

## NUMERICAL SIMULATIONS FOR VARIABLE-ORDER FRACTIONAL NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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**ABSTRACT.** In this paper numerical studies for the variable-order fractional delay differential equations are presented. Adams-Bashforth-Moulton algorithm has been extended to study this problem, where the derivative is defined in the Caputo variable-order fractional sense. Special attention is given to prove the error estimate of the proposed method. Numerical test examples are presented to demonstrate utility of the method. Chaotic behaviors are observed in variable-order one dimensional delayed systems.

### 1. INTRODUCTION

In real world systems, delay is very often encountered in many practical systems, such as control systems [8], lasers, traffic models [19], metal cutting, epidemiology, neuroscience, population dynamics [30], chemical kinetics [13] etc. Delayed fractional differential equations FDEs are correspondingly used to describe such dynamical systems. In recent years, delayed FDEs begin to arouse the attention of many researchers [18, 19, 26, 30, 31]. Simulating these equations is an important technique in the research, accordingly, finding effective numerical methods for the delayed FDEs is a necessary process.

The effective methods and their development for numerically solving fractional differential equations (FDEs) have received increasing attention over the last few years. Several methods based on Caputo or Riemann-Liouville definitions [9] have been proposed and analyzed. For instance, based on the predictor-corrector scheme, Diethelm et al. introduced Adams-Bashforth-Moulton algorithm [16, 17, 31], and mean while some error analysis presented to improve the numerical accuracy. In recent years, the application of the method is extended to more concrete physical and mathematical models [28].

Variable order differential equations, i.e., differential equations where the order of the derivative changes with respect to either the dependent or the independent variables, have not received as much attention as fractional order systems,

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despite of the ability of variable order formulations to model continuous spectral behavior in complex dynamics, see ([1, 7],[10]-[12],[14]-[16],[22],[23],[27]). Many authors have introduced different definitions of variable order differential operators, each of these with a specific meaning to suit desired goals. These definitions such as Riemann-Liouville, Grünwald, Caputo, Riesz ([4]-[6],[25]), and some notes as Coimbra definition [2, 3].

The main aim of this paper is to study numerically the variable-order fractional delay differential equations (FDDEs) by using the Adams- Bashforth-Moulton method. This paper is organized as follows. In Section 1, we give some definitions and mathematical tools of variable-order fractional calculus. In Section 2, we introduce the known Adams- Bashforth-Moulton method, moreover, the effectiveness of the Adams-Bashforth-Moulton method for solving variable-order fractional differential equation is illustrated, and the error analysis is also estimated. In Section 3, numerical test examples are presented. Finally, the conclusion is given in Section 4.

## 2. SOME MATHEMATICAL TOOLS

In this part, we give some definitions of fractional derivative and variable-order derivative [14]-[16],[24],[26]-[29].

**Definition 1** Let  $f \in C_\alpha$  and  $\alpha \geq -1$ , then the (left-sided) Riemann-Liouville integral of order  $\mu$ ,  $\mu > 0$  is given by

$$I_t^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, \quad t > 0.$$

**Definition 2** The (left sided) Caputo fractional derivative of  $f$ ,  $f \in C_{-1}^m$ ,  $m \in \mathbb{N} \cup \{0\}$ , is defined as:

$$\mathcal{D}_t^\mu f(t) = \begin{cases} \frac{d^m}{dt^m} f(t), & \mu = m, \\ I_t^{m-\mu} \frac{d^m}{dt^m} f(t), & m-1 < \mu < m, \quad m \in \mathbb{N}. \end{cases}$$

Note that for  $m-1 < \mu \leq m$ ,  $m \in \mathbb{N}$ ,

$$I_t^\mu \mathcal{D}_t^\mu f(t) = f(t) - \sum_{k=0}^{m-1} \frac{d^k f}{dt^k}(0) \frac{t^k}{k!},$$

$$I_t^\mu t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu}.$$

**Definition 3** Let  $\alpha(t)$  be a positive real number,  $f \in C^m[0, T]$ ,  $t \leq T$ , and  $m = [\max_{0 \leq t \leq T} \{\alpha(t)\}] + 1$ . Then

$$\mathcal{D}^{\alpha(t)} f(t) = \lim_{h_N \rightarrow 0} \frac{1}{h_N^{\alpha(t)}} \sum_{k=0}^N (-1)^k \binom{\alpha(t)}{k} f(t - kh_N), \quad (1)$$

with  $h_N = (t-0)/N$  being called the Grünwald-Letnikov variable-order fractional derivative of order  $\alpha(t)$  of the function  $f$ .

**Definition 4** The Riemann-Liouville variable order derivative is defined as follows:

$$\mathcal{D}_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(m-\alpha(t))} \left( \frac{d^m}{dt^m} \right) \int_0^t (t-\tau)^{m-1-\alpha(\tau)} f(\tau) d\tau, \quad (2)$$

where  $m = [\max_{0 \leq t \leq T} \{\alpha(t)\}] + 1$ ,  $m \in N$  provided the right side is point wise defined on  $t > 0$ .

Let  $\alpha(t) > 0$ , be a continuous and bounded function,  $f(\tau) \in C^m[0, t]$ , and  $0 \leq \tau \leq t$ . Then

$$\mathcal{D}^{\alpha(t)} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha(t))} \int_0^t \frac{f^m(\tau)}{(t-\tau)^{\alpha(\tau)+1-m}} d\tau, & m-1 \leq \alpha(t) < m, \\ \frac{d^m f(t)}{dt^m}, & \alpha(t) = m. \end{cases} \quad (3)$$

is called the Caputo variable-order fractional derivative of  $f(t)$  where  $m = [\max_{0 \leq t \leq T} \{\alpha(t)\}] + 1$  and  $\Gamma(\cdot)$  is the Gamma function.

### 3. THE ADAMS-BASHFORTH-MOULTON METHOD

In the following we apply the Adams-Bashforth-Moulton predictor-corrector method to implement the numerical solution of variable-order nonlinear FDDEs. Let us consider the following variable-order fractional system:

$$\mathcal{D}_t^{\alpha(t)} y(t) = f(y(t), y(t-\tau)), \quad t \in [0, T], \quad 0 < \alpha(t) \leq 1. \quad (4)$$

$$y(t) = g(t), \quad t \in [-\tau, 0], \quad (5)$$

where  $f$  is in general a nonlinear function.

Also, consider a uniform grid  $\{t_n = nh : n = -k, -k+1, \dots, -1, 0, 1, \dots, N\}$  where  $k$  and  $N$  are integers such that  $h = \tau/k$ . Let

$$y_h(t_j) = g(t_j), \quad j = -k, -k+1, \dots, -1, 0, \quad (6)$$

and note that

$$y_h(t_j - \tau) = y_h(jh - kh) = y_h(t_{j-k}), \quad j = 0, 1, 2, \dots, N. \quad (7)$$

Applying  $I_{t_{n+1}}^{\alpha(t_{n+1})}$  on both sides of (4) and using (5), we claim to:

$$y(t_{n+1}) = g(0) + \frac{1}{\Gamma(\alpha(t_{n+1}))} \int_0^{t_{n+1}} (t_{n+1} - \zeta)^{\alpha(t_{n+1})-1} f(y(\zeta), y(\zeta - \tau)) d\zeta. \quad (8)$$

Further the integral in equation (8) is evaluated using product trapezoidal quadrature formula. Then we have the following corrector formula:

$$\begin{aligned} y_h(t_{n+1}) &= g(0) + \frac{h^{\alpha(t_{n+1})}}{\Gamma(\alpha(t_{n+1}) + 2)} f(y_h(t_{n+1}), y_h(t_{n+1} - \tau)) \\ &\quad + \frac{h^{\alpha(t_{n+1})}}{\Gamma(\alpha(t_{n+1}) + 2)} \sum_{j=0}^n a_{j,n+1} f(t_j, y_h(t_j), y_h(t_j - \tau)), \end{aligned}$$

or

$$\begin{aligned} y_h(t_{n+1}) &= g(0) + \frac{h^{\alpha(t_{n+1})}}{\Gamma(\alpha(t_{n+1}) + 2)} f(y_h(t_{n+1}), y_h(t_{n+1-k})) \\ &\quad + \frac{h^{\alpha(t_{n+1})}}{\Gamma(\alpha(t_{n+1}) + 2)} \sum_{j=0}^n a_{j,n+1} f(y_h(t_j), y_h(t_{j-k})), \end{aligned} \quad (9)$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha(t_{n+1})+1} - (n - \alpha(t_{n+1}))(n+1)^{\alpha(t_{n+1})}, & j = 0, \\ (n-j+2)^{\alpha(t_{n+1})+1} + (n-j)^{\alpha(t_{n+1})+1} - 2(n-j+1)^{\alpha(t_{n+1})+1}, & 1 \leq j \leq n, \\ 1, & j = n+1, \end{cases} \quad (10)$$

$$y_h(t_{j-k}) \approx v_{n+1} = \begin{cases} \delta y_{n-m+2} + (1-\delta)y_{n-m+1}, & \text{if } m > 1, \\ \delta y_{n+1}^p + (1-\delta)y_n, & \text{if } m = 1, \end{cases} \quad (11)$$

$0 \leq \delta < 1$ , and the unknown term  $y_h(t_{n+1})$  appears on both sides of (9). Due to nonlinearity of  $f$  equation (9) can't be solved explicitly for  $y_h(t_{n+1})$ , so we replace the term  $y_h(t_{n+1})$  on the right hand side by an approximation  $y_h^p(t_{n+1})$ , which called predictor. The product rectangle rule is used in (8) to evaluate predictor term

$$y_h^p(t_{n+1}) = g(0) + \frac{1}{\Gamma(\alpha(t_{n+1}))} \sum_{j=0}^n b_{j,n+1} f(y_h(t_j), y_h(t_j - \tau)),$$

or

$$y_h^p(t_{n+1}) = g(0) + \frac{1}{\Gamma(\alpha(t_{n+1}))} \sum_{j=0}^n b_{j,n+1} f(y_h(t_j), y_h(t_{j-k})), \quad (12)$$

where

$$b_{j,n+1} = \frac{h^{\alpha(t_{n+1})}}{\alpha(t_{n+1})} ((n-j+1)^{\alpha(t_{n+1})} - (n-j)^{\alpha(t_{n+1})}). \quad (13)$$

#### 4. ERROR ANALYSIS OF THE ALGORITHM

First we introduce the following two lemmas and Lipschitz conditions which will be used in the proof of main theorem.

**Lemma 1**

(a) Let  $z \in C^1[0, T]$ , then

$$\left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha(t_{n+1})-1} z(t) dt - \sum_{j=0}^n b_{j,n+1} z(t_j) \right| \leq \frac{1}{\alpha(t_{n+1})} \|z'\|_{\infty} t_{n+1}^{\alpha(t_{n+1})} h. \quad (14)$$

(b) Let  $z \in C^2[0, T]$ , then

$$\left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha(t_{n+1})-1} z(t) dt - \sum_{j=0}^{n+1} a_{j,n+1} z(t_j) \right| \leq C_{\alpha(t_{n+1})}^{Tr} \|z''\|_{\infty} t_{n+1}^{\alpha(t_{n+1})} h. \quad (15)$$

**Proof.** To prove statement (a), by construction of the product rectangle formula, we find in both cases that the quadrature error has the representation

$$\int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha(t_{n+1})-1} z(t) dt - \sum_{j=0}^n b_{j,n+1} z(t_j) = \sum_{j=0}^n \int_{jh}^{(j+1)h} (t_{n+1} - t)^{\alpha(t_{n+1})-1} (z(t) - z(t_j)) dt.$$

We apply the Mean Value Theorem of differential calculus to the second factor of the integrand on the right-hand side of the above equation

$$\begin{aligned}
 \left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha(t_{n+1})-1} z(t) dt - \sum_{j=0}^n b_{j,n+1} z(t_j) \right| &\leq \|z'\|_\infty \sum_{j=0}^n \int_{jh}^{(j+1)h} (t_{n+1} - t)^{\alpha(t_{n+1})-1} (t - jh) dt \\
 &= \|z'\|_\infty \frac{h^{1+\alpha(t_{n+1})}}{\alpha(t_{n+1})} \sum_{j=0}^n \left( \frac{1}{1 + \alpha(t_{n+1})} [(n+1-j)^{1+\alpha(t_{n+1})} - (n-j)^{1+\alpha(t_{n+1})}] - (n-j)^{\alpha(t_{n+1})} \right) \\
 &= \|z'\|_\infty \frac{h^{1+\alpha(t_{n+1})}}{\alpha(t_{n+1})} \left( \frac{(n+1)^{1+\alpha(t_{n+1})}}{1 + \alpha(t_{n+1})} \right) - \sum_{j=0}^n j^{\alpha(t_{n+1})} \\
 &= \|z'\|_\infty \frac{h^{1+\alpha(t_{n+1})}}{\alpha(t_{n+1})} \left( \int_0^{t_{n+1}} j^{\alpha(t_{n+1})} dt - \sum_{j=0}^n j^{\alpha(t_{n+1})} \right).
 \end{aligned}$$

Here the term in parentheses is simply the remainder of the standard rectangle quadrature formula, applied to the function  $t^{\alpha(t_{n+1})}$  and taken over the interval  $[0, n+1]$ . Since the integrand is monotonic, we may apply some standard results from quadrature theory to find that this term is bounded by the total variation of the integrand, thus

$$\left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha(t_{n+1})-1} z(t) dt - \sum_{j=0}^n b_{j,n+1} z(t_j) \right| \leq \frac{1}{\alpha(t_{n+1})} \|z'\|_\infty (n+1)^{\alpha(t_{n+1})} h^{1+\alpha(t_{n+1})}.$$

Similarly, to prove (b).

Assume that  $f(\cdot)$  in (4) satisfies the following Lipschitz conditions with respect to its variables as follows:

$$\begin{aligned}
 |f(y_1, u) - f(y_2, u)| &\leq L_1 |y_1 - y_2|, \\
 |f(y, u_1) - f(y, u_2)| &\leq L_2 |u_1 - u_2|,
 \end{aligned} \tag{16}$$

where  $L_1, L_2$  are positive constants.

**Theorem 1** Suppose the solution  $y(t) \in C^2[0, T]$  of the initial value problem satisfies the following two conditions:

$$\left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha(t_{n+1})-1} D_t^{\alpha(t_{n+1})} y(t) dt - \sum_{j=0}^n b_{j,n+1} D_t^{\alpha(t_{n+1})} y(t_j) \right| \leq C t_{n+1}^{\gamma_1} h^{\delta_1}, \tag{17}$$

$$\left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha(t_{n+1})-1} D_t^{\alpha(t_{n+1})} y(t) dt - \sum_{j=0}^n a_{j,n+1} D_t^{\alpha(t_{n+1})} y(t_j) \right| \leq C t_{n+1}^{\gamma_2} h^{\delta_2}, \tag{18}$$

with some  $\gamma_1, \gamma_2 \geq 0$ , and  $\delta_1, \delta_2 > 0$ , then for some suitable  $T > 0$ , we have

$$\max_{0 \leq j \leq N} |y(t_j) - y_h(t_j)| = O(h^q), \tag{19}$$

where  $q = \min(\delta_1 + \alpha(t), \delta_2)$ ,  $N = [T/h]$ , and  $C$  is a positive constant.

**Proof.** We will use the mathematical induction to prove the result. Suppose that the conclusion is true for  $j = 0, 1, \dots, n$ , then we have to prove that the inequality also holds for  $j = n+1$ . To do this, we first consider the error of the predictor  $y_{n+1}^p$ .

From (16), we have

$$\begin{aligned} |f(y(t_j), (y(t_j - \tau) - f(y_j, v_j)))| &\leq |f(y(t_j), y(t_j - \tau)) - f(y_j, y(t_j - \tau))| + \\ &\quad |f(y_j, y(t_j - \tau)) - f(y_j, v_j)| \\ &\leq L_1 h^q + L_2 h^q = (L_1 + L_2) h^q. \end{aligned}$$

So

$$\begin{aligned} |y(t_{n+1}) - y_{n+1}^p| &= \frac{1}{\Gamma(\alpha(t_{n+1}))} \left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha(t_{n+1})-1} f(y(t), y(t - \tau)) dt - \right. \\ &\quad \left. \sum_{j=0}^n b_{j,n+1} f(y_j, v_j) \right| \\ &\leq \frac{1}{\Gamma(\alpha(t_{n+1}))} \left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha(t_{n+1})-1} D_t^{\alpha(t_{n+1})} y(t) dt - \sum_{j=0}^n b_{j,n+1} D_t^{\alpha(t_{n+1})} y(t_j) \right| \\ &\quad + \frac{1}{\Gamma(\alpha(t_{n+1}))} \sum_{j=0}^n b_{j,n+1} (|f(y(t_j), y(t_j - \tau)) - f(y_j, y(t_j - \tau))| + \\ &\quad |f(y_j, y(t_j - \tau)) - f(y_j, v_j)|) \\ &\leq \frac{CT_{n+1}^{\gamma_1}}{\Gamma(\alpha(t_{n+1}))} h^{\delta_1} + \frac{1}{\Gamma(\alpha(t_{n+1}))} \sum_{j=0}^n b_{j,n+1} (L_1 + L_2) Ch^q, \end{aligned}$$

and from

$$\sum_{j=0}^n b_{j,n+1} = \sum_{j=0}^n \int_{jh}^{(j+1)h} (t_{n+1} - t)^{\alpha(t_{n+1})-1} dt = \frac{1}{\alpha(t_{n+1})} t_{n+1}^{\alpha(t_{n+1})} \leq \frac{1}{\alpha(t_{n+1})} T^{\alpha(t_{n+1})},$$

we have

$$|y(t_{n+1}) - y_{n+1}^p| \leq \frac{CT^{\gamma_1}}{\Gamma(\alpha(t_{n+1}))} h^{\delta_1} + \frac{CT^{\alpha(t_{n+1})}}{\Gamma(\alpha(t_{n+1}) + 1)} h^q.$$

Since

$$\begin{aligned} \sum_{j=0}^n a_{j,n+1} &\leq \sum_{j=0}^n \frac{h^{\alpha(t_{n+1})}}{\alpha(t_{n+1})(\alpha(t_{n+1}) + 1)} \left[ (k - j + 2)^{\alpha(t_{n+1})+1} - (k - j + 1)^{\alpha(t_{n+1})+1} - \right. \\ &\quad \left. (k - j + 1)^{\alpha(t_{n+1})+1} + (k - j)^{\alpha(t_{n+1})+1} \right], \\ &= \int_0^{t_{n+1}} (t_{n+2} - t)^{\alpha(t_{n+1})} dt - \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha(t_{n+1})} dt, \\ &= \frac{1}{\alpha(t_{n+1})} \int_0^{t_{n+1}} [(t_{n+1} - t)^{\alpha(t_{n+1})}]'(t) dt, \\ &= \frac{1}{\alpha(t_{n+1})} t_{n+1}^{\alpha(t_{n+1})} \leq T^{\alpha(t_{n+1})}, \end{aligned}$$

we have

$$\begin{aligned}
 |y(t_{n+1}) - y_{n+1}| &= \frac{1}{\Gamma(\alpha(t_{n+1}))} \left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha(t_{n+1})-1} f(y(t), y(t - \tau)) dt - \right. \\
 &\quad \left. \sum_{j=0}^n a_{j,n+1} f(y_j, v_j) - a_{n+1,n+1} f(y_{n+1}^p, v_{n+1}) \right|, \\
 &\leq \frac{1}{\Gamma(\alpha(t_{n+1}))} \left[ \left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha(t_{n+1})-1} f(y(t), y(t - \tau)) dt - \right. \right. \\
 &\quad \left. \sum_{j=0}^{n+1} a_{j,n+1} f(y(t_j), y(t_j - \tau)) \right| + \\
 &\quad \sum_{j=0}^n a_{j,n+1} |f(y(t_j), y(t_j - \tau)) - f(y_j, y(t_j - \tau))| + \\
 &\quad \left. a_{n+1,n+1} |f(y(t_{n+1}), y(t_{n+1} - \tau)) - f(y_{n+1}^p, v_{n+1})| \right] \\
 &\leq \frac{1}{\Gamma(\alpha(t_{n+1}))} \left[ CT_{n+1}^{\gamma_2} h^{\delta_2} + Ch^q \sum_{j=0}^n a_{j,n+1} + \right. \\
 &\quad \left. a_{n+1,n+1} L \left( h^q + \frac{CT^{\gamma_1}}{\Gamma(\alpha(t_{n+1}))} h^{\delta_1} + \frac{CT^{\alpha(t_{n+1})}}{\Gamma(\alpha(t_{n+1}) + 1)} h^q \right) \right] \\
 &\leq \frac{1}{\Gamma(\alpha(t_{n+1}))} \left[ CT^{\gamma_2} + \frac{Lh^h}{\alpha(t_{n+1})(\alpha(t_{n+1}) + 1)} + \frac{CLT^{\gamma_1}}{\Gamma(\alpha(t_{n+1}) + 2)} + \right. \\
 &\quad \left. \frac{CLT^{\alpha(t_{n+1})} h^{\alpha(t_{n+1})}}{\alpha(t_{n+1})\Gamma(\alpha(t_{n+1}) + 2)} \right] h^q \\
 &= Ch^q,
 \end{aligned}$$

which completes the proof.

## 5. NUMERICAL TEST EXAMPLES

The purpose of this section is to show that the proposed scheme designed in this paper provides good approximations for variable-order delay fractional differential equations. Throughout this section, we discuss three examples and their numerical solutions.

**Example 1** Consider a fractional variable order DDE:

$$\begin{aligned}
 \mathcal{D}_t^{\alpha(t)} y(t) &= \frac{2y(t-2)}{1+y(t-2)^{9.65}} - y(t), \\
 y(t) &= 0.5, \quad t \leq 0.
 \end{aligned} \tag{20}$$

Figs. 1(a) and 1(b) show the solutions  $y(t)$  and  $y(t-2)$  of system (20) for  $\alpha = 1$  and  $h = 0.1$ , whereas Fig. 1(c) shows phase portrait of the system i.e., plot of  $y(t)$  versus  $y(t-2)$  for the same value of  $\alpha$ . Moreover, we give the numerical solutions of with different values of  $\alpha(t)$ . For the variable-order we choose two different cases for  $\alpha(t)$ . Figs. 2(a), 2(b), 3(c) and 3(b) show the solutions  $y(t)$  and  $y(t-2)$  of the given system for  $\alpha(t) = 0.99 - (0.01/100)t$  and  $\alpha(t) = 0.95 - (0.01/100)t$  respectively. Also, Figs. 2(c) and 3(c) show chaotic portrait for the two values of  $\alpha(t)$ , respectively.

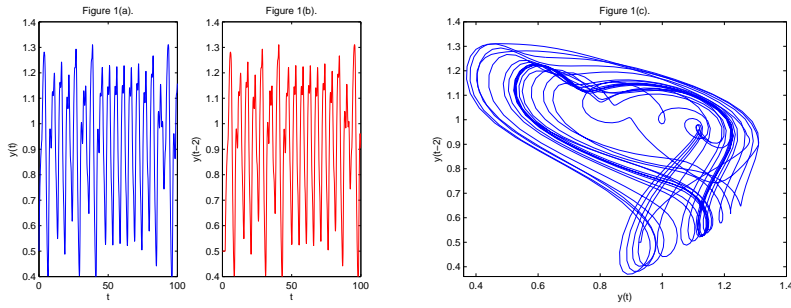


FIGURE 1. The numerical behavior of system (20) and chaotic attractors at  $\alpha = 1$ .

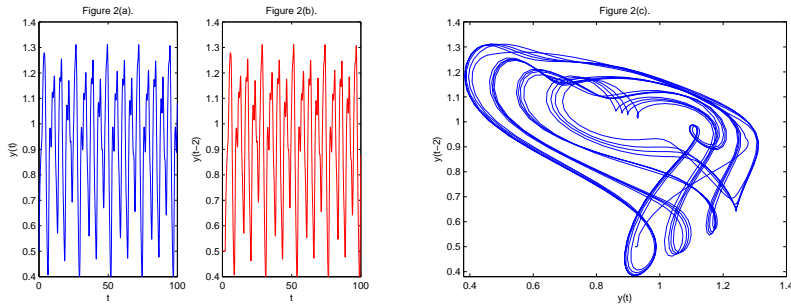


FIGURE 2. The numerical behavior of system (20) and chaotic attractors at  $\alpha(t) = 0.99 - (0.01/100)t$ .

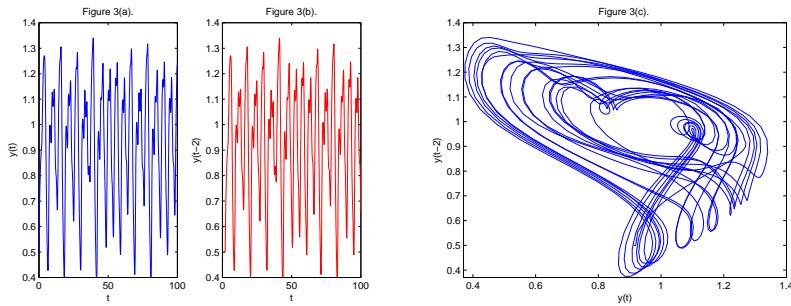


FIGURE 3. The numerical behavior of system (20) and chaotic attractors at  $\alpha(t) = 0.95 - (0.01/100)t$ .

**Example 2** Consider the variable order version of the four years life cycle of a population of lemmings which be an extension of the model given in [20]

$$\begin{aligned} D_t^{\alpha(t)} y(t) &= 3.5y(t) \left( 1 - \frac{y(t-0.74)}{19} \right), \quad y(0) = 19.00001, \\ y(t) &= 19, \quad t < 0. \end{aligned} \quad (21)$$



Fig. 4(a) and 4(b) show the solutions  $y(t)$  and  $y(t - 0.74)$  of system (21) for  $\alpha = 1$  and  $h = 0.1$ , whereas Fig. 4(c) shows phase portrait of  $y(t)$  versus  $y(t - 0.74)$  for the same value of  $\alpha$ . The numerical results for variable-order FDEs at different values of  $\alpha(t)$  are given in Figs. 5 and 6.

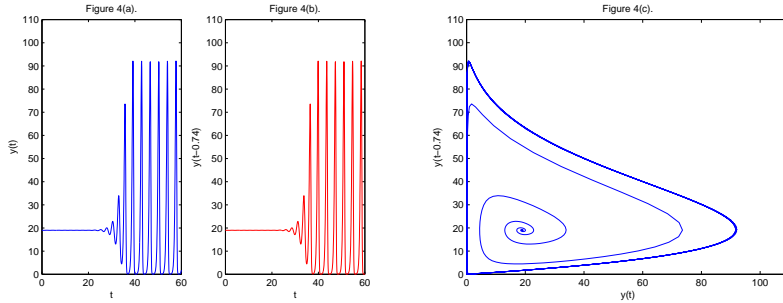


FIGURE 4. The numerical behavior of system (21) and chaotic attractors at  $\alpha = 1$ .

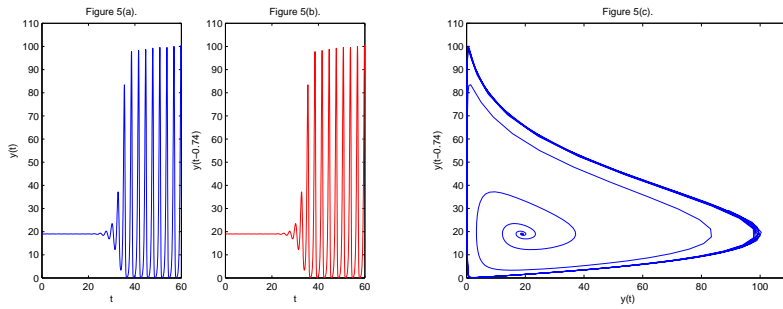


FIGURE 5. The numerical behavior of system (21) and chaotic attractors at  $\alpha(t) = 0.99 - (0.01/100)t$ .

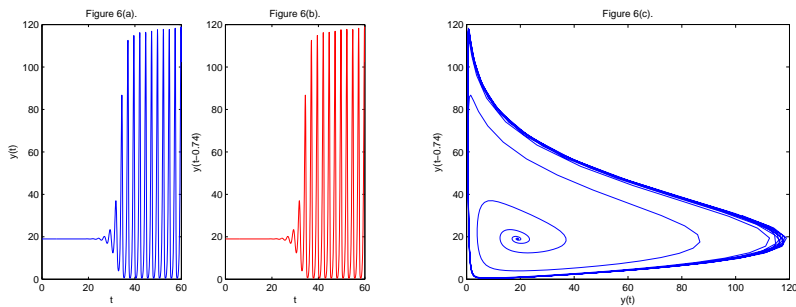


FIGURE 6. The numerical behavior of system (21) and chaotic attractors at  $\alpha(t) = 0.95 - (0.01/100)t$ .

**Example 3** Consider the variable order version of four dimensional enzyme kinetics with an inhibitor molecule which be an extension of the model given in [21]

$$\begin{aligned} \mathcal{D}_t^{\alpha(t)} y_1(t) &= 10.5 - \frac{y_1(t)}{1 + 0.0005y_4^3(t-4)}, \\ \mathcal{D}_t^{\alpha(t)} y_2(t) &= \frac{y_1(t)}{1 + 0.0005y_4^3(t-4)} - y_2(t), \\ \mathcal{D}_t^{\alpha(t)} y_3(t) &= y_2(t) - y_3(t), \\ \mathcal{D}_t^{\alpha(t)} y_4(t) &= y_3(t) - 0.5y_4(t), \\ y(t) &= [60, 10, 10, 20]^T, \quad t \leq 0. \end{aligned} \quad (22)$$

Fig. 7(a), shows the solutions  $y_i(t)$ ,  $i = 1, \dots, 4$  for  $\alpha = 1$ , whereas 7(b) and 7(c) show the numerical results in case of variable-order at  $\alpha(t) = 0.99 - (0.01/100)t$  and  $\alpha(t) = 0.95 - (0.01/100)t$ , respectively.

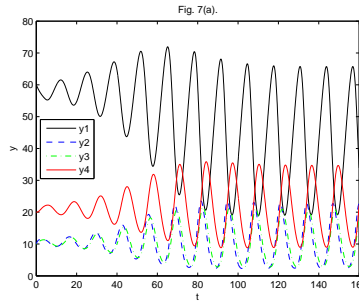


FIGURE 7. The numerical behavior of system (22) at  $\alpha = 1$ .

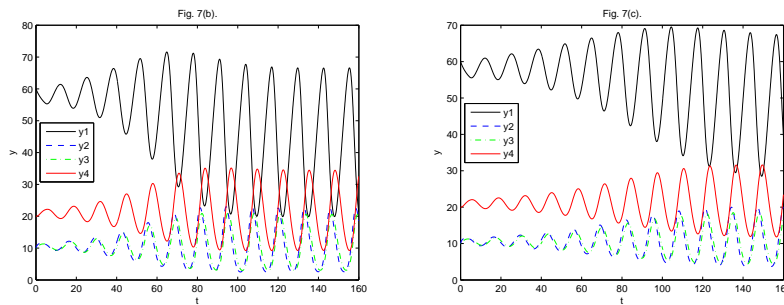


FIGURE 8. The numerical behavior of system (22) at  $\alpha(t) = 0.99 - (0.01/100)t$  (left) and  $\alpha(t) = 0.95 - (0.01/100)t$  (right).

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