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ON THE SOLUTIONS OF IMPLICIT ARBITRARY ORDERS DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. In this paper, we use the fixed point theorem by Arino-Gautier and Penot to discuss the solvability of some implicit arbitrary orders differential equations in Banach spaces and we convert the mentioned equations to the form of functional integral equations to establish the existence of pseudo-solutions to a Cauchy problem of differential equation of arbitrary orders in Banach spaces.

1. Introduction and Preliminaries

The topic of fractional calculus (derivatives and integrals of arbitrary orders) is enjoying growing interest not only in Mathematics but also in Physics, Engineering, and Mathematical Biology (see for example [5]-[7]).
The very first approach via weak topology follows by Szép [19]. Then more ideas are taken from from papers by Kubiaczyk, Szufla [10] or Kubiaczyk [11], see for examples [1]-[3], [12], [13] and [15]-[18].
In [15] and [16] the existence of weak solutions for the initial value problem of the arbitrary (fractional) orders differential equation

$$\frac{dx}{dt} = f(t, D^{\alpha}x(t)),\ x(0) = x_0,\ t\ \in [0,1]$$

in the reflexive Banach space E have been considered, for the first time, by Salem and El-Sayed.

Let E be a Banach space with norm $\| \ . \ \|$ and dual E^*. Moreover, let $E_\omega = (E, \omega)$ denote the space E with its weak topology. By $\mathcal{C} = C[I, E]$ the Banach space of strongly continuous functions $x : I \to E$ with $||x||_C \ = \ \sup \ ||x(t)||_E,\ t \in I = [0, T]$.

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For the properties of the fractional Pettis-integral in Banach spaces (see [15] and [18]). Let $x : I \to E$, then $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in I$ if for every $\phi \in E^*$, $\phi(x(.))$ is continuous (measurable) at t_0.
It is evident that in reflexive Banach spaces, both Pettis integrable functions and weakly continuous functions are weakly measurable. If x is weakly continuous on I, then x is strongly measurable (see [9]), hence weakly measurable. Moreover the weakly measurable function $x(.)$ is Pettis integrable on I if and only if $\phi(x(.))$ is Lebesgue integrable on I; for every $\phi \in E^*$ ([4]).
Now, we have some propositions which will be used in the sequel (see [15], [19]).

Proposition 1. *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

The following result follows directly from the Hahn-Banach theorem.

Proposition 2. *Let E be a normed space with $x_0 \neq 0$. Then there exists a $\phi \in E^*$ with*
$\| \phi \| = 1$ *and* $\phi(x_0) = \| x_0 \|$.

Definition 1. [2, 3, 18] *A function $x(.)$ is said to be pseudo-differentiable on I to a function $y(.)$ if for every $\phi \in E^*$, there exists a null set $N(\phi)$ (i.e. N is depending on ϕ and $mes(N(\phi) = 0)$) such that the real function $t \to \phi(x(t))$ is differentiable a.e. on I and*

$$\phi(x'(t)) = \phi(y(t)),\ t \in I \backslash N(\phi).$$

The function $y(.)$ is called a pseudo-derivative of $x(.)$

Proposition 3. [14] *Let $y(.) : I \to E$ be a weakly measurable function.*

(A) *If $y(.)$ is Pettis integrable on I, then the indefinite Pettis integral*

$$x(t) = \int_0^t y(s)ds,\ t \in I,$$

is absolutely continuous on I and $y(.)$ is a pseudo-derivative of $x(.)$.

(B) *If $x(.)$ is an absolutely continuous function on I and it has a pseudo-derivative $y(.)$ on I, then $y(.)$ is Pettis integrable on I and*

$$x(t) = x(0) + \int_0^t y(s)ds,\ t \in I.$$

Also, we are in a position to recall a fixed point theorem being an extension of results from [1].

Theorem 1. *Let E be a Banach space with Q_r a nonempty, closed, convex, and a weakly compact subset of $C[I, E]$. Assume that $A : Q_r \to Q_r$ is weakly-weakly sequentially continuous. Then A has a fixed point in Q_r.*

Let $\alpha \in (0, 1)$. Here, we have studied the existence of weakly differentiable and pseudo-solutions as well as the initial value problem of the implicit arbitrary (fractional) orders nonlinear differential equation

$$x'(t) = f(t, x(t), D^\alpha x(t)),\ x(0) = x_0,\ t \in (0, T] \tag{1}$$

in the Banach space E. Operating by $I^{1-\alpha}$ on both sides of the differential equation (1), we obtain

$$D^\alpha x(t) = I^{1-\alpha} f(t, x(t), D^\alpha x(t)). \tag{2}$$

Lemma 1. *Let y be a function in $C[I, E_\omega]$. Then the equation*

$$D^\alpha x(t) = y(t)$$

has the solution $x \in C[I, E_\omega]$ defined by

$$x(t) = x_0 + I^\alpha y(t) = x_0 + I^1 f(t, x_0 + I^\alpha y(t), y(t)). \tag{3}$$

From the above lemma, we have the following lemma.

Lemma 2. *Let $f(t,x,y) : I \times E \times E \to E$ be a weakly continuous function. Then the problem (2) is equivalent to the equation of obtaining*

$$y(t) = I^\beta f(t, x_0 + I^\alpha y(t), y(t)),\ t \in I = [0,T],\ \beta = 1-\alpha. \tag{4}$$

and if $y \in C[I, E_\omega]$ is the solution of this equation, then $x(t) = x_0 + I^\alpha y(t)$ is a solution of (1).

Proof. Let $y \in C[I, E_\omega]$ be a solution to (4) and $\phi \in E^*$. As in ([18], lemma (9), it follows that $I^\alpha y$ exists and $\phi(x)$ is continuous for every $\phi \in E^*$. We also have

$$\begin{aligned}
\lim_{t\to 0} \phi(x(t)) &= \lim_{t\to 0}[x_0 + (I^\alpha \phi y)(t)] \\
&= x_0 + \lim_{t\to 0}(I^\alpha \phi y)(t) \\
&= x_0 + \lim_{t\to 0} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi y(s)\, ds = x_0
\end{aligned}$$

Thus $\phi(x(0)) = x_0$ for every $\phi \in E^*$; that is, $x(0) = x_0$. As in ([18], lemma (16), we also have

$$D^\alpha x(t) = D^\alpha(x_0 + I^\alpha y(t)) = D^\alpha I^\alpha y(t) = y(t)$$

Here, we restrict ourselves to the case of left-sided fractional Pettis-integrals I_+^α (shortly denote it by I^α). □

In this paper, for clarity of proof, we restrict ourselves to the case of reflexive spaces, but it is easy to extend our results for nonreflexive spaces by putting the contraction hypothesis with respect to some measure of weak noncompactness and by using appropriate fixed point theorem ([2]). Nevertheless, all auxiliary results in this paper are not restricted to reflexive spaces.

2. The main results

2.1. **Functional integral equation.**

Let us start by defining what we mean by a weak solution of the integral equation (4)

Definition 2. *By a weak solution to (4) we mean to finding a function $y \in \mathcal{C}$ such that for all $\phi \in E^*$*

$$\phi(y(t)) \ = \ \phi(I^\beta f(t, x_0 + I^\alpha y(t), y(t))),\ t \in I.$$

The following hypotheses will be used in the sequel.

(i) For each $t \in I,\ f_t = f(t,.,.) : I \times E \times E \to E$ is weakly-weakly sequentially continuous i.e. the function $x \to f(t,x,.)$ and $y \to f(t,.,y)$ are weakly sequentially continuous.

(ii) For each $x, y \in C[I, E_\omega],\ f(., x(.), y(.))$ is weakly continuous on I.

(iii) For any $\phi \in E^*$, there exists the function $a : I \to E$ is bounded and measurable such that

$$|\phi(f(t,x,y))| \leq a(t) + b \parallel x \parallel,\ b > 0,\ for\ a.e.\ t \in I,\ x, y \in E.$$

Where $a = \sup\{a(t) : t \in I\}$.

Theorem 2. *Let the assumptions (i)-(iii) be satisfied, If* $bT^{\alpha+\beta} < \Gamma(\alpha+1)\Gamma(\beta+1)$, *then the nonlinear integral equation (4) has at least one weak solution* $y \in \mathcal{C}$.

Proof.
Let $A : \mathcal{C} \to \mathcal{C}$ be an operator defined by

$$Ay(t) = I^{\beta} f(t, x_0 + I^{\alpha} y(t), y(t)),\ t \in I = [0, T].$$

For any $y \in \mathcal{C},\ I^{\alpha} y \in C[I, E_{\omega}],\ f(., x_0 + I^{\alpha} y(.), y(.))$ is weakly continuous on I (assumption (ii)) and $f(t, ., .)$ is weakly-weakly sequentially continuous (assumption (i)). By (Lemma 19 in [18]) $f(., x_0 + I^{\alpha} y(.), y(.))$ is Pettis integrable for all $t \in I$, consequently, it is fractionally Pettis integrable on I, and thus the operator A makes sense. Note that A is well defined. To see this, let $t_1,\ t_2 \in I,\ t_2 > t_1$, without loss of generality, assume $Ay(t_2) - Ay(t_1) \neq 0$ we get $\parallel Ay(t_2) - Ay(t_1) \parallel = \phi(Ay(t_2) - Ay(t_1))$. Thus

$$\parallel Ay(t_2) - Ay(t_1) \parallel$$

$$\begin{aligned}
&= \phi(I^{\beta}f(t_2,x_0+I^{\alpha}y(t_2),y(t_2)) - I^{\beta}f(t_1,x_0+I^{\alpha}y(t_1),y(t_1)))\\
&= \phi(\int_0^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}f(s,x_0+I^{\alpha}y(s),y(s))\ ds\\
&- \int_0^{t_1}\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}f(s,x_0+I^{\alpha}y(s),y(s))\ ds)\\
&\leq \ |\int_0^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\phi(f(s,x_0+I^{\alpha}y(s),y(s)))\ ds\\
&- \int_0^{t_1}\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\phi(f(s,x_0+I^{\alpha}y(s),y(s)))\ ds\ \ |\\
&\leq \ |\int_0^{t_1}(\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)})\phi(f(s,x_0+I^{\alpha}y(s),y(s)))\ ds\ |\\
&+ \ |\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\phi(f(s,x_0+I^{\alpha}y(s),y(s)))\ ds\ \ |.\\
&\leq \int_0^{t_1}|\ \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\ |\ |\ \phi(f(s,x_0+I^{\alpha}y(s),y(s)))\ |\ \ ds\\
&+ \int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\ |\ \phi(f(s,x_0+I^{\alpha}y(s),y(s)))\ |\ \ ds.\\
&\leq \int_0^{t_1}|\ \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\ |\ (a(s)+b[x_0+\parallel I^{\alpha}y(s)\parallel])\ ds\\
&+ \int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}(a(s)+b[x_0+\parallel I^{\alpha}y(s)\parallel])\ ds.\\
&\leq (a+bx_0)(\int_0^{t_1}|\ \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\ |\ \ ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\ ds)\\
&+ \frac{brT^{\alpha}}{\Gamma(\alpha+1)}(\int_0^{t_1}|\ \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}-\frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)}\ |\ \ ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\ ds)\\
&\leq (\frac{a+bx_0}{\Gamma(\beta+1)}+\frac{b\parallel y\parallel_0 T^{\alpha}}{\Gamma(\alpha+1)\Gamma(\beta+1)})[2(t_2-t_1)^{\beta}+|\ t_2^{\beta}-t_1^{\beta}\ |].
\end{aligned}$$

Hence

$$\parallel Ay(t_2)-Ay(t_1)\parallel\leq(\frac{a+bx_0}{\Gamma(\beta+1)}+\frac{b\parallel y\parallel_0 T^{\alpha}}{\Gamma(\alpha+1)\Gamma(\beta+1)})[2(t_2-t_1)^{\beta}+|\ t_2^{\beta}-t_1^{\beta}\ |]. \quad (5)$$

Thus $Ay\in\mathcal{C}$ and the operator A maps $\mathcal{C}$ into itself.
Now, define the set Q_r by

$$Q_r=\{y\in\mathcal{C}:\parallel y\parallel_{\mathcal{C}}\leq\ r,$$

$$\forall t_1,\ t_2\in I,\ \parallel y(t_2)-y(t_1)\parallel\leq(\frac{a+bx_0}{\Gamma(\beta+1)}+\frac{b\parallel y\parallel_0 T^{\alpha}}{\Gamma(\alpha+1)\Gamma(\beta+1)})[2(t_2-t_1)^{\beta}+|\ t_2^{\beta}-t_1^{\beta}\ |]\}$$

Note that Q_r is closed, convex and equi-continuous subset of $\mathcal{C}$.
We shall show that A satisfies the assumptions of the fixed point theorem by Arino-Gautier-Penot [1].
The operator A maps Q_r into itself. To see this, take $y\in Q_r,\ \parallel I^{\alpha}y\parallel\leq\frac{rT^{\alpha}}{\Gamma(\alpha+1)}$.

The inequality (5) imply that

$$\| Ay(t_2) - Ay(t_1) \| \leq (\frac{a+bx_0}{\Gamma(\beta+1)} + \frac{b \| y \|_0 \, T^\alpha}{\Gamma(\alpha+1)\Gamma(\beta+1)})[2(t_2-t_1)^\beta + | \, t_2^\beta - t_1^\beta \, |].$$

Now, without loss of generality; assume $I^\beta f(t, x_0 + I^\alpha y(t), y(t)) \neq 0$, then there exists $\phi \in E^*$ with $\| \phi \| = 1$ and $\| I^\beta f(t, x_0 + I^\alpha y(t), y(t)) \| = \phi(I^\beta f(t, x_0 + I^\alpha y(t), y(t)))$ and $\phi(I^\beta f(t, x_0 + I^\alpha y(t), y(t))) = I^\beta \phi(f(t, x_0 + I^\alpha y(t), y(t)))$. Thus

$$\begin{aligned}
\| Ay(t) \| &= \phi(Ay(t)) = \phi(I^\beta f(t, x_0 + I^\alpha y(t), y(t))) \\
&= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \mid \phi(f(s, x_0 + I^\alpha y(s), y(s))) \mid \; ds \\
&\leq \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} (a(s) + b[x_0 + \| I^\alpha y(s) \|]) \; ds \\
&\leq \frac{(a+bx_0)T^\beta}{\Gamma(\beta+1)} + \frac{brT^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}.
\end{aligned}$$

From the last estimate, we deduce that

$$r = (\frac{(a+bx_0)T^\beta}{\Gamma(\beta+1)})(1 - \frac{bT^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)})^{-1}$$

Therefore, $\| Ay \|_{\mathcal{C}} = \sup_{t \in I} \| Ay(t) \| \leq r$. Hence $A : Q_r \to Q_r$.

To show that the operator $A : Q_r \to Q_r$ is weakly sequentially continuous, let $\{y_n\}$ be sequence in Q_r converges weakly to y on I, since Q_r is closed then $y \in Q_r$. Fix $t \in I$ By the Lebesgue Dominated Convergence Theorem ([8]) we have $I^\alpha y_n(s) \to I^\alpha y(s)$ in E_ω, we have $f(t, x_0 + I^\alpha y_n(t), y_n(t)))$ converges weakly to $f(t, x_0 + I^\alpha y(t), y(t)))$; hence again the Lebesgue Dominated Convergence Theorem (see assumption (iii)) for Pettis integral yields $A : Q_r \to Q_r$ is weakly sequentially continuous.

Since all conditions of a fixed point theorem [1] are satisfied, then the operator A has at least one fixed point $y \in Q_r$, then the nonlinear integral equation of fractional order (4) has at least one weak solution $y \in \mathcal{C}$.

2.2. **Initial Value Problem.**

Definition 3. *(Pseudo-solutions) A solution to the Initial value problem (1) is an absolutely continuous function $x : I \to E$ with $x(t) = x_0 + \int_0^t x'(s) \, ds$, where x' is a Pettis integrable mapping weakly equivalent to $f(., x(.), D^\alpha x(.))$. Then x' is the a.e derivative of x and x is a pseudo solution.*

Theorem 3. *Let the assumption of Theorem 2 be satisfied, then the initial value problem (1) has a pseudo-solution.*

Proof. Since $f(., x_0 + I^\alpha y(.), y(.))$ is Pettis integrable function, then the solution

$$x(t) = x_0 + I^1 f(t, x_0 + I^\alpha y(t), y(t)), \; t \in I$$

of (1) is an absolutely continuous (proposition [14]). Thus for any $\phi \in E^*$ we have

$$\frac{d}{dt}\phi(x(t)) = \phi(f(t, x_0 + I^\alpha y(t), y(t))), \; a.e \; t \in I.$$

$$x'(t) = f(t, x_0 + I^\alpha y(t), y(t)) = f(t, x(t), D^\alpha x(t)).$$

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