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SOME RESULTS ON UNIQUENESS OF ENTIRE FUNCTIONS OF q-SHIFT DIFFERENCE POLYNOMIALS

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Abstract. This paper is devoted to the uniqueness problem on entire functions whose product of q-shift difference polynomials sharing weight z. We obtain some results that extend the results of Nintu Mandal and Abhijit Shaw [25].

1. Introduction

Let $\mathbb{C}$ be a open complex plane and two functions f and g are non-constant and meromorphic in $\mathbb{C}$. In this article we use standard notations of Value Distribution Theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$ ([7], [10], [22]). For $a \in \mathbb{C} \cup \{\infty\}$, if f and g have the same a-point with same multiplicities then f and g share a CM. If we do not take the multiplicities into account then f and g share the value a IM. A meromorphic function a is said to be a small function of f provided $T(r,a) = S(r,f)$ i.e., $T(r,a) = O\{T(r,f)\}$ as $r \to \infty$, $r \notin E$. For our convenience we means that $\mathbb{S}(f)$ contains all constant functions and $\widehat{\mathbb{S}} = \mathbb{S}(f) \cup \{\infty\}$. Let f and g be two meromorphic functions defined in the complex plane and a be a value in the extended complex plane.

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Now we say that f and g share that value a CM (counting multiplicities) if the zeros of $f-a$ and $g-a$ coincide in location and multiplicity and say that f and g share the value a IM if zeros of $f-a$ and $g-a$ coincide only in location but not in multiplicity. The counting function of zeros of $f-a$ where m-fold zero is counted m-times if $m \leq p$ and p times if $m > p$ is denoted by $N_p(r, a; f)$ where $p \in \mathbb{Z}^+$. We define difference operators for a meromorphic function by $\triangle_c f(z) = f(z+c) - f(z)$, $(c \neq 0)$ and $\triangle_q f(z) = f(qz) - f(z)$, $(q \neq 0)$.

Definition 1 Let $p(z) = \sum_{i=0}^{m} a_i z^i$ be a non-zero polynomial, where $a_i (i = 0,\ 1,\ 2,\ 3, \ldots, m)$ are complex constants and $a_m \neq 0$. Let m_1 is the numbers of single zeros of $p(z)$ and m_2 is the number of multiple zeros of $p(z)$ and $\Gamma_1,\ \Gamma_2$ defined by $\Gamma_1 = m_1 + m_2,\ \Gamma_2 = m_1 + 2m_2$ respectively. We denote $\gamma = gcd\ (\gamma_0, \gamma_1, \ldots, \gamma_m)$ where $\gamma_i = m+1$, if $a_i = 0$, $\gamma_i = i+1$, if $a_i \neq 0$.
Definition 2 ([8], [9]) Let p be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_p(a; f)$ the set of all a-point of f where an a-point of multiplicity m is counted m times if $m \leq p$ and $p+1$ times if $m > p$. If $E_p(a; f) = E_p(a; g)$ we say that f, g share the value a with weight p.

In 2007, Laine and Yang [11] studied zero distributions of difference polynomials of entire functions and obtained the following results.

Theorem 1 [11] Let f be a transcendental entire function of finite order and ζ be a non-zero complex constant. Then for $n \geq 2$, $f^n f(z+\zeta)$ assumes every non-zero value a in $\mathbb{C}$ infinitely often.
The uniqueness result corresponding to Theorem [1] given by Qi, Yang and Liu [15].

Theorem 2 [15] Let f and g be two transcendental entire functions of finite order, and ζ be a non-zero complex constant, and let $n \geq 6$ be an integer. If $f^n(z)f(z+\zeta)$ and $g^n(z)g(z+\zeta)$ share 1 CM, then either $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.

Theorem 3 [20] Let f be a transcendental entire function of finite order, and ζ be a fixed non-zero complex constant. Then for $n > \Gamma_1$, $P(f(z))f(z+\zeta) - \omega(z) = 0$ has infinitely many solutions, where $\omega(z) \in \mathbb{S}(f) \setminus \{0\}$.

Theorem 4 [20] Let f and g be two transcendental entire functions of finite order, ζ be a non-zero complex constant and $n > 2\Gamma_2 + 1$ be an integer. If $p(f)f(z+\zeta)$ and $p(g)g(z+\zeta)$ share 1 CM, then one of the following results hold:
i) $f = tg$, where $t^\gamma = 1$;
ii) f and g satisfy the algebraic equation $\Phi(f,g) = 0$, where $\Phi(\lambda_1, \lambda_2) = p(\lambda_1)\lambda_1(z+\zeta) - p(\lambda_2)\lambda_2(z+\zeta)$;
iii) $f = e^\xi$, $g = e^\Psi$, where ξ and Ψ are two polynomials and $\xi + \Psi = d$, d is a complex constant satisfying $a_n^2 e^{(n+1)d} = 1$.

In 2010, Zhang and Korhonen [23] obtained the following result on value distribution of q-shift difference polynomials of meromorphic functions.

Theorem 5 [23] Let f be a transcendental meromorphic (resp. entire) function of zero order and q be a non-zero complex constant. Then for $n \geq 6$ (resp. $n \geq 2$) $f(z)^n f(qz)$ assume every non-zero value $c \in \mathbb{C}$ infinitely often.

Theorem 6 [23] Let f and g be two transcendental meromorphic functions of zero order. Suppose that q is a non-zero complex constant and $n \geq 6$ is an integer. If $f(z)^n(f(z)-1)f(qz)$ and $g(z)^n(g(z)-1)g(qz)$ share 1 CM, then $f \equiv g$.

In 2015, Xu, Liu and Cao [19] obtained the following result for a q-shift of a meromorphic function.

Theorem 7 [19] Let f be a zero order transcendental meromorphic (resp. entire) function, $q \in \mathbb{C} \setminus \{0\}$, $\zeta \in \mathbb{C}$. Then for any positive integer $n > \Gamma_1 + 4$ (resp. for entire $n > \Gamma_1$), $p(f(z))f(qz+\zeta) = \Phi(z)$ has infinitely many solutions, where $\Phi(z) \in \mathbb{S}(f) \setminus \{0\}$.

Theorem 8 [19] Let f and g be two transcendental entire functions of zero order and let $q \in \mathbb{C} \setminus \{0\}$, $\zeta \in \mathbb{C}$. If $p(f(z))f(qz+\zeta)$ and $p(g(z))g(qz+\zeta)$ share 1 CM and $n > 2\Gamma_2 + 1$ be an integer, then one of the following result holds:
i) $f = tg$ for a constant t such that $t^\gamma = 1$;
ii) $f$and g satisfy the algebraic equation $\Phi(f,g) = 0$, where $\Phi(\lambda_1, \lambda_2) = p(\lambda_1)\lambda_1(qz+\zeta) - p(\lambda_2)\lambda_2(qz+\zeta)$;
iii) $fg = d$, where d is a complex constant satisfying $a_n^2 d^{n+1} \equiv 1$.

In 2001, I. Lahiri ([8], [9]) proved the following result by considering the concept of weighted sharing.

Theorem 9 [8] [9] Let f and g be two transcendental entire functions of zero order and let $q \in \mathbb{C} \setminus \{0\}$, $\zeta \in \mathbb{C}$. If $E_l(1; p(f(z))f(qz+\zeta)) = E_l(1; p(g(z))g(qz+\zeta))$ and l, m, n are integers satisfy one of the following conditions:
i) $l \geq 3$; $n > 2\Gamma_2 + 1$;
ii) $l = 2$; $n > \Gamma_1 + 2\Gamma_2 + 2 - \alpha$;
iii) $l = 1$; $n > 2\Gamma_1 + 2\Gamma_2 + 3 - 2\alpha$;
iv) $l = 0$; $n > 3\Gamma_1 + 2\Gamma_2 + 4 - 3\alpha$.
Then the conclusions of Theorem 1 holds, where $\alpha = min\{\Theta(0,f), \Theta(0,g)\}$.

In 2016, Sahoo and Biswas [18] proved the following Theorem.

Theorem 10 [18] Let f and g be two transcendental entire functions of zero order and let $q \in \mathbb{C} \setminus \{0\}$, $\zeta \in \mathbb{C}$. If $E_l(1; (p(f(z))f(qz+\zeta))^{(k)}) = E_l(1; (p(g(z))g(qz+\zeta))^{(k)})$ and l, m, n are integers satisfy one of the following conditions:
i) $l \geq 2$; $n > 2\Gamma_2 + 2km_2 + 1$;
ii) $l = 1$; $n > \frac{1}{2}(\Gamma_1 + 4\Gamma_2 + 5km_2 + 3)$;
iii) $l = 0$; $n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4$;
Then one of the following results holds:
i) $f = tg$ for a constant t such that $t^\gamma = 1$;
ii) f and g satisfy the algebraic equation $\Phi(f,g) = 0$, where $\Phi(\lambda_1, \lambda_2) = p(\lambda_1)\lambda_1(qz+\zeta) - p(\lambda_2)\lambda_2(qz+\zeta)$;
iii) $fg = d$, where d is a complex constant satisfying $a_n^2 d^{n+1} \equiv 1$.

In 2020, Nintu Mandal and Abhijit Shaw [25] proved the following result.

Theorem 11 [25] Let f and g be two transcendental entire functions of zero order and let $q \in \mathbb{C} \setminus \{0\}$, $\zeta \in \mathbb{C}$. If $E_l(z; (p(f)f(qz+\zeta))^{(k)}) = E_l(z; (p(g)g(qz+\zeta))^{(k)})$ and l, m, n are integers satisfy one of the following conditions:
i) $l \geq 2$; $n > 2\Gamma_2 + 2km_2 + 1$;
ii) $l = 1$; $n > \frac{1}{2}(\Gamma_1 + 4\Gamma_2 + 5km_2 + 3)$;
iii) $l = 0$; $n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4$;
Then one of the following results holds:
i) $f = tg$ for a constant t such that $t^\gamma = 1$;
ii) f and g satisfy the algebraic equation $\Phi(f,g) = 0$, where $\Phi(\lambda_1, \lambda_2) =$

$p(\lambda_1)\lambda_1(qz+\zeta) - p(\lambda_2)\lambda_2(qz+\zeta)$;
iii)$f(z) = \mu_1 e^{\frac{u}{2}z^2+vz}$ and $g(z) = \frac{\mu_1}{\mu}e^{-}(\frac{u}{2}z^2+vz)$ where μ_1, μ, u, v are complex constant and not equal to zero. If $A = (-1)a_n^2\mu^{n+1}$, then $u^2 = \frac{1}{A(n+q^2)^2}$ and $v^2 = \frac{\zeta^2 q^2}{(n+q)^2(n+q^2)^2}$.

Question 1. What happens if $(p(f)f(qz+\zeta))^{(k)}$ is replaced by $(p(f)f^n(z)\prod_{j=1}^{d} f(q_jz+\zeta_j)^{v_j})^{(k)}$ in Theorem 1?

To answer the above question affirmatively, we prove the following results which is the main results of this article.

Theorem 12. Let f and g be two transcendental entire functions of zero order and let $q \in \mathbb{C}\setminus\{0\}$, $\zeta \in \mathbb{C}$. If $E_l\Big(z;(p(f)f^n(z)\prod_{j=1}^{d} f(q_jz+\zeta_j)^{v_j})^{(k)}\Big) = E_l\Big(z;(p(g)g^n(z)\prod_{j=1}^{d} g(q_jz+\zeta_j)^{v_j})^{(k)}\Big)$ and l, m, n are integers satisfy one of the following conditions:
i) $l \geq 2$; $n > 2\Gamma_1 + 2km_2 + 2d + 2 - \sigma - m$;
ii) $l = 1$; $n > \frac{1}{2}(\Gamma_1 + 4\Gamma_2 + 5km_2 + 3\sigma + 2m)$;
iii) $l = 0$; $n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 5d + 5 - m - \sigma$;
Then one of the following results holds:
i) $f = tg$ for a constant t such that $t^\gamma = 1$;
ii) f and g satisfy the algebraic equation $\Phi(f,g) = 0$, where

$$\Phi(\lambda_1,\lambda_2) = \lambda_1^n p(\lambda_1)\prod_{j=1}^{d} f(q_j\lambda_1+\zeta_j)^{s_j} - \lambda_2^n p(\lambda_2)\prod_{j=1}^{d} g(q_j\lambda_2+\zeta_j)^{s_j};$$

iii) $f(z) = \mu_1 e^{\frac{u}{2}z^2+vz}$ and $g(z) = \frac{\mu_1}{\mu}e^{-}(\frac{u}{2}z^2+vz)$ where μ_1, μ, u, v are complex constant and not equal to zero. If $A = (-1)a_m^2 e^{(n+m+\sigma)c} = (-1)a_m^2\mu^{n+m+\sigma}$, then $u^2 = \frac{1}{A[(m+n)+\sigma q_j^2]^2}$ and $v^2 = \frac{{\zeta_j}^2 q_j^2\sigma^2}{A[(m+n)+\sigma q_j^2]^2[(m+n)+\sigma q_j]^2}$.

2. Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows:

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big).$$

Lemma 1. [22] Let f be a non-constant meromorphic function, and $P(f) = \sum_{i=0}^{m} a_i f^i$, where $a_0,\ a_1,\ a_2,\ldots,a_m$ are complex constants and $a_m \neq 0$. Then

$$T(r, P(f)) = mT(r, f) + S(r, f).$$

Lemma 2. [24] Let f be a non-constant meromorphic function, and $p,\ k \in \mathbb{Z}^+$. Then

$$N_p\Big(r, \frac{1}{f^{(k)}}\Big) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}\Big(r, \frac{1}{f}\Big) + S(r, f). \tag{1}$$

$$N_p\Big(r, \frac{1}{f^{(k)}}\Big) \leq k\overline{N}(r, f) + N_{p+k}\Big(r, \frac{1}{f}\Big) + S(r, f). \tag{2}$$

Lemma 3. [9] Let f and g be two non-constant meromorphic functions. If $E_2(1; f) = E_2(1; g)$, then one of the following relation holds:
i) $T(r) \leq N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{g}) + N_2(r, f) + N_2(r, g) + S(r)$;
ii) $f = g$;
iii) $fg = 1$.
where $T(r) = max\{T(r, g), T(r, f)\}$ and $S(r) = o\{T(r)\}$.

Lemma 4. [1] Let F and G be two non-constant meromorphic functions such that $E_1(1; F) = E_1(1; G)$, and $H \not\equiv 0$, then

$$\begin{aligned} T(r, F) \leq\ & N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + N_2(r, F) + N_2(r, G) + \frac{1}{2}\overline{N}(r, \frac{1}{F}) + \frac{1}{2}\overline{N}(r, F) \\ & + S(r, F) + S(r, G); \end{aligned}$$

and we can deduce same result for $T(r, G)$.

Lemma 5. [1] Let F and G be two non constant meromorphic functions which are share 1 IM and $H \not\equiv 0$, then

$$\begin{aligned} T(r, F) \leq\ & N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + N_2(r, F) + N_2(r, G) + 2\overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) \\ & + 2\overline{N}(r, F) + \overline{N}(r, G) + S(r, F) + S(r, G). \end{aligned}$$

Lemma 6. [19] Let f be a transcendental meromorphic function of order zero and $q,\ \zeta$ two non-zero complex constants. Then

$$T(r, f(qz + \zeta)) = T(r, f(z)) + S(r, f);$$
$$N(r, f(qz + \zeta)) = N(r, f(z)) + S(r, f);$$

$$N\Big(r, \frac{1}{f(qz+\zeta)}\Big) = N\Big(r, \frac{1}{f(z)}\Big) + S(r,f);$$

$$\overline{N}(r, f(qz+\zeta)) = \overline{N}(r, f(z)) + S(r,f);$$

$$\overline{N}\Big(r, \frac{1}{f(qz+\zeta)}\Big) = \overline{N}\Big(r, \frac{1}{f(z)}\Big) + S(r,f).$$

Lemma 7. [19] Let f be a transcendental meromorphic function of order zero and $q(\neq 0)$, ζ two non-zero complex constants. Then $F_1 = f^n P(f) \prod_{j=1}^{d} f(q_j z + \zeta_j)^{v_j}$ and $G_1 = g^n P(g) \prod_{j=1}^{d} g(q_j z + \zeta_j)^{v_j}$. $F = \frac{(F_1)^{(k)}}{z}$, $G = \frac{(G_1)^{(k)}}{z}$. Then $T(r, F_1) = (n+m+\sigma)T(r,f) + S(r,f)$. In addition, if it is a transcendental entire function of zero order, then

$$T(r,F_1) = T(r, f^n P(f) \prod_{j=1}^{d} f(q_j z+\zeta_j)^{v_j}) = (n+m+\sigma)T(r,f) + S(r,f)$$

where $\sigma = \sum_{j=1}^{d} v_j$.

Lemma 8. Let f and g be two entire functions q, ζ complex constants and $q \neq 0$; n, k are two positive integers and let $F_1 = f^n p(f) \prod_{j=1}^{d} f(q_j z + \zeta_j)^{v_j}$ and $G_1 = g^n p(g) \prod_{j=1}^{d} g(q_j z + \zeta_j)^{v_j}$. $F = \frac{F_1^{(k)}}{z}$, $G = \frac{G_1^{(k)}}{z}$.

If there exists two non-zero constants c_1 and c_2 such that $\overline{N}(r, c_1; F) = \overline{N}(r, \frac{1}{G})$ and $\overline{N}(r, c_2; G) = \overline{N}(r, \frac{1}{F})$, then $n \leq 2\Gamma_1 + 2km_2 + 2d + 2 - \sigma - m$.

Proof. Let $F_1 = f^n p(f) \prod_{j=1}^{d} f(q_j z + \zeta_j)^{v_j}$ and $G_1 = g^n p(g) \prod_{j=1}^{d} g(q_j z + \zeta_j)^{v_j}$ and $F = \frac{F_1^{(k)}}{z}$, $G = \frac{G_1^{(k)}}{z}$.

By the Second main theorem of Nevanlinna, we have

$$T(r,F) \leq \overline{N}(r, \frac{1}{F}) + \overline{N}(r, c_1; F) + S(r,F) \leq \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + S(r,F). \tag{3}$$

Using equations (1), (2), (3), Lemma 2, Lemma 2 and Lemma 2, we get

$$\begin{aligned}(n+m+\sigma)T(r,f) &\leq T(r,F) - \overline{N}\left(r,\frac{1}{F}\right) + N_{k+1}\left(r,\frac{1}{F_1}\right) + S(r,f)\\ &\leq \overline{N}\left(r,\frac{1}{G}\right) + N_{k+1}\left(r,\frac{1}{F_1}\right) + S(r,f)\\ &\leq \overline{N}_{k+1}\left(r,\frac{1}{F_1}\right) + N_{k+1}\left(r,\frac{1}{G_1}\right) + S(r,f) + S(r,g)\\ &\leq \overline{N}_{k+1}\left(r,\frac{1}{f^n}\right) + \overline{N}_{k+1}\left(r,\frac{1}{g^n}\right) + N_{k+1}\left(r,\frac{1}{p(f)}\right)\\ &+ N_{k+1}\left(r,\frac{1}{p(g)}\right) + N_{k+1}\left(r,\frac{1}{\prod_{j=1}^{d} f(q_jz+\zeta_j)^{v_j}}\right)\\ &+ N_{k+1}\left(r,\frac{1}{\prod_{j=1}^{d} g(q_jz+\zeta_j)^{v_j}}\right) + S(r,f) + S(r,g).\end{aligned}$$

This implies,

(4)

$$(n+m+\sigma)T(r,f) \leq (1+m_1+m_2+km_2+d)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$$

Similarly,

(5)

$$(n+m+\sigma)T(r,g) \leq (1+m_1+m_2+km_2+d)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g).$$

In view of equation (4) and (5), we have,

$$(n+m+\sigma-2m_1-2m_2-2km_2-2d-2)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g).$$

which gives $n \leq 2\Gamma_1 + 2km_2 + 2d + 2 - \sigma - m$. This proves the Lemma.

Lemma 9. Let f and g be two transcendental entire functions of zero order and let $q \in \mathbb{C} \setminus \{0\}$, $\zeta \in \mathbb{C}$. If $f^n p(f) \prod_{j=1}^{d} f(q_jz+\zeta_j)^{v_j} = g^n p(g) \prod_{j=1}^{d} g(q_jz+\zeta_j)^{v_j}$. Then one of the following results holds:

i) $f = tg$ for a constant t such that $t^{\gamma} = 1$;
ii) f and g satisfy the algebraic equation $\Phi(f, g) = 0$, where

$$\Phi(\lambda_1, \lambda_2) = \lambda_1^n p(\lambda_1) \prod_{j=1}^{d} \lambda_1(q_j z + \zeta_j)^{v_j} - \lambda_2^n p(\lambda_2) \prod_{j=1}^{d} \lambda_2(q_j z + \zeta_j)^{v_j}.$$

Proof. This lemma can be proved easily in the line of the proof of the Theorem 1 [19].

3. Proof of the Main Result

Let $F_1 = f^n p(f) \prod_{j=1}^{d} f(q_j z + \zeta_j)^{v_j}$ and $G_1 = g^n p(g) \prod_{j=1}^{d} g(q_j z + \zeta_j)^{v_j}$ then $E_1(z; F_1^{(k)}) = E_1(z; G_1^{(k)})$. Again let $F = \frac{F_1^{(k)}}{z}$ and $G = \frac{G_1^{(k)}}{z}$. Then F and G are transcendental meromorphic functions satisfy $E_1(1; F) = E_1(1; G)$. Now with help of lemma 2 and using (1) we have

$$\begin{aligned} N_2\Big(r, \frac{1}{F}\Big) &\le N_2\Big(r, \frac{1}{\frac{F_1^{(k)}}{z}}\Big) + S(r, f) \\ &\le T(r, F_1^{(k)}) - T(r, F_1) + N_{k+2}(r, \frac{1}{F_1}) + S(r, f) \\ &\le T(r, F) - (n + m + \sigma)T(r, f) + N_{k+2}(r, \frac{1}{F_1}) + S(r, f), \end{aligned}$$

Hence,

$$(n + m + \sigma)T(r, f) \le T(r, F) - N_2(r, \frac{1}{F}) + N_{k+2}(r, \frac{1}{F_1}) + S(r, f). \tag{6}$$

we can show from (2)

$$\begin{aligned} N_2(r, \frac{1}{F}) &\le N_2(r, \frac{1}{F^{(k)}}) + S(r, f). \\ &\le N_{k+2}(r, \frac{1}{F_1}) + S(r, f). \end{aligned} \tag{7}$$

Now following three cases will be discuss separately.
Case I. Let $l \ge 2$. If possible we assume that (i) of lemma 2 holds. We can deduce from (6) with help of (7)

$$\begin{aligned} (n + m + \sigma)T(r, f) &\le N_2(r, \frac{1}{G}) + N_2(r, F) + N_2(r, G) + N_{k+2}(r, \frac{1}{F_1}) \\ &\quad + S(r, f) + S(r, g) \\ &\le N_{k+2}(r, \frac{1}{F_1}) + N_{k+2}(r, \frac{1}{G_1}) + S(r, f) + S(r, g). \end{aligned}$$

(8)
$$(n+m+\sigma)T(r,f) \leq (1+m_1+2m_2+km_2+d)(T(r,f)+T(r,g))+S(r,f)+S(r,g).$$

Same we can show for $T(r,g)$ i.e.,
(9)
$$(n+m+\sigma)T(r,g) \leq (1+m_1+2m_2+km_2+d)(T(r,f)+T(r,g))+S(r,f)+S(r,g).$$

We can obtain from (8) and (9)

$$(n+m+\sigma-2m_1-4m_2-2km_2-2d-2)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g).$$

which contradict the fact $n > 2\Gamma_2 + 2km_2 + 2d - \sigma - m + 2$. Then by Lemma 2 we claim that either $FG = 1$ or $F = G$.
Let $FG = 1$. Then,
(10)
$$\left(f^n p(f) \prod_{j=1}^{d} f(q_j z + \zeta_j)^{v_j}\right)^{(k)} \left(g^n p(g) \prod_{j=1}^{d} g(q_j z + \zeta_j)^{v_j}\right)^{(k)} = z^2.$$

If possible, let $p(z) = 0$ has m roots α_1, α_2, α_3,…,α_m with multiplicity n_1, n_2, n_3,…,n_m. Then we have $n_1 + n_2 + n_3 + \ldots + n_m = m$. Now

$$\begin{aligned}(11)\qquad &\left[a_m(f-\alpha_1)^{n_1}\ldots(f-\alpha_m)^{n_m} f^n \prod_{j=1}^{d} f(q_j z + \zeta_j)^{v_j})\right]^{(k)} \\ &\left[a_n(g-\alpha_1)^{n_1}\ldots(g-\alpha_m)^{n_m} g^n \prod_{j=1}^{d} g(q_j z + \zeta_j)^{v_j})\right]^{(k)} = z^2.\end{aligned}$$

Since f and g are entire functions from (11), we see that $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$. Also, we can say that α_1, α_2,…, α_m are picard's exceptional values. By picard's theorem of entire function, we have at least three picard's exceptional values of f and if $m \geq 2$ and $\alpha_i \neq 0 (i = 1, 2, \ldots, m)$, then we obtain a contradiction. Next we assume that $p(z) = 0$ has only one root. Then $p(f) = a_m(f-a)^m$ and $p(g) = a_m(g-a)^m$, where a is any complex constant. Now from (10) we can write
(12)
$$[f^n a_m(f-a)^m \prod_{j=1}^{d} f(q_j z+\zeta_j)^{v_j}]^{(k)} [g^n a_m(g-a)^m \prod_{j=1}^{d} g(q_j z+\zeta_j)^{v_j}]^{(k)} = z^2.$$

By picard's theorem and as f and g are transcendental entire functions , then we can say that $f - a = 0$ and $g - a = 0$ do not have zeros. Then, we obtain that $f(z) = e^{\alpha(z)} + a$ and $g(z) = e^{\beta(z)} + a$, $\alpha(z)$, $\beta(z)$ being non constant polynomials . From (12), we also see that

$\prod_{j=1}^{d} f(q_j z+\zeta_j)^{v_j} \neq 0$ and $\prod_{j=1}^{d} g(q_j z+\zeta_j)^{v_j} \neq 0$ and therefore $a = 0$.
Thus $f(z) = e^{\alpha(z)}$, $g(z) = e^{\beta(z)}$, $p(z) = a_m z^m$ and

(13)
$$[a_m e^{n\alpha(z)} e^{m\alpha(z)} \prod_{j=1}^{d} e^{(q_j z+\zeta_j)^{v_j}}]^{(k)} [a_m e^{n\beta(z)} e^{m\beta(z)} \prod_{j=1}^{d} e^{(q_j z+\zeta_j)^{v_j}}]^k = z^2.$$

If $k = 0$, then from (13) we have

$$a_m^2 e^{(n+m)(\alpha(z)+\beta(z))} \prod_{j=1}^{d} e^{(\alpha+\beta)(q_j z+\zeta_j)^{v_j}} = z^2.$$

which is a contradiction as for no value of $\alpha(z)$ and $\beta(z)$ we can compare both side.
If $k = 1$, then from (13) we have

$$[a_m e^{m\alpha(z)+n\alpha(z)+\sigma\alpha(q_j z+\zeta_j)}(m\alpha'(z) + n\alpha'(z) + \sigma q_j \alpha'(q_j z+\zeta_j))]$$

(14)
$$\times [a_m e^{m\beta(z)+n\beta(z)+\sigma\beta(q_j z+\zeta_j)}(m\beta'(z) + n\beta'(z) + \sigma q_j \beta'(q_j z+\zeta_j))] = z^2,$$

i.e

$$a_m^2 e^{(m+n)(\alpha+\beta)+\sigma\alpha(q_j z+\zeta_j)+\sigma\beta(q_j z+\zeta_j)}$$
$$\times [(m+n)\alpha'(z) + \sigma q_j \alpha'(q_j z+\zeta_j))(m+n)\beta'(z) + \sigma q_j \beta'(q_j z+\zeta_j))] = z^2,$$

Now the relation can be hold if $\alpha+\beta = c$; c is complex constant. Then $\alpha' + \beta' = 0$, i.e. $\beta' = -\alpha'$. Then from (14) we have

(15)
$$(-1)a_m^2 e^{(n+m+\sigma)c}(m\alpha'(z) + n\alpha'(z) + \sigma q_j \alpha'(q_j z+\zeta_j))^2 = z^2.$$

Now if $\alpha'(z)$be one degree polynomials, i.e $\alpha'(z) = uz+v$,then $\alpha'(q_j z+\zeta_j) = uq_j z + u\zeta + v$. Let $A = (-1)a_m^2 e^{(n+m+\sigma)c} = (-1)a_m^2 \mu^{(n+m+\sigma)}$, where $\mu = e^c$. Then we can show from (15) that $u^2 = \frac{1}{A[(m+n)+\sigma q_j^2]^2}$ and $v^2 = \frac{{\zeta_j}^2 q_j^2 \sigma^2}{A[(m+n)+\sigma q_j^2]^2[(m+n)+\sigma q_j]^2}$. Now $\alpha' = uz + v$ i.e $\alpha = \frac{u}{2}z^2 + vz + w$, then $f(z) = \mu_1 e^{\frac{u}{2}z^2+vz}$ and $g(z) = \dfrac{\mu}{\mu_1} e^{-(\frac{u}{2}z^2+vz)}$ where $\mu_1 = e^w$.

If $k \geq 2$, then we get

$$[a_m e^{(n+m)\alpha(z)+\sigma\alpha(q_j z+\zeta_j)}]^{(k)} = z^2 a_m e^{(n+m)\alpha(z)+\sigma\alpha(q_j z+\zeta_j)} p(\alpha'\alpha'_\zeta, ..., \alpha^{(k)}\alpha^{(k)}_{\zeta_j})$$

where $\alpha_{\zeta_j} = \alpha(q_j z+\zeta_j)$. Obviously, $p(\alpha'\alpha'_\zeta, ..., \alpha^{(k)}\alpha^{(k)}_{\zeta_j})$ has infinitely many zeros, and which contradict with (13).

Now let $F = G$. Then

$$\frac{\left(f^n p(f)\prod_{j=1}^{d} f(q_j z+\zeta_j)^{v_j}\right)^{(k)}}{z} = \frac{\left(g^n p(g)\prod_{j=1}^{d} g(q_j z+\zeta_j)^{v_j}\right)^{(k)}}{z}$$

That is

$$\left(f^n p(f)\prod_{j=1}^{d} f(q_j z+\zeta_j)^{vj}\right)^{(k)} = \left(g^n p(g)\prod_{j=1}^{d} g(q_j z+\zeta_j)^{v_j}\right)^{(k)}$$

Integrating one time we have

$$\left(f^n p(f)\prod_{j=1}^{d} f(q_j z+\zeta_j)^{vj}\right)^{(k-1)} = \left(g^n p(g)\prod_{j=1}^{d} g(q_j z+\zeta_j)^{v_j}\right)^{(k-1)} +\eta_{k-1}$$

where η_{k-1} is a constant. If $\eta_{k-1} \neq 0$ using Lemma 2 we say that $n \leq 2\Gamma_1+2km_2+2d-\sigma-m+2$, which contradict with the fact that $n > 2\Gamma_2+2km_2+2d-\sigma-m+2$. $(\Gamma_2 \geq \Gamma_1)$. Hence $\eta_{k-1} = 0$. Now repeating the process upto k - times, we can establish $f^n p(f)\prod_{j=1}^{d} f(q_j z+\zeta_j)^{v_j} = g^n p(g)\prod_{j=1}^{d} g(q_j z+\zeta_j)^{v_j}$. Hence by Lemma 2 we have either $f = tg$ for a constant t such that $t^\gamma = 1$, or f and g satisfy the algebraic equation $\Phi(f, g) = 0$ where,

$$\Phi(\lambda_1,\lambda_2) = \lambda_1^n p(\lambda_1)\prod_{j=1}^{d} f(q_j\lambda_1+\zeta_j)^{v_j} - \lambda_2^n p(\lambda_2)\prod_{j=1}^{d} g(q_j\lambda_2+\zeta_j)^{v_j}$$

Case II. Let $l=1$ and $H \not\equiv 0$. Using Lemma 2 and equation (7) we can establish from (6)

$$\begin{aligned}(n+m+\sigma)T(r,f) &= N_2(r,\frac{1}{G})+N_2(r,F)+N_2(r,G)+\frac{1}{2}\overline{N}(r,\frac{1}{F})\\ &+\frac{1}{2}\overline{N}(r,F)+N_{k+2}(r,\frac{1}{F_1})+S(r,f)+S(r,g)\\ &\le N_{k+2}(r,\frac{1}{F_1})+N_{k+2}(r,\frac{1}{G_1})+\frac{1}{2}N_{k+1}(r,\frac{1}{F_1})\\ &+S(r,f)+S(r,g)\\ &\le (1+m_1+(k+2)m_2+d)T(r,f)\\ &+\frac{1}{2}(1+m_1+(k+1)m_2+d)T(r,f)\\ &+(1+m_1+(k+2)m_2+d)T(r,g)+S(r,f)+S(r,g)\\ &\le \frac{1}{2}(3m_1+(3k+5)m_2+3d+3)T(r,f)\\ &+(1+m_1+(k+2)m_2+d)T(r,g)+S(r,f)+S(r,g)\\ &\le \frac{1}{2}(5m_1+(5k+9)m_2+5d+5)T(r)+S(r).\end{aligned}$$

where $T(r)$ and $S(r)$ two inequalities, defined in Lemma 2. Similarly we can show that

$$(n+m+\sigma)T(r,g) \le \frac{1}{2}(5m_1+(5k+9)m_2+5d+5)T(r)+S(r),$$

we have from two inequalities,

$$\left(n-\frac{5m_1+(5k+9)m_2+5d+5-2m-2\sigma}{2}\right)T(r)\le S(r)$$

which contradict the fact $n > \frac{\Gamma_1+4\Gamma_2+5km_2+3\sigma+2m}{2}$.
Now, let $H\equiv 0$, i.e., $(\frac{F''}{F'}-\frac{2F'}{F-1})-(\frac{G''}{G'}-\frac{2G'}{G-1})=0$. After two times integration we have

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{16}$$

where A, B are constants and $A\neq 0$. From (14) it is clear that F, G share the value 1 CM and then they share (1,2) and hence $\left(f^np(f)\prod_{j=1}^{d}f(q_jz+\zeta_j)^{v_j}\right)^{(k)}$ and $\left(g^np(g)\prod_{j=1}^{d}g(q_jz+\zeta_j)^{v_j}\right)^{(k)}$ share

$(z, 2)$. Hence we have $n > 2\Gamma_2 + 2km_2 + 1$. Now we study the following cases.
Subcase I. Let $B \neq 0$ and $A = B$. Then from (14) we get

$$\frac{1}{F-1} = \frac{BG}{G-1}, \tag{17}$$

If $B = -1$, then from (17), $FG = 1$ i.e.,
$\left(f^n p(f) \prod_{j=1}^{d} f(q_j z + \zeta_j)^{v_j}\right)^{(k)} \left(g^n p(g) \prod_{j=1}^{d} g(q_j z + \zeta_j)^{v_j}\right)^{(k)} = z^2$ then we obtain the same result as in Case I.

Now if $B \neq -1$. Then from (17), we have, $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and then, $\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r, \frac{1}{F})$.
Now from the second main theorem of Nevalinna, we get using (1) and (3) that

$$\begin{aligned} T(r,G) &= \overline{N}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{1+B}; G) + \overline{N}(r,G) + S(r,G) \\ &\leq \overline{N}(r, \frac{1}{F} + \overline{N}(r, \frac{1}{G}) + \overline{N}(r,G) + S(r,G) \\ &\leq N_{k+1}(r, \frac{1}{F_1}) + T(r,G) + N_{k+1}(r, \frac{1}{G_1}) - (n+m+\sigma)T(r,g) + S(r,g). \end{aligned}$$

This gives,

$$(n+m+\sigma)T(r,g) \leq (1+m_1+(k+1)m_2+d)(T(r,f)+T(r,g))+S(r,g),$$

we can show same result for $T(r,f)$ i.e.,

$$(n+m+\sigma)T(r,f) \leq (1+m_1+(k+1)m_2+d)(T(r,f)+T(r,g))+S(r,f),$$

Thus, we obtain

$$(n+m+\sigma-2-2m_1-2(k+1)m_2-2d)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),$$

a contradiction as $n > 2\Gamma_1 + 2km_2 + 2d - m - \sigma + 2$.

Subcase II. Let $A \neq 0$ and $B = 0$. Now from (16) we have $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \neq 1$, we have $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r, \frac{1}{G})$ and $\overline{N}(r, 1-A; G) = \overline{N}(r, \frac{1}{F})$. Then by Lemma 2, we have $n \leq 2\Gamma_1 + 2km_2 + 2d - \sigma - m + 2$, which is a contradiction. Thus $A = 1$ and $F = G$, then the result follows from the Case I.

Subcase III. Let $A \neq 0$ and $A \neq B$. Then from (16), we obtain

$F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and therefore $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r, \frac{1}{F})$. Proceeding similarly as in Subcase I, we can get a contradiction.

Case III. Let $l = 0$ and $H \not\equiv 0$, we can establish from (6) after using Lemma 2 and (7)

$$\begin{aligned}(n+m+\sigma)T(r,f) &= N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) + 2\overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}\\ &+ 2\overline{N}(r,F) + \overline{N}(r,G) + N_{k+2}(r,\frac{1}{F_1}) + S(r,f) + S(r,g)\\ &\leq N_{k+2}(r,\frac{1}{F_1}) + N_{k+2}(r,\frac{1}{G_1}) + 2N_{k+1}(r,\frac{1}{F_1})\\ &+ N_{k+1}(r,\frac{1}{G_1}) + S(r,f) + S(r,g)\\ &\leq (3+3m_1+(3k+4)m_2+3d)T(r,f)\\ &+ (2m_1+(2k+3)m_2+2+2d)T(r,g)\\ &+ S(r,f) + S(r,g)\\ &\leq (5m_1+(5k+7)m_2+5d+5)T(r) + S(r),\end{aligned}$$

Similarly it follows that $(n+m+\sigma)T(r,g) \leq (5m_1+(5k+7)m_2+5d+5)T(r)+S(r)$. From the above two inequalities we have $(n+m+\sigma-5m_1-(5k+7)m_2-5d-5)T(r) \leq S(r)$, which contradict with our assumption that $n > 3\Gamma_1+2\Gamma_2+5km_2+5d-m-\sigma+5$. Therefore $H = 0$ and then proceeding in similar manner as Case II, we get the results. This complete the proof of the theorem.

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