b-a}{2}\right)\right) f_-'(b)\, dt$$

$$= \frac{1}{\alpha(\alpha+1)}((\alpha+1)\left(\left(b-h\frac{b-a}{2}\right)-x\right)(b-x)^{\alpha} - (b-x)^{\alpha+1}) f_-'(b).$$

Finally, if we subtract (3.10) from (3.11), and use the representation (3.2), we deduce the desired inequality (3.7). □

Corollary 2. *Under the assumptions of theorem 4 and with $\alpha = 1$, $h = 0$ one has,*

$$\int_a^b f(t)dt - (b-a) f(x) \leq \frac{1}{2}\left((b-x)^2 f_-'(b) - (a-x)^2 f_+'(a)\right).$$

Remark 5. *If we take $x = \frac{a+b}{2}$ in above corollary, we get*

$$0 \leq \int_a^b f(t)dt - (b-a) f(\frac{a+b}{2}) \leq \frac{(b-a)}{8} \left(f_-'(b) - f_+'(a) \right).$$

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