EXISTENCE OF SOLUTION FOR PERTURBED FRACTIONAL HAMILTONIAN SYSTEMS

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ABSTRACT. The main goal of this work is to prove the existence of two non-trivial solutions for a perturbed fractional Hamiltonian systems given by

$$- {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u(t)) - L(t)u(t) + \nabla W(t, u(t)) = f(t),$$

where $\alpha \in (1/2, 1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and ∇W is the gradient of W at u.

1. INTRODUCTION

In this paper, we shall be concerned with the existence of two solutions for the perturbed fractional Hamiltonian systems

$$- {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u(t)) - L(t)u(t) + \nabla W(t, u(t)) = f(t) \tag{1}$$

where $t \in \mathbb{R}$ and $u \in \mathbb{R}^n$.

Fractional differential equations both ordinary and partial ones are applied in mathematical modeling of processes in physics, mechanics, control theory, biochemistry, bioengineering and economics. Therefore the theory of fractional differential equations is an area intensively developed during last decades [1], [10], [15], [21], [29]. The monographs [11], [16], [18], enclose a review of methods of solving which are an extension of procedures from differential equations theory.

Recently, a great attention has been focused on the study of differential equations including both - left and right fractional derivatives. Apart from their possible applications, equations with left and right fractional derivatives are an interesting and new field in fractional differential equations theory. Some works in this topic can be founded in papers [3], [4], [12], [24], [25], [26], [27], [28], [30] and their references.

If $f = 0$ in (1), in [24] the author considers the following fractional Hamiltonian systems

$$ {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u(t)) + L(t)u(t) = \nabla W(t, u(t)) \tag{2}$$

where $\alpha \in (1/2, 1)$, $t \in \mathbb{R}$, $u \in \mathbb{R}^n$, $L \in C(\mathbb{R}, \mathbb{R}^{n \times n})$ is a symmetric matrix valued function for all $t \in \mathbb{R}$, $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $\nabla W(t, u(t))$ is the gradient of W at u. Assuming that L and W satisfy the following hypotheses:

(L) $L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$, and there exists an $l \in C(\mathbb{R}, (0, \infty))$ such that $l(t) \to +\infty$ as $t \to \infty$ and

$$(L(t)x, x) \geq l(t)|x|^2, \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n. \tag{3}$$

(W_1) $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, and there is a constant $\mu > 2$ such that

$$0 < \mu W(t, x) \leq (x, \nabla W(t, x)), \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \setminus \{0\}.$$

(W_2) $|\nabla W(t, x)| = o(|x|)$ as $x \to 0$ uniformly with respect to $t \in \mathbb{R}$.

(W_3) There exists $\overline{W} \in C(\mathbb{R}^n, \mathbb{R})$ such that

$$|W(t, x)| + |\nabla W(t, x)| \leq |\overline{W(x)}| \text{ for every } x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}.$$

It showed that (2) has at least one nontrivial solution via Mountain pass theorem. Very recently Zhang and Yuan [30], using the genus properties of critical point theory, generalized the result of [24] and established some new criterion to guarantee the existence of infinitely many solutions of (2) for the case that $W(t, u)$ is subquadratic as $|u| \to +\infty$. Explicitly, L satisfies (L) and the potential $W(t, u)$ is supposed to satisfy the following conditions:

$(HS)_1$ $W(t, 0) = 0$ for all $t \in \mathbb{R}$, $W(t, u) \geq a(t)|u|^\theta$ and $|\delta W(t, u)| \leq b(t)|u|^{\theta-1}$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^n$, where $\theta < 2$ is a constant, $a : \mathbb{R} \to \mathbb{R}^+$ is a bounded continuous function and $b : \mathbb{R} \to \mathbb{R}^+$ is a continuous function such that $b \in L^{\frac{2}{2-\theta}}(\mathbb{R})$;

$(HS)_2$ There is a constant $1 < \sigma \leq \theta < 2$ such that

$$(W(t, u), u) \leq \sigma W(t, u) \quad \text{for all } t \in \mathbb{R} \text{ and } u \in \mathbb{R}^n \setminus \{0\};$$

$(HS)_3$ $W(t, u)$ is even in u, i.e. $W(t, u) = W(t, -u)$ for all $t \in \mathbb{R}$ and $u \in \mathbb{R}^n$.

In this paper we extend the result of Torres [24] and Zhang and Yuan [30] to the case $f \neq 0$. For that purpose throughout the paper, $(.,.) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the standar scalar product in $\mathbb{R}^n$ and $|.|$ is the induced norm. Let

$$X^\alpha = \left\{ u \in H^\alpha(\mathbb{R}, \mathbb{R}^n) : \int_{\mathbb{R}} [|_{-\infty} D_t^\alpha u(t)|^2 + (L(t)u(t), u(t))]dt < \infty \right\}.$$

X^α is a Hilbert space under the norm

$$\|u\|_{X^\alpha} = \left(\int_{\mathbb{R}} [|_{-\infty} D_t^\alpha u(t)|^2 + (L(t)u(t), u(t))]dt \right)^{1/2}.$$

From (L) it follows that there is a constant $C_e > 0$ such that for every $u \in X^\alpha$,

$$\|u\|_\alpha \leq C_\epsilon \|u\|_{X^\alpha}, \tag{4}$$

see [24] for more details. Set

$$M = \max_{|u|=1} \overline{W}(u).$$

We will also assume that

(W_f) $M < \frac{1}{2C_\alpha^2 C_e^2}$ and $f : \mathbb{R} \to \mathbb{R}^n$ is a continuous square integrable function such that

$$\|f\|_{L^2} < \frac{1}{2C_\alpha^2 C_e^2} - M,$$

where $C_\alpha > 0$ (see Theorem 2.1, below).

We are going to prove the following theorem.

Theorem 1.1 Suppose that $(L) - (W_3)$ and (W_f) hold, then (1) possesses at least two nontrivial solution in X^α.

Under this assumption, the problem of existence of solutions is much more delicate, because extra difficulties arise in studying the properties of the corresponding action functional $I : X^\alpha \to R$.

The problem here is as follows. We are given two sequences of almost critical points in X^α. The first one, obtained by Ekeland's variational principle, is contained in a small ball centered at 0. Using the mountain pass geometry of the action functional, the existence of the second sequence is established. Both sequences are weakly convergent in X^α. The question is whether their weak limits are equal to each other or they define two geometrically distinct solutions of (1). The PS-condition is enough to obtain two solutions.

The assumption (L) ensures the PS-condition at each level. In fact one needs the PS-condition only at two levels and therefore it is tempting to seek for weaker compactness assumptions.

The rest of the paper is organized as follows: In section §2, we describe the Liouville-Weyl fractional calculus and we introduce the fractional space that we use in our work and some proposition are proven which will aid in our analysis. In section §3, we will prove Theorem 1.

2. PRELIMINARY RESULTS

2.1. **Liouville-Weyl Fractional Calculus.** In this section we introduce some basic definitions of fractional calculus which are used further in this paper. For more details we refer the reader to [10].

The Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ are defined as

$$_{-\infty}I_x^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x - \xi)^{\alpha-1} u(\xi) d\xi \tag{5}$$

$$_x I_\infty^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (\xi - x)^{\alpha-1} u(\xi) d\xi \tag{6}$$

The Liouville-Weyl fractional derivative of order $0 < \alpha < 1$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals

$$_{-\infty}D_x^\alpha u(x) = \frac{d}{dx} {}_{-\infty}I_x^{1-\alpha} u(x) \tag{7}$$

$$_x D_\infty^\alpha u(x) = -\frac{d}{dx} {}_x I_\infty^{1-\alpha} u(x) \tag{8}$$

The definitions (7) and (8) may be written in an alternative form:

$$_{-\infty}D_x^\alpha u(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{u(x) - u(x - \xi)}{\xi^{\alpha+1}} d\xi \tag{9}$$

$$_x D_\infty^\alpha u(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{u(x) - u(x + \xi)}{\xi^{\alpha+1}} d\xi \tag{10}$$

We establish the Fourier transform properties of the fractional integral and fractional differential operators. Recall that the Fourier transform $\widehat{u}(w)$ of $u(x)$ is

defined by

$$\widehat{u}(w) = \int_{-\infty}^{\infty} e^{-ix.w} u(x) dx.$$

Let $u(x)$ be defined on $(-\infty, \infty)$. Then the Fourier transform of the Liouville-Weyl integral and differential operator satisfies

$$\widehat{_{-\infty}I_x^\alpha u(x)}(w) = (iw)^{-\alpha}\widehat{u}(w) \tag{11}$$

$$\widehat{_xI_\infty^\alpha u(x)}(w) = (-iw)^{-\alpha}\widehat{u}(w) \tag{12}$$

$$\widehat{_{-\infty}D_x^\alpha u(x)}(w) = (iw)^{\alpha}\widehat{u}(w) \tag{13}$$

$$\widehat{_xD_\infty^\alpha u(x)}(w) = (-iw)^{\alpha}\widehat{u}(w) \tag{14}$$

2.2. **Fractional Derivative Spaces.** In this section we introduce some fractional spaces for more detail see [8]. Let $\alpha > 0$. Define the semi-norm

$$|u|_{I_{-\infty}^\alpha} = \|_{-\infty}D_x^\alpha u\|_{L^2}$$

and norm

$$\|u\|_{I_{-\infty}^\alpha} = \left(\|u\|_{L^2}^2 + |u|_{I_{-\infty}^\alpha}^2\right)^{1/2}, \tag{15}$$

and let

$$I_{-\infty}^\alpha(\mathbb{R}, \mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R}, \mathbb{R}^n)}^{\|.\|_{I_{-\infty}^\alpha}}.$$

Now we define the fractional Sobolev space $H^\alpha(\mathbb{R}, \mathbb{R}^n)$ in terms of the Fourier transform. Let $0 < \alpha < 1$, let the semi-norm

$$|u|_\alpha = \||w|^\alpha\widehat{u}\|_{L^2} \tag{16}$$

and norm

$$\|u\|_\alpha = \left(\|u\|_{L^2}^2 + |u|_\alpha^2\right)^{1/2},$$

and let

$$H^\alpha(\mathbb{R}, \mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R}, \mathbb{R}^n)}^{\|.\|_\alpha}.$$

We note a function $u \in L^2(\mathbb{R}, \mathbb{R}^n)$ belong to $I_{-\infty}^\alpha(\mathbb{R}, \mathbb{R}^n)$ if and only if

$$|w|^\alpha\widehat{u} \in L^2(\mathbb{R}, \mathbb{R}^n). \tag{17}$$

Especially

$$|u|_{I_{-\infty}^\alpha} = \||w|^\alpha\widehat{u}\|_{L^2}. \tag{18}$$

Therefore $I_{-\infty}^\alpha(\mathbb{R}, \mathbb{R}^n)$ and $H^\alpha(\mathbb{R}, \mathbb{R}^n)$ are equivalent with equivalent semi-norm and norm. Analogous to $I_{-\infty}^\alpha(\mathbb{R}, \mathbb{R}^n)$ we introduce $I_\infty^\alpha(\mathbb{R}, \mathbb{R}^n)$. Let the semi-norm

$$|u|_{I_\infty^\alpha} = \|_xD_\infty^\alpha u\|_{L^2}$$

and norm

$$\|u\|_{I_\infty^\alpha} = \left(\|u\|_{L^2}^2 + |u|_{I_\infty^\alpha}^2\right)^{1/2}, \tag{19}$$

and let

$$I_\infty^\alpha(\mathbb{R}, \mathbb{R}^n) = \overline{C_0^\infty(\mathbb{R}, \mathbb{R}^n)}^{\|.\|_{I_\infty^\alpha}}.$$

Moreover $I_{-\infty}^\alpha(\mathbb{R}, \mathbb{R}^n)$ and $I_\infty^\alpha(\mathbb{R}, \mathbb{R}^n)$ are equivalent, with equivalent semi-norm and norm [8]. We recall the Sobolev Lemma.

Theorem 2.1 [24] If $\alpha > \frac{1}{2}$, then $H^\alpha(\mathbb{R}, \mathbb{R}^n) \subset C(\mathbb{R}, \mathbb{R}^n)$ and there is a constant $C = C_\alpha$ such that

$$\sup_{x \in \mathbb{R}} |u(x)| \leq C\|u\|_\alpha. \tag{20}$$

Remark 2.1 If $u \in H^{\alpha}(\mathbb{R}, \mathbb{R}^n)$, then $u \in L^q(\mathbb{R}, \mathbb{R}^n)$ for all $q \in [2, \infty]$, since

$$\int_{\mathbb{R}} |u(x)|^q dx \leq \|u\|_{\infty}^{q-2} \|u\|_{L^2}^2.$$

We introduce a new fractional spaces. Let

$$X^{\alpha} = \left\{ u \in H^{\alpha}(\mathbb{R}, \mathbb{R}^n) | \quad \int_{\mathbb{R}} |_{-\infty} D_t^{\alpha} u(t)|^2 + (L(t)u(t), u(t)) dt < \infty \right\}.$$

The space X^{α} is a Hilbert space with the inner product

$$\langle u, v \rangle_{X^{\alpha}} = \int_{\mathbb{R}} (_{-\infty} D_t^{\alpha} u(t), \ _{-\infty} D_t^{\alpha} v(t)) + (L(t)u(t), v(t)) dt$$

and the corresponding norm

$$\|u\|_{X^{\alpha}}^2 = \langle u, u \rangle_{X^{\alpha}}.$$

Lemma 2.1 [24] Suppose L satisfies (L). Then X^{α} is continuously embedded in $H^{\alpha}(\mathbb{R}, \mathbb{R}^n)$.

Lemma 2.2 [24] Suppose L satisfies (L). Then the imbedding of X^{α} in $L^2(\mathbb{R}, \mathbb{R}^n)$ is compact.

Lemma 2.3 [24]

$$W(t, u) \geq W(t, \frac{u}{|u|})|u|^{\mu}, \quad |u| \geq 1 \tag{21}$$

and

$$W(t, u) \leq W(t, \frac{u}{|u|})|u|^{\mu}, \quad |u| \leq 1 \tag{22}$$

Remark 2.2 By Lemma 2.3, we have

$$W(t, u) = o(|u|^2) \text{ as } u \to 0 \text{ uniformly in } t \in \mathbb{R}. \tag{23}$$

In addition, by (W_2), we have, for any $u \in \mathbb{R}^n$ such that $|u| \leq M_1$, there exists some constant $d > 0$ (dependent on M_1) such that

$$|\nabla W(t, u(t))| \leq d|u(t)|. \tag{24}$$

Lemma 2.4 [24] Suppose that (L), (W_1)-(W_2) are satisfied. If $u_k \rightharpoonup u$ in X^{α}, then $\nabla W(t, u_k) \to \nabla W(t, u)$ in $L^2(\mathbb{R}, \mathbb{R}^n)$.

Now we introduce more notations and some necessary definitions. Let $\mathfrak{B}$ be a real Banach space, $I \in C^1(\mathfrak{B}, \mathbb{R})$, which means that I is a continuously Frchet-differentiable functional defined on $\mathfrak{B}$. Recall that $I \in C^1(\mathfrak{B}, \mathbb{R})$ is said to satisfy the (PS) condition if any sequence $\{u_k\}_{k \in \mathbb{N}} \in \mathfrak{B}$, for which $\{I(u_k)\}_{k \in \mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$, possesses a convergent subsequence in $\mathfrak{B}$.

Moreover, let B_r be the open ball in $\mathfrak{B}$ with the radius r and centered at 0 and ∂B_r denote its boundary. We obtain the existence of homoclinic solutions of (2) by use of the following well-known Mountain Pass Theorems, see [20].

Theorem 2.2 (Mountain Pass Theorem) Let $\mathfrak{B}$ be a real Banach space and $I \in C^1(\mathfrak{B}, \mathbb{R})$ satisfying (PS) condition. Suppose that $I(0) = 0$ and

    i. There are constants $\rho, \beta > 0$ such that $I|_{\partial B_{\rho}} \geq \beta$, and

    ii. There is and $e \in \mathfrak{B} \setminus \overline{B_{\rho}}$ such that $I(e) \leq 0$.

Then I possesses a critical value $c \geq \beta$. Moreover c can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s)),$$

where
$$\Gamma = \{\gamma \in C([0,1], \mathfrak{B}) : \;\; \gamma(0) = 0, \;\; \gamma(1) = e\}.$$

3. Proof of Main Theorem

Now we are going to establish the corresponding variational framework to obtain the existence of solutions for (1). Define the functional $I : X^\alpha \to \mathbb{R}$ by

$$
\begin{aligned}
I(u) &= \int_{\mathbb{R}} \left[\frac{1}{2} |_{-\infty}D_t^\alpha u(t)|^2 + \frac{1}{2}(L(t)u(t), u(t)) - W(t, u(t)) + (f(t), u(t)) \right] dt \\
&= \frac{1}{2} \|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, u(t))dt + \int_{\mathbb{R}} (f(t), u(t))dt. \quad\quad (25)
\end{aligned}
$$

Then $I \in C^1(X^\alpha, \mathbb{R})$ and it is easy to check that

$$
I'(u)v = \int_{\mathbb{R}} [(_{-\infty}D_t^\alpha u(t), _{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t)) + (f(t), v(t))] \, dt \quad (26)
$$

for all $u, v \in X^\alpha$, which yields that

$$
I'(u)u = \|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} (\nabla W(t, u(t)), u(t))dt + \int_{\mathbb{R}} (f(t), u(t)). \quad\quad (27)
$$

In order to prove Theorem 1.1 we use the mountain pass theorem and Ekeland's variational principle [14], [20]. The proof will be divided into a sequence of lemmas.
Lemma 3.1 Suppose that $(L) - (W_3)$ and (W_f) holds. Then I satisfies the PS-condition.
Proof. Assume that $(u_k)_{k \in \mathbb{N}} \in X^\alpha$ is a sequence such that

$$
I(u_k) \to c \quad \text{and} \quad I'(u_k) \to 0 \quad \text{as} \;\; k \to +\infty. \quad\quad (28)
$$

We have to show that $\{u_k\}_{k \in \mathbb{N}}$ possesses a convergent subsequence in X^α. We firstly prove that $\{u_k\}_{k \in \mathbb{N}}$ is bounded in X^α. By (25), (27) we get

$$
\begin{aligned}
I(u_k) - \frac{1}{\mu}I'(u_k)u_k &= \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_k\|_{X^\alpha}^2 \\
&\quad - \int_{\mathbb{R}} \left[W(t, u_k(t)) - \frac{1}{\mu}(\nabla W(t, u_k(t)), u_k(t)) \right] dt \\
&\quad + \left(1 - \frac{1}{\mu} \right) \int_{\mathbb{R}} (f(t), u_k(t))dt. \quad\quad (29)
\end{aligned}
$$

From (W_1) and Lemma 2.1, it follows that

$$
I(u_k) - \frac{1}{\mu}I'(u_k)u_k \geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_k\|_{X^\alpha}^2 - \left(C_e - \frac{C_e}{\mu} \right) \|f\|_{L^2} \|u_k\|_{X^\alpha},
$$

where C_e denote the continuous embedding constant given by Lemma 2.1. On the other hand, by (28), there is $k_0 \in \mathbb{N}$ such that for $k \geq k_0$,

$$
c + 1 + \|u_k\|_{X^\alpha} \geq I(u_k) - \frac{1}{\mu}I'(u_k)u_k.
$$

In consequence, since $\mu > 2$, $\{u_k\}_{k \in \mathbb{N}}$ is bounded in X^α. Since X^α is a Hilbert space, passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u$ in X^α

and hence, by Lemma 2.2, $u_k \to u$ in $L^2(\mathbb{R}, \mathbb{R}^n)$. It follows from the definition of I that

$$
\begin{aligned}
\|u_k - u\|_{X^\alpha}^2 &= (I'(u_k) - I'(u))(u_k - u) \\
&\quad + \int_{\mathbb{R}} [\nabla W(t, u_k) - \nabla W(t, u)](u_k - u)dt. \tag{30}
\end{aligned}
$$

Since $u_k \to u$ in $L^2(\mathbb{R}, \mathbb{R}^n)$, by Lemma 2.3, $\nabla W(t, u_k(t)) \to \nabla W(t, u(t))$ in $L^2(\mathbb{R}, \mathbb{R}^n)$. Hence

$$
\int_{\mathbb{R}} (\nabla W(t, u_k(t)) - \nabla W(t, u(t)), u_k(t) - u(t))dt \to 0,
$$

as $k \to +\infty$. So (30) implies

$$
\|u_k - u\|_{X^\alpha} \to 0 \quad \text{as} \quad k \to +\infty.
$$

$\square$

Lemma 3.2 Suppose that $(L) - (W_3)$ and (W_f) holds. There are $\rho > 0$ and β such that $I(u) \geq \beta$ for $\|u\|_{X^\alpha} = \rho$.

Proof. Let $\rho = \frac{1}{C_\alpha C_e}$. Assume that $u \in X^\alpha$ and $\|u\|_\alpha \leq \rho$. By Theorem 2.1 and (4)

$$
\|u\|_\infty \leq C_\alpha C_e \|u\|_{X^\alpha} \leq 1
$$

By (22) and (W_3), we get

$$
\begin{aligned}
I(u) &= \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, u(t))dt + \int_{\mathbb{R}} (f(t), u(t))dt \\
&\geq \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, \frac{u}{|u|})|u|^\mu dt - \|f\|_{L^2}\|u\|_{L^2} \\
&\geq \frac{1}{2}\|u\|_{X^\alpha}^2 - M\|u\|_\infty^{\mu-2} \int_{\mathbb{R}} |u|^2 dt - C_\alpha C_e \|f\|_{L^2}\|u\|_{X^\alpha} \\
&\geq \frac{1}{2}\|u\|_{X^\alpha}^2 - M C_\alpha^2 C_e^2 \|u\|_{X^\alpha}^2 - C_\alpha C_e \|f\|_{L^2}\|u\|_{X^\alpha}. \tag{31}
\end{aligned}
$$

Hence, if $\|u\|_{X^\alpha} = \rho$, we have

$$
I(u) \geq \frac{1}{2C_\alpha^2 C_e^2} - M - \|f\|_{L^2} = \beta > 0,
$$

by (W_f). $\square$

Lemma 3.3 Suppose that $(L) - (W_3)$ and (W_f) holds. There is $e \in X^\alpha \setminus \overline{B(0, \rho)}$ such that $I(e) \leq 0$, where $B(0, \rho)$ is a ball in X^α of radius ρ centered at 0 and ρ is given by Lemma 3.2.

Proof. Fix $u \in X^\alpha$ such that $|u(t)| = 1$ for all $t \in [0, 1]$ and assume that $\sigma \geq 1$. Then by (21)

$$
\begin{aligned}
I(\sigma u) &= \frac{\sigma^2}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, \sigma u(t))dt + \sigma \int_{\mathbb{R}} (f(t), u(t))dt \\
&\leq \frac{\sigma^2}{2}\|u\|_{X^\alpha}^2 - \sigma^\mu \int_0^1 W\left(t, \frac{u(t)}{|u(t)|}\right)|u(t)|^\mu dt + \sigma \int_{\mathbb{R}} (f(t), u(t))dt \\
&\leq \frac{\sigma^2}{2}\|u\|_{X^\alpha}^2 - \sigma^\mu m \int_0^1 |u(t)|^\mu dt + \sigma \int_{\mathbb{R}} (f(t), u(t))dt,
\end{aligned}
$$

where

$$
m = \min_{t \in [0,1], |u|=1} W(t, u).
$$

Since $\mu > 2$, $I(\sigma u) \to -\infty$ as $\sigma \to \infty$. Hence there is $\sigma \geq 1$ such that $\|\sigma u\|_{X^\alpha} > \rho$ and $I(\sigma u) \leq 0$. $\square$

Proof of Theorem 1.1 Since $I(0) = 0$ and I satisfies Lemmas 3.1 - 3.3, it follows by the mountain pass Theorem that I has a critical value c given by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X^\alpha) : \ \gamma(0) = 0, \ I(\gamma(1)) < 0\}.$$

By definition, it follows that $c \geq \beta > 0$. By (31), I is bounded from below on $\overline{B(0, \rho)}$. Let

$$c_1 = \inf_{\|u\|_{X^\alpha} \leq \rho} I(u). \tag{32}$$

Since $I(0) = 0$, $c_1 \leq 0$. Thus $c_1 \leq c$. By Ekeland's variational principle, there is a minimizing sequence $\{w_k\}_{k \in \mathbb{N}} \subset \overline{B(0, \rho)}$ such that

$$I(w_k) \to c_1 \quad \text{and} \quad I'(w_k) \to 0,$$

as $k \to \infty$. From Lemma 3.1, c_1 is a critical value of I. Consequently, I has at least two critical points. $\square$

References

[1] O. Agrawal, J. Tenreiro Machado and J. Sabatier, Fractional derivatives and their application: Nonlinear dynamics, Springer-Verlag, Berlin, 2004.
[2] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical points theory and applications, J. Func. Anal. Vol. 14, 349-381, 1973.
[3] T. Atanackovic and B. Stankovic, On a class of differential equations with left and right fractional derivatives, ZAMM., Vol. 87, 537-539, 2007.
[4] D. Baleanu and J. Trujillo, On exact solutions of a class of fractional Euler-Lagrange equations, Nonlinear Dyn., Vol. 52, 331-335, 2008.
[5] D. Benson, S. Wheatcraft and M. Meerschaert, Application of a fractional advection-dispersion equation, Water Resour. Res. Vol. 36, 1403-1412, 2000.
[6] D. Benson, S. Wheatcraft and M. Meerschaert, The fractional-order governing equation of Lévy motion, Water Resour. Res. Vol. 36, 1413-1423, 2000.
[7] D. Benson, R. Schumer and M. Meerschaert, Fractional dispersion, Lévy motion, and the MADE tracer test, Transp. Porous Med. Vol. 42, 211-240, 2001.
[8] V. Ervin and J. Roop, Variational formulation for the stationary fractional advection dispersion equation, Numer. Meth. Part. Diff. Eqs, Vol. 22, 58-76, 2006.
[9] R. Herrmann, Fractional calculus: An introduction for physicists 2 ed., World Scientific Publishing, 2014.
[10] R. Hilfer, Applications of fractional calculus in physics, World Scientific, Singapore, 2000.
[11] A. Kilbas, H. Srivastava and J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies, vol 204, Amsterdam, 2006.
[12] M. Klimek, Existence and uniqueness result for a certain equation of motion in fractional mechanics, Bull. Polish Acad. Sci. Tech. Sci., Vol. 58, N 4, 573-581, 2010.
[13] F. Jiao and Y. Zhou, Existence results for fractional boundary value problem via critical point theory, Intern. Journal of Bif. and Chaos, Vol. 22, N 4, 1-17, 2012.
[14] J. Mawhin and M. Willen, Critical point theory and Hamiltonian systems, Applied Mathematical Sciences 74, Springer, Berlin, 1989.
[15] R. Metsler and J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, J. Phys. A, Vol. 37, 161-208, 2004.
[16] K. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, Wiley and Sons, New York, 1993.
[17] W. Omana and M. Willem, Homoclinic orbits for a class of Hamiltonian systems, Differ. Integr. Equ., Vol. 5, No 5, 1115-1120, 1992.

[18] I. Podlubny, Fractional differential equations, Academic Press, New York, 1999.

[19] P. Rabinowitz and K. Tanaka, Some result on connecting orbits for a class of Hamiltonian systems, Math. Z., Vol. 206, 473-499, 1991.

[20] P. Rabinowitz, Minimax method in critical point theory with applications to differential equations, CBMS Amer. Math. Soc., No 65, 1986.

[21] J. Sabatier, O. Agrawal and J. Tenreiro Machado, Advances in fractional calculus. Theoretical developments and applications in physics and engineering, Springer-Verlag, Berlin, 2007.

[22] S. Samko, A. Kilbas and O. Marichev, Fractional integrals and derivatives: Theory and applications, Gordon and Breach, New York, 1993.

[23] C. Stuart, Bifurcation into spectral gaps, Socit Mathmatique de Belgique, 1995.

[24] C. Torres, Existence of solution for fractional Hamiltonian systems, Electronic Jour. Diff. Eq., Vol 2013, No 259, 1-12, 2013.

[25] C. Torres, Mountain pass solution for a fractional boundary value problem, Journal of Fractional Calculus and Applications, Vol. 5, No 1, 1-10, 2014.

[26] C. Torres, Existence of a solution for fractional forced pendulum, Journal of Applied Mathematics and Computational Mechanics, Vol. 13, No 1, 125-142, 2014.

[27] C. Torres, Fractional Hamiltonian systems with Liouville-Weyl fractional derivatives, preprint.

[28] C. Torres, Ground state solution for a class of differential equations with left and right fractional derivatives, arXiv:1308.4215v1 [math.AP] 20 Aug 2013.

[29] B. West, M. Bologna and P. Grigolini, Physics of fractal operators, Springer-Verlag, Berlin, 2003.

[30] Z. Zhang and R. Yuang, Variational approach to solutions for a class of fractional Hamiltonian systems, Math. Methods Appl. Sci., DOI:10.1002/mma.2941.

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