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ANALYSIS OF A FRACTIONAL-ORDER PREDATOR-PREY MODEL WITH HARVEST INCORPORATING AN ALLEE EFFECT

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ABSTRACT. The objective of this work is to model a population management in which two species intervene. There is an interdependence predator-prey relationship between them. It will be assumed, as a descriptive biological model, the Lotka-Volterra one with the application of an Allee effect in one of the dynamics of the species. This incorporation into the system make the model more realistic. In view of the advantages, the model will be in its fractional version, that is, where derivatives of Caputo of fractional order are considered, with the fractional order in (0,1]. Existence and uniqueness of the model solution will be explicitly proved, and a non-negative invariance of the solution and the stability of the resulting equilibrium will be studied. In order to solve the problem, the use of fractional numerical techniques will be of fundamental importance and absolutely essential to conclude the analysis. Some examples of great importance will be shown.

1. INTRODUCTION

Predicting the future of a population number is one of the most important factors needed for the good management of it. This has been treated by several known methods, one of them being the development of a mathematical model which describes the population growth. The model generally takes the form of a differential equation, or a system of differential equations, according to the complexity of the underlying properties of the population.

A classical Lotka-Volterra predator-prey mathematical model is a system of nonlinear first-order differential equations that studies the growth of two biological populations occupying the same environment. One species, predators, feed on the

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other species, preys, which in turn feed on a third food widely available in that environment, [15].

In this work, a predator-prey model with harvest incorporating an Allee effect will be studied. The harvest will affect predators, this may be from capture by humans. The so-called Allee effect will modify the growth of preys. The currently most used growth models are those that have a sigmoid solution in time, including Gompertz and Verhulst's logistic equations. The logistic equation is commonly used in population growth models, disease propagation epidemic, and social networks [6]. These equations are based on the assumption that the density has a negative effect on the per-capita growth rate. However, some species often incorporate among themselves in their search for food and to escape from their predators. For example fish and birds often form schools and flocks as a defense against their predators. Some parasitic insect aggregate so that they can overcome the defense mechanism of a host. A number of social species such as ants, termites, bees, etc., have developed complex cooperative behavior involving division of labor, altruism, etc. Such cooperative processes have a positive feedback influence since individuals have been provided a greater chance to survive and reproduce as density increase. Aggregation, associated cooperative and social characteristics among members of a species were extensively studied in animal populations by Allee [4]. The phenomenon in which reproduction rate of individuals decrease when density drops below a certain critical level is now known as the Allee effect, [12] and [27].

On the other hand, motivated by its applications in different scientific areas (electricity, magnetism, mechanics, fluid dynamics, medicine, etc. [5], [8], [11], [20] and [22]), fractional calculus is in development, which has led to great growth in its study in recent decades. The fractional derivative is a nonlocal operator [26], this makes fractional differential equations good candidates for modeling situations in which it is important to consider the history of the phenomenon studied [1], [2], [3], [10], [17], [18], [21] and [24], unlike the models with classical derivative where this is not taken into account. There are several definitions of fractional derivatives [16] and [22]. The most commonly used are the Riemann-Liouville fractional derivative and the Caputo fractional derivative. It is important to remark that while the Riemann-Liouville fractional derivatives [25] are historically the most studied approach to fractional calculus, the Caputo [13] and [14] approach to fractional derivatives is the most popular among physicists and scientists, because the differential equations defined in terms of Caputo derivatives require regular initial and boundary conditions. Furthermore, differential equations with Riemann-Liouville derivatives require nonstandard fractional initial and boundary conditions that lead, in general, to singular solutions, thus limiting their application in physics and science [19] and [20].

In the present work, a predator-prey model with harvest incorporating an Allee effect will be analyzed, in presence of Caputo derivatives. The paper is organized as follows: some basic definitions of fractional derivatives and fractional differential equations are shown in Section 2. Results are presented in Section 3, the model description is in Section 3.1, existence and uniqueness of the problem solution is proved in Section 3.2, its non-negative invariance is demonstrated in Section 3.3, stability of the model is studied in Section 3.4 and an example is shown in Section 3.5. Finally, conclusions are presented in Section 4.

2. Mathematical tools

2.1. **Introduction to fractional calculus.**
In this section, we present some definitions and properties of the Caputo fractional derivatives. For more details on the subject and applications, we refer the reader to [16], [25] and [26].

Definition 1. *The Gamma function, $\Gamma : (0, \infty) \to \mathbb{R}$, is defined by*

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s}\, ds. \tag{1}$$

Definition 2. *The Riemann-Liouville fractional integral operator of order $\alpha \in \mathbb{R}_0^+$ is defined in $L^1[a, b]$ by*

$$ {}_aI_t^\alpha[f] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds. \tag{2}$$

Definition 3. *If $\frac{d^n f}{dt^n} \in L^1[a, b]$, the Caputo fractional derivative of order $\alpha \in \mathbb{R}_0^+$ is defined by*

$$ {}_a^CD_t^\alpha[f] = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-1-\alpha} \frac{d^n}{ds^n} f(s)\, ds \tag{3}$$

Now some different properties of the Caputo derivative will be seen.

Proposition 1. *If K is an arbitrary constant then ${}_a^CD_t^\alpha[K] = 0$.*

Theorem 1. *Let $f(t) \in AC[0, T]$, $T > 0$, where $AC[0, T]$ denotes the set of absolutely continuous functions on $[0, T]$. Then for $0 < \alpha \leq 1$:*

$$f(t) = f(0) + \frac{1}{\Gamma(\alpha+1)} {}_0^CD_\xi^\alpha[f] \cdot t^\alpha, \tag{4}$$

with $0 \leq \xi \leq t$, $\forall t \in [0, T]$.

Remark 1. *Theorem 1 is known as the generalized mean value theorem. When $\alpha = 1$, it reduces to the classical mean value theorem.*

Corollary 1. *Suppose that $f(t) \in AC[0, T]$, $T > 0$ and ${}_0^CD_t^\alpha[f] \in C(0, T]$ for $0 < \alpha \leq 1$. If ${}_0^CD_t^\alpha[f] \geq 0\,({}_0^CD_t^\alpha[f] > 0), \forall t \in (0, T)$, then $f(t)$ is non-decreasing (increasing) and if ${}_0^CD_t^\alpha[f] \leq 0\,({}_0^CD_t^\alpha[f] < 0), \forall t \in (0, T)$, then $f(t)$ is non-increasing (decreasing) for all $t \in [0, T]$.*

Remark 2. *Proposition 1, Theorem 1 and Corollary 1, show that the Caputo fractional derivatives are similar to the classical derivatives in these senses.*

2.2. **Fractional initial value problems.**

Definition 4. *Let $f : \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$, $t > 0$, $u_0 \in \mathbb{R}^n$, and $0 < \alpha \leq 1$. A fractional initial value problem is defined by*

$$\begin{cases} {}_0^CD_t^\alpha[u] = f(t, u(t)), \\ u(0) = u_0. \end{cases} \tag{5}$$

Theorem 2. *Consider $0 < \alpha \leq 1$, $u_0 \in \mathbb{R}$, $K > 0$, $h^* > 0$.*
Define $G := [0, h^*] \times [u_0 - K, u_0 + K]$ and let the function $f : G \to \mathbb{R}$ be continuous. Furthermore, define $M := \sup_{(t,z)\in G} |f(x, z)|$ and

$$h := \begin{cases} h^* & \text{if } M = 0 \\ min\{h^*, (\frac{K\Gamma(\alpha+1)}{M})^{\frac{1}{\alpha}}\} & \text{else.} \end{cases} \tag{6}$$

Then, there exists a function $u \in C[0,h]$ solving the fractional initial value problem (5).

Theorem 3. *Assume the hypotheses of Theorem 2. Moreover assume that $f : G \to \mathbb{R}$ satisfies a Lipschitz condition with respect to the second variable, that is, there exists $L > 0$ independent of t, u_1 and u_2 such that*

$$||f(t,u_1) - f(t,u_2)|| \leq L||u_1 - u_2||, \tag{7}$$

for all (t,u_1) y $(t,u_2) \in G$. Then, denoting h as in Theorem 2, there exists a uniquely defined function $u \in C[0,h]$ solving the fractional initial value problem (5).

Since the exact solution of fractional differential equations can be very difficult to find, an introduction about stability analysis and the fractional Adams numerical method will be useful.

2.3. **Fractional stability analysis.**

Definition 5. *The fractional initial value problem (5) is said to be autonomous if it is defined by*

$$\begin{cases} {}_0^C D_t^\alpha [u] = f(u(t)), \\ u(0) = u_0, \end{cases} \tag{8}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$.

When talking about stability, one is interested in the behavior of the solutions of (8) when $t \to \infty$. Therefore, the equilibrium points are defined as the solutions u_{eq} of

$${}_0^C D_t^\alpha [u] = 0 \iff f(u_{eq}) = 0. \tag{9}$$

To study the stability of each point, the jacobian matrices J of f evaluated at the equilibrium points $J(u_{eq})$ are considered. Then, the eigenvalues λ_{eq} of $J(u_{eq})$ are calculated for each equilibrium point. Finally, conclusions are drawn following the Theorem below, see [2] and [16].

Theorem 4. *Consider λ_{eq} the nonzero eigenvalues of $J(u_{eq})$, jacobian matriz of f asociated to each equilibrium point u_{eq}.*

- *If $|arg(\lambda_{eq})| \geq \frac{\alpha\pi}{2}$ for all λ_{eq} and all eigenvalues with $|arg(\lambda_{eq})| = \frac{\alpha\pi}{2}$ have a geometric multiplicity that coincides with their algebraic multiplicity then u_{eq} is locally stable.*
- *If $|arg(\lambda_{eq})| > \frac{\alpha\pi}{2}$ for all λ_{eq}, then u_{eq} is locally asymptotically stable.*
- *If $|arg(\lambda_{eq})| < \frac{\alpha\pi}{2}$ for some λ_{eq}, then u_{eq} is locally unstable.*

Remark 3. *In the previous theorem, $arg(\lambda_{eq})$ is considered as the main argument of the complex number λ_{eq}, that is $arg(\lambda_{eq}) \in (-\pi, \pi]$. Also, when any of the eigenvalues is zero, the stability of the equilibrium point is said to be a degenerate case.*

2.4. **Fractional numerical method.** To perform numerical implementations a predictor-corrector method is used. Fractional Forward Euler method is utilized to get u_{n+1}^P (predictor), and then fractional Trapezoidal Rule is used to get u_{n+1} (corrector), which leads to fractional Adams method, [7] and [23]. A regular partition of $[0,t]$ is considered, as $t_0 = 0 < t_1 < ... < t_{n+1} = t$ with $t_{j+1} - t_j = \Delta t$. The method approximates the solution $u(t)$ by interpolating the points $(t_j, u_{n+1}(j))$,

$\forall j = 0, ..., n+1$, where $u_{n+1}(j)$ is the $j-$th component of the vector u_{n+1} obtained through the following recursion:

$$\begin{cases} u_{n+1}^P = & \frac{u_0}{j!} + \sum_{j=0}^{n} b_{j,n+1} f(u_j), \\ u_{n+1} = & \frac{u_0}{j!} + \sum_{j=0}^{n} a_{j,n+1} f(u_j) + a_{n+1,n+1} f(u_{n+1}^P), \end{cases} \tag{10}$$

where

$$b_{j,n+1} = \frac{\Delta t^\alpha}{\Gamma(\alpha+1)} \left[(n-j+1)^\alpha - (n-j)^\alpha\right] \tag{11}$$

are the coefficients of the fractional Forward Euler method and

$$a_{j,n+1} = \frac{\Delta t^\alpha}{\Gamma(\alpha+2)} \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha & j=0 \\ (n-j+2)^{\alpha+1} - 2(n-j+1)^{\alpha+1} + (n-j)^{\alpha+1} & 1 \le j \le n \\ 1 & j = n+1 \end{cases} \tag{12}$$

are the coefficients of the fractional Trapezoidal Rule method.

3. Results

The scope of this section is to present a fractional differential equation model of two species in competition. There is an interdependence predator-prey relationship between them. It will be assumed, as a descriptive biological model, the Lotka-Volterra one with the application of an Allee effect in one of the dynamics of the species. Existence and uniqueness of the model solution will be explicitly proved and the stability of the resulting equilibrium will be studied. Finally, some examples will be shown.

3.1. **Model description.**

In this work, the following model will be analyzed, which is a particular case of (8) for two dimensions:

$$\begin{cases} {}_0^C D_t^\alpha [x] = & rx(t)\left(1 - \frac{x(t)}{K}\right)(x(t)-m) - bx(t)y(t), \\ {}_0^C D_t^\alpha [y] = & cx(t)y(t) - dy(t) - ey(t), \\ x(0) = x_0, & y(0) = y_0, \end{cases} \tag{13}$$

where $0 < \alpha \le 1$ and $t > 0$.

In these equations, $x(t)$ represents the number of preys, $y(t)$ represents the number of predators at time t and the parameters r, K, m, b, c, d, e are all positive.

First equation models the growth of preys. First addend corresponds to their growth, where an effect called Allee is considered in which the following parameters intervene: intrinsic growth rate r , carrying capacity K and Allee effect threshold m , which means the minimum population density for the growth of certain species, below which the population dies out (the population growth rate is positive only within the range $m < x < K$ and is negative outside this range). Second addend represents the decrease in preys to be captured, [10] and [9].

In the equation that models predators, it can be seen that the first addend corresponds to their growth by capturing preys, the second addend represents the natural mortality of predators, while the third addend represents their harvest.

3.2. **Existence and uniqueness of the problem solution.** In this section the study of the existence and uniqueness of the solution for system (13) will be seen.

Theorem 5. *There exists an unique solution $u(t) = (x(t), y(t))$ for system (13) with $t \geq 0$.*

Proof. Theorem 3 is considered in region the $G := [0, h^*] \times \Omega$, where the set Ω is defined by $\Omega = [x_0 - H, x_0 + H] \times [y_0 - H, y_0 + H]$.

Define $f : G \to \mathbb{R}^2$ as $f(t, u) = (f_1(t, u), f_2(t, u))$ with

$$f_1(t,u) = rx\left(1 - \frac{x}{K}\right)(x - m) - bxy$$

and

$$f_2(t,u) = cxy - dy - ey,$$

f is continuous in G.

Furthermore, f satisfies Lipschitz condition (7). In effect, let $u = (x, y), \bar{u} = (\bar{x}, \bar{y}) \in \Omega$, then considering $||f(t,u) - f(t,\bar{u})|| = |f_1(t,u) - f_1(t,\bar{u})| + |f_2(t,u) - f_2(t,\bar{u})|$, it is obtained

$$\begin{aligned}
||f(t,u) - f(t,\bar{u})|| &= |rx\left(1 - \tfrac{x}{K}\right)(x - m) - bxy - r\bar{x}\left(1 - \tfrac{\bar{x}}{K}\right)(\bar{x} - m) + b\bar{x}\bar{y}| \\
&+ \ |cxy - dy - ey - c\bar{x}\bar{y} + d\bar{y} + e\bar{y}| \\
&= |r(x^2 - \bar{x}^2) + rm(\bar{x} - x) + \tfrac{r}{K}(\bar{x}^3 - x^3) - \tfrac{r}{K}m(\bar{x}^2 - x^2) \\
&+ \ b(\bar{x}\bar{y} - xy)| + \ |c(xy - \bar{x}\bar{y}) + d(\bar{y} - y) + e(\bar{y} - y)| \\
&\leq r2H|x - \bar{x}| + rm|x - \bar{x}| + 3H^2\tfrac{r}{K}|x - \bar{x}| + 2H\tfrac{r}{K}m|x - \bar{x}| \\
&+ \ bH|\bar{y} - y| + \ bH|x - \bar{x}| + cH|x - \bar{x}| + cH|y - \bar{y}| + d|y - \bar{y}| \\
&+ \ e|y - \bar{y}| \\
&= (r2H + rm + 3H^2\tfrac{r}{K} + 2H\tfrac{r}{K}m + bH + cH)|x - \bar{x}| \\
&+ \ (bH + cH + d + e)|y - \bar{y}| \\
&\leq L||u - \bar{u}||
\end{aligned}$$

where $L = max\{r2H + rm + 3H^2\frac{r}{K} + 2H\frac{r}{K}m + bH + cH, bH + cH + d + e\}$.

It follows from Theorem 3 that there is a unique solution $u \in C[0, h]$ of system (13) with initial condition $u_0 = (x_0, y_0)$.

□

3.3. **Non-negative invariance.** Denote $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$.

Theorem 6. *The solution for (13) with $t \geq 0$ remains in $\mathbb{R}^2_+$.*

Proof. In order to make the demonstration, Corollary 1 is considered. Assume problem (13) with $(x_0, y_0) \in \mathbb{R}^2_+$. With the objective of seeing how the solution evolves, different initial condition are proposed.

Case $x_0 = 0$, $y_0 > 0$. In this situation, from problem (13):

$$\begin{aligned}
{}_0^C D^\alpha_{0+}[x] &= 0, \\
{}_0^C D^\alpha_{0+}[y] &= (-d - e)y_0 < 0.
\end{aligned}$$

This implies $x(t) = 0\,\forall t \in [0,T]$ and $y(t)$ decreases asymptotically to zero. If there exists $t^* \in (0,T]$ in which $y(t^*) = 0$,

$$ {}_{0}^{C}D_{t^*}^{\alpha}\,[y] = 0. $$

This implies $y(t) = 0\,\forall t \in [t^*,T]$, then $y(t) \nless 0$, $\forall t \in [0,T]$. In conclusion, $x(t) \geq 0$, $y(t) \geq 0\,\forall t \in [0,T]$.

Case $x_0 > 0$, $y_0 = 0$. In this situation, from problem (13):

$$ \begin{array}{rcl} {}_{0}^{C}D_{0+}^{\alpha}\,[x] & = & rx_0\left(1 - \frac{x_0}{K}\right)(x_0 - m), \\ {}_{0}^{C}D_{0+}^{\alpha}\,[y] & = & 0. \end{array} $$

This implies $y(t) = 0\,\forall t \in [0,T]$ and $x(t)$ will depend on the value of x_0. If $x_0 > K$ then ${}_{0}^{C}D_{0+}^{\alpha}\,[x] < 0$, this means $x(t)$ decreases asymptotically to K and remains positive. If $x_0 = K$ then ${}_{0}^{C}D_{0+}^{\alpha}\,[x] = 0$ then $x(t) = K > 0\,\forall t \in [0,T]$. If $m < x_0 < K$ then ${}_{0}^{C}D_{0+}^{\alpha}\,[x] > 0$, which means $x(t)$ creases asymptotically to K and remains positive. If $x_0 = m$ then ${}_{0}^{C}D_{0+}^{\alpha}\,[x] = 0$, then $x(t) = m > 0\,\forall t \in [0,T]$. Finally if $0 < x_0 < m$ then ${}_{0}^{C}D_{0+}^{\alpha}\,[x] < 0$, this implies $x(t)$ decreases asymptotically to zero, where, as the previous case, we can infer that $x(t) \nless 0$, $\forall t \in [0,T]$. In conclusion, $x(t) \geq 0$, $y(t) \geq 0\,\forall t \in [0,T]$.

Case $x_0 = 0$, $y_0 = 0$. The system does not evolve.

Case $x_0 > 0$, $y_0 > 0$. If ${}_{0}^{C}D_{0+}^{\alpha}\,[x] \geq 0$ then $x(t) \geq 0\,\forall t \in [0,T]$. Let's suppose ${}_{0}^{C}D_{0+}^{\alpha}\,[x] < 0$, then $x(t)$ decreases. If there exists $t^* \in (0,T]$ such us $x(t^*) = 0$ then we can follow the ideas of the first Case to conclude that $x(t) \geq 0$, $y(t) \geq 0\,\forall t \in [0,T]$. Analogously for $y(t)$.

Therefore the solution for (13) with $t \geq 0$ remains in $\mathbb{R}_+^2$.

□

3.4. **Stability analysis of the problem solution.**

3.4.1. *Equilibrium points.* The local stability of system (13) will be discussed below. Let us observe that the equation is autonomous and, for this reason, we are in a position to use what we have seen in Section 2.3.

Setting ${}_{0}^{C}D_{t}^{\alpha}\,[x]$ and ${}_{0}^{C}D_{t}^{\alpha}\,[y]$ equal to zero, the following 4 equilibrium points are obtained:

(1) Trivial state $P_1(0,0)$, which corresponds to the extinction of the species.
(2) Axial state $P_2(K,0)$, which corresponds to the fact that preys reache the carrying capacity and there is no presence of predators.
(3) Axial state $P_3(m,0)$, which corresponds to the fact that preys stabilize at the threshold of Allee and there is no presence of predators.
(4) Coexistence of the species state $P_4\left(\frac{d+e}{c}, \frac{r}{b}\left(1 - \frac{d+e}{cK}\right)\left(\frac{d+e}{c} - m\right)\right)$.

Remark 4. *P_4 only makes sense when $cm - d \leq e \leq cK - d$, because $x(t)$ and $y(t)$ must be non negatives. Also, it can be observed when $e = cm - d$ then $P_4 = P_3$ and when $e = cK - d$ then $P_4 = P_2$. Therefore, only the case in which $cm - d < e < cK - d$ will be analyzed to guarantee the coexistence of the species.*

3.4.2. *Linearization.* The jacobian matrix of system (13) at a point $P(x,y)$ is the following:

$$J(x,y)=\begin{pmatrix} r\left(1-\frac{x}{K}\right)(x-m)-\frac{r}{K}x\,(x-m)+rx\left(1-\frac{x}{K}\right)-by & -bx \\ cy & cx-(d+e) \end{pmatrix}. \tag{14}$$

Using Theorem 4, the stability of equilibrium points P_1, P_2, P_3 and P_4 is analyzed.

- Jacobian matrix (14) evaluated in $P_1(0,0)$ is

$$J_{P_1}=\begin{pmatrix} -rm & 0 \\ 0 & -(d+e) \end{pmatrix}. \tag{15}$$

Its eigenvalues are $\lambda_1=-rm$ and $\lambda_2=-(d+e)$. Being $r,m,d,e>0$, $\lambda_1,\lambda_2\in\mathbb{R}^-$ results $|arg(\lambda_1)|=|arg(\lambda_2)|=\pi$. From $0<\alpha\le 1\Rightarrow 0<\alpha\frac{\pi}{2}\le\frac{\pi}{2}<\pi\ \ \forall\alpha\in(0,1]$, then $|arg(\lambda_i)|>\alpha\frac{\pi}{2}\ \forall\alpha\in(0,1],\ i=1,2$.

Theorem 7. *The equilibrium P_1 of system (13) is locally asymptotically stable.*

- Jacobian matrix (14) evaluated in $P_2(K,0)$ is

$$J_{P_2}=\begin{pmatrix} r(m-K) & -bK \\ 0 & cK-(d+e) \end{pmatrix}. \tag{16}$$

Its eigenvalues are $\lambda_1=r(m-K)$ and $\lambda_2=cK-(d+e)$. Being $m<K$, $\lambda_1\in\mathbb{R}^-$ results $|arg(\lambda_1)|=\pi>\alpha\frac{\pi}{2}\ \forall\alpha\in(0,1]$. On the other hand, since $e<cK-d$ is considered, $\lambda_2\in\mathbb{R}^+$ and then $|arg(\lambda_2)|=0<\alpha\frac{\pi}{2}\ \forall\alpha\in(0,1]$.

Theorem 8. *The equilibrium P_2 of system (13) is locally unstable.*

- Jacobian matrix (14) evaluated in $P_3(m,0)$ is

$$J_{P_3}=\begin{pmatrix} rm(1-\frac{m}{K}) & -bm \\ 0 & cm-(d+e) \end{pmatrix}. \tag{17}$$

Its eigenvalues are $\lambda_1=rm(1-\frac{m}{K})$ and $\lambda_2=cm-(d+e)$. Being $m<K$, $\lambda_1\in\mathbb{R}^+$ results $|arg(\lambda_1)|=0<\alpha\frac{\pi}{2}\ \forall\alpha\in(0,1]$.

Theorem 9. *The equilibrium P_3 of system (13) is locally unstable.*

- Jacobian matrix (14) evaluated in $P_4\left(\frac{d+e}{c},\frac{r}{b}\left(1-\frac{d+e}{cK}\right)\left(\frac{d+e}{c}-m\right)\right)$ is

$$J_{P_4}=\begin{pmatrix} \frac{r}{K}\frac{d+e}{c}\left[-\left(\frac{d+e}{c}-m\right)+\left(K-\frac{d+e}{c}\right)\right] & -b\frac{d+e}{c} \\ \frac{r}{K}\frac{c}{b}\left(\frac{d+e}{c}-m\right)\left(K-\frac{d+e}{c}\right) & 0 \end{pmatrix}. \tag{18}$$

The following characteristic equation is obtained

$$\lambda^2+\lambda\frac{r}{K}\frac{d+e}{c}\left[\left(\frac{d+e}{c}-m\right)-\left(K-\frac{d+e}{c}\right)\right]+c\frac{d+e}{c}\frac{r}{K}\left(\frac{d+e}{c}-m\right)\left(K-\frac{d+e}{c}\right)=0. \tag{19}$$

The roots of this polynomial are:

$$\lambda_{1,2}=\frac{1}{2}\left\{\frac{r}{K}\frac{d+e}{c}\left[-\left(\frac{d+e}{c}-m\right)+\left(K-\frac{d+e}{c}\right)\right]\pm\sqrt{\Delta}\right\}, \tag{20}$$

where

$$\Delta = \left[\frac{r}{K}\frac{d+e}{c}\left[\left(\frac{d+e}{c}-m\right)-\left(K-\frac{d+e}{c}\right)\right]\right]^2 - 4c\frac{d+e}{c}\frac{r}{K}\left(\frac{d+e}{c}-m\right)\left(K-\frac{d+e}{c}\right). \quad (21)$$

From $cm-d<e<cK-d$, then
$4c\frac{d+e}{c}\frac{r}{K}\left(\frac{d+e}{c}-m\right)\left(K-\frac{d+e}{c}\right)>0$, and $\Delta < \left[\frac{r}{K}\frac{d+e}{c}\left[\left(\frac{d+e}{c}-m\right)-\left(K-\frac{d+e}{c}\right)\right]\right]^2$.

Different cases are analyzed according to the value of e.

- Case $cm-d<e<\frac{c}{2}(K+m)-d$: it can be deduced that

$$\frac{r}{K}\frac{d+e}{c}\left[-\left(\frac{d+e}{c}-m\right)+\left(K-\frac{d+e}{c}\right)\right]>0. \quad (22)$$

  * If $\Delta \geq 0$, $\lambda_1, \lambda_2 \in \mathbb{R}^+$ then $|arg(\lambda_1)| = |arg(\lambda_2)| = 0 < \alpha\frac{\pi}{2}$ $\forall\alpha \in (0,1]$.
  * If $\Delta < 0$ then $\lambda_1, \lambda_2 \in \mathbb{C}$ are conjugated, with $Re(\lambda_1) = Re(\lambda_2) > 0$ from (22) and $|arg(\lambda_1)| = |arg(\lambda_2)| =$ $= \left|tan^{-1}\left(\frac{\sqrt{-\Delta}}{\frac{r}{K}\frac{d+e}{c}\left[\left(\frac{d+e}{c}-m\right)-\left(K-\frac{d+e}{c}\right)\right]}\right)\right| \in (0, \frac{\pi}{2})$, so its stability will depend on the value of α.

- Case $e=\frac{c}{2}(K+m)-d$: it can be deduced that

$$\frac{r}{K}\frac{d+e}{c}\left[-\left(\frac{d+e}{c}-m\right)+\left(K-\frac{d+e}{c}\right)\right]=0, \quad (23)$$

  and it follows that $\Delta < 0$. Therefore, $Re(\lambda_1) = Re(\lambda_2) = 0$ then $|arg(\lambda_1)| = |arg(\lambda_2)| = \frac{\pi}{2} > \alpha\frac{\pi}{2}$ $\forall\alpha \in (0,1)$ and $|arg(\lambda_1)| = |arg(\lambda_2)| = \frac{\pi}{2} = \alpha\frac{\pi}{2}$ for $\alpha = 1$.

- Case $\frac{c}{2}(K+m)-d<e<cK-d$: it can be deduced that

$$\frac{r}{K}\frac{d+e}{c}\left[-\left(\frac{d+e}{c}-m\right)+\left(K-\frac{d+e}{c}\right)\right]<0. \quad (24)$$

  * If $\Delta \geq 0$, λ_1, $\lambda_2 \in \mathbb{R}^-$ then $|arg(\lambda_1)| = |arg(\lambda_2)| = \pi > \alpha\frac{\pi}{2}$ $\forall\alpha \in (0,1]$.
  * If $\Delta < 0$ then $\lambda_1, \lambda_2 \in \mathbb{C}$ are conjugated, with $Re(\lambda_1) = Re(\lambda_2) < 0$ from (24) and $|arg(\lambda_1)| = |arg(\lambda_2)| > \frac{\pi}{2} \geq \alpha\frac{\pi}{2}$ $\forall\alpha \in (0,1]$.

Theorem 10. *The equilibrium P_4 of system (13) is:*

- *locally stable if*
  * $cm-d<e<\frac{c}{2}(K+m)-d$, $\Delta<0$ *and* $\alpha \leq \frac{2}{\pi}tan^{-1}\left(\frac{\sqrt{-\Delta}}{\frac{r}{K}\frac{d+e}{c}\left[\left(\frac{d+e}{c}-m\right)-\left(K-\frac{d+e}{c}\right)\right]}\right)$.
  * $e=\frac{c}{2}(K+m)-d$ *and* $\alpha \in (0,1]$.
  * $\frac{c}{2}(K+m)-d<e<cK-d$ *and* $\alpha \in (0,1]$.

- *locally asymptotically stable if*
  * $cm-d<e<\frac{c}{2}(K+m)-d$, $\Delta<0$ *and* $\alpha < \frac{2}{\pi}tan^{-1}\left(\frac{\sqrt{-\Delta}}{\frac{r}{K}\frac{d+e}{c}\left[\left(\frac{d+e}{c}-m\right)-\left(K-\frac{d+e}{c}\right)\right]}\right)$.
  * $e=\frac{c}{2}(K+m)-d$ *and* $\alpha \in (0,1)$.

  * $\frac{c}{2}(K+m)-d<e<cK-d$ *and* $\alpha\in(0,1]$.

- *locally unstable if*
  * $cm-d<e<\frac{c}{2}(K+m)-d$, $\Delta\geq 0$ *and* $\alpha\in(0,1]$.
  * $cm-d<e<\frac{c}{2}(K+m)-d$, $\Delta<0$ *and* $\alpha>\frac{2}{\pi}tan^{-1}\left(\frac{\sqrt{-\Delta}}{\frac{r}{K}\frac{d+e}{c}\left[\left(\frac{d+e}{c}-m\right)-\left(K-\frac{d+e}{c}\right)\right]}\right)$.

3.5. **Example.**

$$\begin{cases} {}_0^CD_t^\alpha[x] = & 0.5x(t)\left(1-\frac{x(t)}{3}\right)(x(t)-0.5)-0.7x(t)y(t), \\ {}_0^CD_t^\alpha[y] = & 0.35x(t)y(t)-0.35y(t)-ey(t). \end{cases} \tag{25}$$

Using Theorems 7, 8, 9 and 10, a stability analysis will be performed. It will be studied the sensibility to the harvest parameter e and the fractional order α.

Different graphics of the problem solutions are presented. Fractional Adams method in Section 2.4 will be used. For the examples, the value of e will be preset and the values of α will be varied.

Case $e=0.1$

For this value, $cm-d<e<\frac{c}{2}(K+m)-d$ with $\Delta<0$. The following equilibrium points are obtained: $P_1(0,0)$, $P_2(3,0)$, $P_3(0.5,0)$ and $P_4(1.2857,0.3207)$.

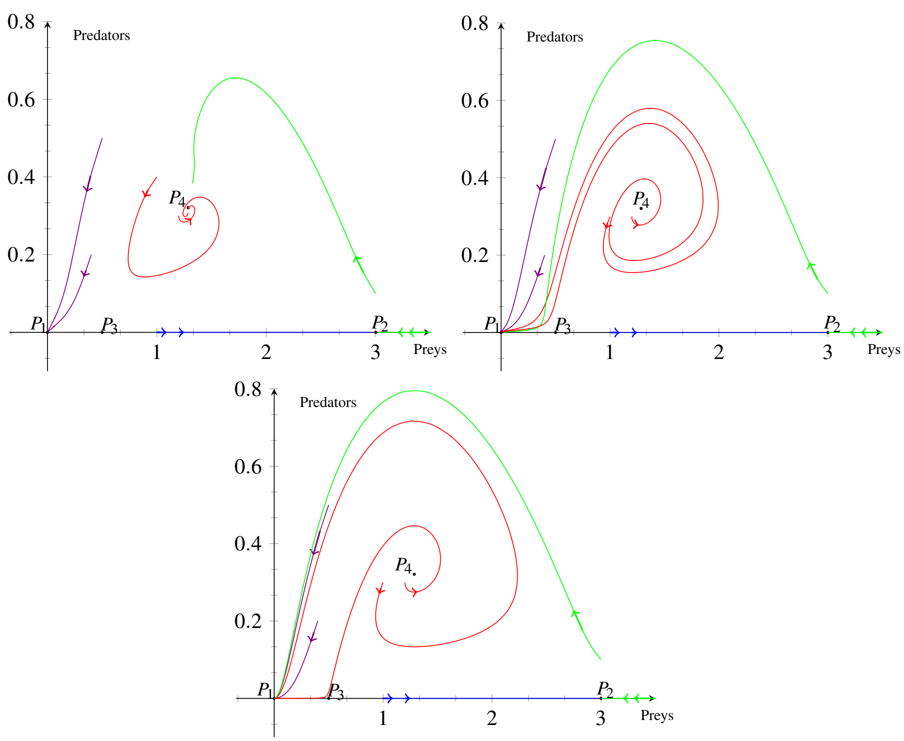

FIGURE 1. Solutions with $e=0.1$ for $\alpha=0.6$ (upper left), $\alpha=0.9$ (upper right) and $\alpha=1$ (lower).

Figure 1 corresponds to the approximate solutions of problem (25) with $\alpha = 0.6$, $\alpha = 0.9$ and $\alpha = 1$. In all of them, the trajectories whose initial conditions are close to P_1 tend to P_1, resulting locally asymptotically stable as it has been seen. This means, if less than Allee's threshold prey or a sufficiently large number of predators is considered then it makes sense for the species to go extinct. Regarding the trajectories whose initial conditions are close to P_2, the non-existence of predators makes them tend to P_2, that is, preys reach the carrying capacity. However, if the initial condition consists of a positive number of predators, the trajectories move away from P_2, resulting in a locally unstable equilibrium. Regarding the trajectories whose initial conditions are close to P_3, it can be observed that all of them move away, resulting in a locally unstable equilibrium. Finally, the difference with respect to P_4 in the images is remarkable. When $\alpha = 0.6$, trajectories starting near P_4 approach it, resulting in a locally asymptotically stable equilibrium. This represents the coexistence of the species, which makes sense due to the great "time delay" in the evolution of growth of predators. When $\alpha = 0.9$, the trajectories move away from P_4 being a locally unstable equilibrium, both species becoming extinct. This behavior also makes sense because, being α large enough, the evolution of predators is not affected by a very large "time delay", making them grow almost in the usual way. Besides the harvest of these is relatively low and therefore the large number of predators makes both species extinct. When $\alpha = 1$, the graphic is similar to $\alpha = 0.9$, except that the trajectories do not intersect and reach equilibrium faster.

Case $e = 0.2625$

For this value, $e = \frac{c}{2}(K+m) - d$. The following equilibrium points are obtained: $P_1(0,0)$, $P_2(3,0)$, $P_3(0.5,0)$ and $P_4(1.75, 0.3720)$.

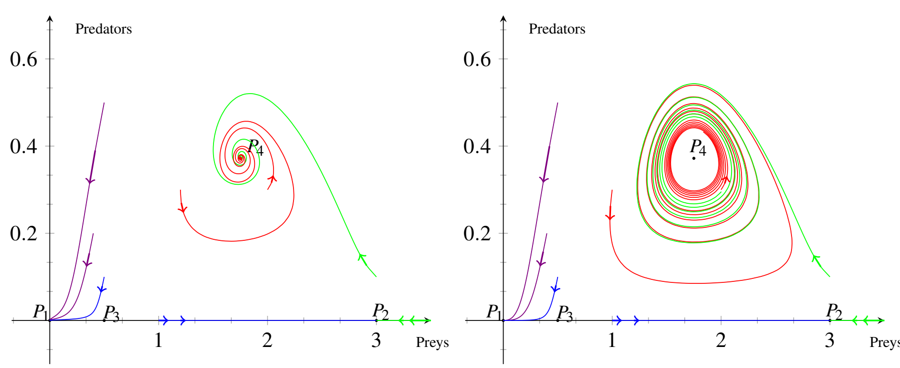

FIGURE 2. Solutions with $e = 0.2625$ para $\alpha = 0.9$ (left) and $\alpha = 1$ (right).

Figure 2 corresponds to the approximate solutions of problem (25) with $\alpha = 0.9$ and $\alpha = 1$. In both images, the trajectories that begin with values close to the equilibrium point P_1 tend to the same point over time. This shows the asymptotic stability mentioned in Theorem 7. Also, the trajectories that begin close to the equilibrium points P_2 and P_3 tend to move away from those points, similar to their behavior in Figure 1. With respect to those trajectories that begin close to the equilibrium point P_4, different behaviors can be observed in each image. In case $\alpha = 0.9$ it is seen that these trajectories approach the point P_4 over time, getting closer to it, coinciding with the asymptotic stability mentioned in Theorem 10. On the other hand, in the case $\alpha = 1$ it is seen that the trajectories with the same initial conditions as in the previous image tend to reach the equilibrium P_4 but do not get close enough. This is a non-asymptotic stability, also seen in Theorem 10. Here an important difference can be noted: while in the classical model the coexistence of species is represented by centers, in the fractional model it is represented by asymptotically stable spirals, enabling to describe other types of predator-prey situations.

Case $e = 0.5$
For this value, $\frac{c}{2}(K+m) - d < e < cK - d$. The following equilibrium points are obtained: $P_1(0,0)$, $P_2(3,0)$, $P_3(0.5,0)$ y $P_4(2.4286, 0.2624)$.

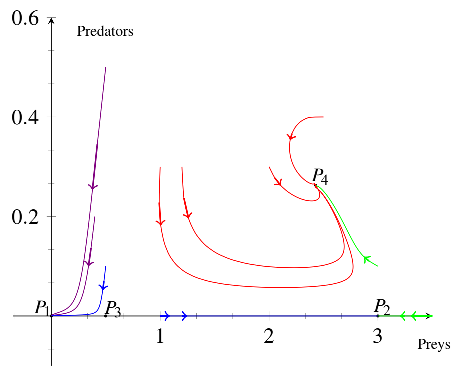

FIGURE 3. Solutions with $e = 0.5$ for $\alpha = 0.9$.

Figure 3 corresponds to the approximate solutions of problem (25) with $\alpha = 0.9$, and it will be the same behavior for all values of $0 < \alpha \leq 1$. Again, the trajectories close to the points P_1, P_2 and P_3 act in a similar way to Figure 1 and Figure 2. With respect to P_4 it can be seen that it is locally asymptotically stable as predicted by Theorem 10. However, the way in which the trajectories reach that equilibrium changes. In the previous graphics the trajectories were in a spiral shape, while now they are not. This behavior is a consequence of the fact that the eigenvalues found in the associated jacobian matrix are real, unlike the other cases in which they were complex.

4. Conclusions

In this article, a predator-prey model has been analysed with an Allee effect on the growth of preys and with harvest of predators. This incorporation into the system made the model more realistic. A fractional order model was considered, in view of the advantages that this entails. It was explicitly proved that the solution of this problem exists, it is unique and it is non-negative invariant. Due to the difficulty of solving nonlinear fractional systems of differential equations, it was necessary to resort to other methods.

An analysis of the stability of the solutions was done, observing that it depended on the harvest coefficient of the predators. When the harvest is low enough, the stability of the solution that represents the coexistence of the species depends on the fractional order chosen for their evolution. When an intermediate harvest is considered, there is a large difference between the fractional case and the classical case. The coexistence solution is only stable for $\alpha = 1$, becoming asymptotically stable for $0 < \alpha < 1$. This situation allows modeling other types of behavior between species, different from the classical model. When the harvest is large enough, the fractional and classical models behave similarly. To sum up, it can be said that differential equations of fractional order are at least as stable as those of integer order.

All this analytical work could be verified from the numerical treatment. For this work, the Adams method was used in its fractional version and some examples of great importance were shown.

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References

[1] S. Abbas, M. Banerjee, and S. Momani. Dynamical analysis of fractional-order modified logistic model. Computers & Mathematics with Applications, 62(3), pp.1098–1104, 2011.
[2] E. Ahmed, A. El-Sayed, and H. A. El-Saka. On some routh-hurwitz conditions for fractional order differential equations and their applications in lorenz, rossler, chua and chen systems. Physics Letters A, 358(1), pp.1–4, 2006.
[3] E. Ahmed, A. El-Sayed, and H. A. El-Saka. Equilibrium points, stability and numerical solutions of fractional-order predator–prey and rabies models. Journal of Mathematical Analysis and Applications, 325(1), pp.542–553, 2007.
[4] W. C. Allee. Co-operation among animals. American Journal of Sociology, 7(3), pp.386–398, 1931.
[5] R. Almeida, A. B. Malinowska, and D. F. Torres. A fractional calculus of variations for multiple integrals with application to vibrating string. Journal of Mathematical Physics, 51(3), pp.033503, 2010.
[6] Z. Amarti, N. Nurkholipah, N. Anggriani, and A. Supriatna. Numerical solution of a logistic growth model for a population with allee effect considering fuzzy initial values and fuzzy parameters. IOP (Institute of Physics) Conference Series: Materials Science and Engineering, 2018.
[7] D. Baleanu, K. Diethelm, E. Scalas, and J. Trujillo. Fractional calculus: models and numerical methods, volume 3. World Scientific, 2012.
[8] M. Barrios and G. Reyero. An Euler-Lagrange Equation only Depending on Derivatives of Caputo for Fractional Variational Problems with Classical Derivatives. Statistics, Optimization & Information Computing, 8(2), pp.590–601, 2020.
[9] M. Barrios, G. Reyero, and M. Tidball. A fractional order cubic differential equation. Applications to natural resource management. arXiv preprint arXiv:2104.04092, 2021.
[10] M. Barrios, G. Reyero, and M. Tidball. Harvest management problem with a fractional logistic equation. Mathematica Pannonica New Series, 27 /NS 1/, no. 1, pp.21–34, 2021.
[11] M. Barrios, G. Reyero, and M. Tidball. Necessary conditions to a fractional variational problem. Statistics, Optimization & Information Computing, 10(2), 2022.
[12] D. S. Boukal and L. Berec. Single-species models of the allee effect: extinction boundaries, sex ratios and mate encounters. Journal of Theoretical Biology, 218(3), pp.375–394, 2002.
[13] M. Caputo. Linear models of dissipation whose q is almost frequency independent—ii. Geophysical Journal International, 13(5), pp.529–539, 1967.
[14] M. Caputo and F. Mainardi. Linear models of dissipation in anelastic solids. La Rivista del Nuovo Cimento (1971-1977), 1(2), pp.161–198, 1971.
[15] C. W. Clark. The optimal management of renewable resources, volume 2. Mathematical Bioeconomics, 1990.

[16] K. Diethelm. The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type. Springer Science & Business Media, 2010.
[17] K. Diethelm, N. J. Ford, and A. D. Freed. A predictor-corrector approach for the numerical solution of fractional differential equations. Nonlinear Dynamics, 29(1), pp.3–22, 2002.
[18] A. Ferrari and E. Santillan Marcus. Study of a fractional-order model for HIV infection of CD4+ T-cells with treatment. Journal of Fractional Calculus and applications, 11(2), pp.12–22, 2020.
[19] R. Herrmann. Fractional calculus: an introduction for physicists. World scientific, 2014.
[20] R. Hilfer et al. Applications of fractional calculus in physics, volume 35. World scientific Singapore, 2000.
[21] M. Javidi and N. Nyamoradi. Dynamic analysis of a fractional order prey–predator interaction with harvesting. Applied mathematical modelling, 37(20-21), pp.8946–8956, 2013.
[22] A. I. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. Theory and applications of fractional differential equations, vol. 204. Elsevier, 2006.
[23] C. Li and F. Zeng. Numerical methods for fractional calculus, vol. 24. CRC Press, 2015.
[24] H.-L. Li, L. Zhang, C. Hu, Y.-L. Jiang, and Z. Teng. Dynamical analysis of a fractional-order predator-prey model incorporating a prey refuge. Journal of Applied Mathematics and Computing, 54(1-2), pp.435–449, 2017.
[25] K. Oldham and J. Spanier. The fractional calculus theory and applications of differentiation and integration to arbitrary order. Elsevier, 1974.
[26] I. Podlubny. Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Elsevier, 1998.
[27] S. Ruan. Delay differential equations in single species dynamics. In Delay differential equations and applications, pp.477–517. Springer, 2006.

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