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SUBORDINATION RESULT FOR A CLASS OF ANALYTIC FUNCTIONS WITH MISSING COEFFICIENTS

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Abstract. In the present investigation we consider a new class of functions $\mathcal{R}_n(\alpha,\gamma)$ and prove some subordination result. Our result gives result of Owa and Ma [4].

1. Introduction and Preliminaries

Let $\mathcal{A}_n$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \ (n \in \mathbb{N}) \tag{1}$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. A domain $D \subset \mathbb{C}$ is convex if the line segment joining any two points in D lies entirely in D. A univalent function $f \in \mathcal{A}_n$ is convex if $f(\Delta)$ is convex. Analytically, a univalent function $f \in \mathcal{A}_n$ is convex if and only if

$$\mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0. \tag{2}$$

Further, a function $f(z)$ in the class $\mathcal{A}_n$ is said to be close-to-convex of order $\alpha(0 \le \alpha < 1)$ in the unit disk Δ if there exists a convex function $g(z) \in \mathcal{A}_n$ such that

$$\mathfrak{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha \ (z \in \Delta) \tag{3}$$

The concept of close-to-convex functions was introduced by Kaplan [2].

Definition 1.1. Let $0 \le \gamma \le 1,\ 0 \le \alpha < 1$. A function $f(z) \in \mathcal{A}_n$ is said to be in the class $\mathcal{R}_n(\alpha,\gamma)$ if and only if satisfies

$$\left|(1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - 1\right| < 1 - \alpha \ (z \in \Delta). \tag{4}$$

Note that $f(z) \in \mathcal{R}_n(\alpha,\gamma)$ gives

$$\mathfrak{Re}\left\{(1-\gamma)\frac{f(z)}{z} + \gamma f'(z)\right\} > \alpha. \tag{5}$$

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Obviously $f(z)=z$ belongs to the class $\mathcal{R}_n(\alpha,\gamma)$. The class $\mathcal{R}_n(\alpha,\gamma)$ is a subclass of class $\mathcal{P}_\gamma^\tau(\beta)$ defined by Swaminathan [6]. If $\gamma=1$, we get the class $\mathcal{R}_n(\alpha)$ defined in Owa and Ma [4]. For $\gamma=0$ we get the class $\mathcal{A}_n(\alpha)$ defined by Owa and Hu [5].

An analytic function f is subordinate to an analytic function g, written as $f(z)\prec g(z)$ $(z\in\mathbb{U})$, if there is an analytic function w defined on Δ with $w(0)=0$ and $|w(z)|<1$, $z\in\Delta$ such that $f(z)=g(w(z))$. In particular, if g is univalent in Δ then we have the following equivalence:

$$f(z)\prec g(z) \iff f(0)=g(0) \text{ and } f(\Delta)\subset g(\Delta).$$

In order to prove our main result we need following lemma due to Miller and Mocanu [3], see also Jack [1].

Lemma 1.2. Let the function

$$w(z)=b_nz^n+b_{n+1}z^{n+1}+...(n\in\mathbb{N}) \tag{6}$$

be analytic in Δ with $w(z)$ is not identically zero. If $z_0=r_0e^{i\theta_0}$ $(r_0<1)$ and

$$|w(z_0)|=\max\left\{|w(z);|z|\le|z_0|\right\}, \tag{7}$$

then

$$z_0w'(z_0)=mw(z_0).$$

where m is real and $m\ge n\ge 1$.

2. Main Results

Theorem 2.1. *Let the function $f(z)$ defined by (1) be in the class $\mathcal{R}_n(\alpha,\gamma)$. Then*

$$\frac{f(z)}{z}\prec 1+\frac{(1-\alpha)z}{1+\gamma n}. \tag{8}$$

Proof. It is clear that the result is true if $f(z)=z$. Then, we assume that $f(z)\ne z$. Define the analytic function $w(z)$ in the unit disk Δ by

$$\frac{f(z)}{z}=1+\frac{(1-\alpha)w(z)}{1+\gamma n}, \tag{9}$$

then we see that

$$w(z)=b_nz^n+b_{n+1}z^{n+1}+...(n\in\mathbb{N}). \tag{10}$$

Obviously $w(0)=0$ and $w(z)$ is not identically zero since $f(z)$ is not identically equal to z. Now, we need only to prove that $|w(z)|<1$ for all $z\in\Delta$. If not so, there exists a point $z_0\in\Delta$ such that

$$\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1.$$

Therefore, applying our Lemma 1.2, we have

$$z_0w'(z_0)=mw(z_0), \tag{11}$$

where m is real and $m\ge n\ge 1$. Using (9),

$$f'(z)=1+\frac{(1-\alpha)\left[zw'(z)+w(z)\right]}{1+\gamma n}. \tag{12}$$

Now using (9) and (12) we see that

$$(1-\gamma)\frac{f(z_0)}{z_0}+\gamma f'(z_0)-1=\frac{1-\alpha}{1+\gamma n}\left[w(z_0)+\gamma z_0w'(z_0)\right].$$

Applying (11), we have

$$(1-\gamma)\,\frac{f(z_0)}{z_0}+\gamma f'(z_0)-1=\frac{(1-\alpha)w(z_0)}{1+\gamma n}\left[1+m\gamma\right].$$

Thus

$$\left|(1-\gamma)\,\frac{f(z_0)}{z_0}+\gamma f'(z_0)-1\right|=\frac{(1-\alpha)(1+m\gamma)}{1+\gamma n}\geq 1-\alpha.$$

This contradicts that $f(z)$ belongs to the class $\mathcal{R}_n(\alpha,\gamma)$. Therefore, we complete the proof of theorem.□

It follows from theorem the following

Remark 2.2 For $\gamma=1$ in Theorem 2.1, we get the result obtained by Owa and Ma [4] in Theorem 1.

Corollary 2.3. *If the function $f(z)$ defined by (1) is in the class $\mathcal{R}_n(\alpha,\gamma)$, then*

$$\left|Arg\left(\frac{f(z)}{z}\right)\right|\leq \sin^{-1}\left(\frac{1-\alpha}{1+n\gamma}\right).$$

The bound is best possible for the function $f(z)$ defined by

$$f(z)=z+\frac{(1-\alpha)}{1+n\gamma}z^{n+1}\in\mathcal{R}_n(\alpha,\gamma).$$

Theorem 2.4. *Let the function $f(z)$ defined by (1) be in the class $\mathcal{R}_n(\alpha,\gamma)$. Then*

$$|a_k|\leq\frac{1-\alpha}{1-\gamma(1-k)}\quad k>n, \tag{13}$$

and

$$\sum_{k=n+1}^{\infty}\frac{1-\gamma(1-k)}{1-\alpha}\,|a_k|^2<1. \tag{14}$$

Proof. Using (1) we can write the condition (4) as follows

$$\left|\sum_{k=n+1}^{\infty}\frac{1-\gamma(1-k)}{1-\alpha}a_k z^{k-1}\right|<1\ (z\in\Delta) \tag{15}$$

then we see that

$$\sum_{k=n+1}^{\infty}\frac{1-\gamma(1-k)}{1-\alpha}a_k z^{k-1}$$

is the bounded function, hence it has the coefficients bounded by 1. Therefore, we have

$$\left|\frac{1-\gamma(1-k)}{1-\alpha}a_k\right|\leq 1\quad k>n,$$

and we immediately obtain the estimation (13), while (14) also follows immediately from another known property of bounded functions. □

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