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DISCRETE ADOMIAN DECOMPOSITION METHOD FOR SOLVING A CLASS OF NONLINEAR FREDHOLM VOLTERRA INTEGRAL EQUATION IN TWO DIMENSIONS

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Abstract. Discrete Adomian decomposition method (DADM) arises when the quadrature rules are used to approximate the integrals which can not be computed analytically using the traditional Adomian decomposition method (ADM). In this paper, DADM is used to solve a class of nonlinear Fredholm Volterra integral equation with degenerate or non-degenerate kernels. The main advantage of DADM is that the computation of the solution need not to solve nonlinear algebraic system of equations like Nystrom method or projection methods. Another advantage is that the coefficient matrices are not changed during the computation of all components. Finally, convergence of the technique is discussed and the error is estimated.

1. Introduction

Many problems of mathematical physics are reduced to the solution of two-dimensional integral equations in the nonlinear case. These type of integral equations have rarely been studied to solve numerically and primary works in this area have been done in the last two decades (see [1] -[5]). Papers ([1] -[5]) reduce the solution of the nonlinear integral equation to the solution of a nonlinear system of algebraic equations. The iteration methods, for example Newton's method, for solving such cumbersome nonlinear system is usually sensitive to the selection of initial guess. DADM can overcome this obstacle and solve nonlinear integral equation, see section 2.

The topic of ADM, introduced by Adomian ([6] ,[7]), has been rapidly growing in recent years. ADM possesses a great potential in solving different kinds of functional equations. Application of ADM to different types of integral equations was discussed by many authors for example (see [8] -[12]). In this paper we consider the two dimensional nonlinear Fredholm-Volterra integral equation

$$u(x,t) = f(x,t) + \lambda_1 \int_0^t k_1(t,\tau)\, f_1(u(x,\tau))\, d\tau + \lambda_2 \int_a^b k_2(x,s)\, f_2(u(s,t))\, ds. \quad (1)$$

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Equation (1) is called Fredholm integral equation with respect to the position and Volterra with respect to the time. This type of equation appears in many problems of mathematical physics, theory of elasticity, contact problems, and mixed problems of mechanics of continuous media (see [13] -[15]). Several numerical methods for obtaining the approximate solution of equation (1) with continuous kernel are known ([2], [16] - [19]). The interested reader should consult the fine expositions by Linz [20], Goldberg [21], Atkinson [22, 23], Delves and Mohammed [24] for numerical methods and consult the book by Tricomi [25] for information concerning analytical solution methods. In this work we assume $f(x,t)$ is bounded $\forall x \in [a,b], t \in [0,T]$, the kernel of Volterra term is bounded such that $|k_1(t,\tau)| \leq M_1$, $\forall\ 0 \leq \tau \leq t \leq T < \infty$ and the kernel of the Fredholm term is bounded such that $|k_2(x,s)| \leq M_2$, $\forall\ a \leq x$ and $s \leq b$. The nonlinear terms $f_1(u)$ and $f_2(u)$ are Lipschitzian with $|f_1(u) - f_1(z)| \leq L_1 |u-z|$, $|f_2(u) - f_2(w)| \leq L_2 |u-w|$ and have Adomian polynomials representations

$$f_1(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \cdots, u_n),\ f_2(u) = \sum_{n=0}^{\infty} B_n(u_0, u_1, u_2, \cdots, u_n). \tag{2}$$

The author in [26, 27] deduced a new formula to the Adomian's polynomials A_n and B_n which can be written in the form

$$\begin{aligned} A_n &= f_1(S_n) - \sum_{i=0}^{n-1} A_i, \\ B_n &= f_2(S_n) - \sum_{i=0}^{n-1} B_i, \end{aligned} \tag{3}$$

where the partial sum $S_n = \sum_{i=0}^{n} u_i$, $A_0 = f_1(u_0)$ and $B_0 = f_2(u_0)$. Formula (3) is called an accelerated Adomian polynomials and it was used successfully in [11] for solving a class of nonlinear fractional differential equations and in [28] for solving a class of nonlinear partial differential equations. Formula (3) has the advantage of absence of any derivative terms in the recursion, thereby allowing for ease of computation. Applying ADM to (1) yields

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t), \tag{4}$$

where the components $u_i(x,t), i \geq 0$ are computed using the following recursive relations

$$u_0(x,t) = f(x,t), \tag{5}$$

$$u_{m+1}(x,t) = \lambda_1 \int_0^t k_1(t,\tau) A_m(x,\tau) d\tau + \lambda_2 \int_a^b k_2(x,s) B_m(s,t) ds,\ m \geq 0. \tag{6}$$

The computation of each component $u_i(x,t), i \geq 1$ requires the computation of integrals in equation (6). If the evaluation of these integrals analytically are possible, ADM can be applied in a simple manner. In case where the evaluation of any integral in (6) is analytically impossible, DADM can be directly applied, please see the details in section 2. In section 3, convergence analysis will be introduced including the sufficient condition that guarantees a unique solution to problem (1) (see Theorem 1), convergence of ADM will be discussed (see Theorem 2), the maximum absolute truncation error of the Adomian's series solution (4) will be estimated (see Theorem 3) and equivalence between DADM and ADM will be introduced (see Theorem 4). Finally, to verify the theoretical results, some numerical examples are presented in section 4.

2. Discrete Adomian Decomposition Method

DADM is a numerical version of ADM. In paper [29], DADM is used to solve linear and nonlinear Fredholm integral equation. DADM arises when the quadrature rules are used to approximate the integrals which can not be computed analytically. Consider any numerical integration scheme to approximate definite integral by the following formula ([30] -[31])

$$\int_a^b g(s)ds \approx \sum_{j=0}^{n} w_{n,j}\; g(s_{n,j}), \tag{7}$$

where $g(s)$ is continuous function on $[a,b]$, $s_{n,j} = a + jh$ are the nodes of the quadrature rule, $h = (b-a)/n$ and $w_{n,j}$, $j = 0, 1, 2, ..., n$ are the weight functions. The idea is to discretize the independent variables x and t just before applying the quadrature rule. This gives an opportunity to evaluate the integrals in equation (6) numerically but, of course, at the discretization points of the independent variables. Thus, the discrete version of equations (5) and (6) take the form

$$\widetilde{u}_0(s_{n,i}, s_{n,j}) = f(s_{n,i}, s_{n,j}), \quad and \tag{8}$$

$$\begin{aligned}\widetilde{u}_{m+1}(s_{n,i}, s_{n,j}) = \lambda_1 \sum_{r=0}^{j} w_{n,r}\; k_1(s_{n,j}, s_{n,r})\; A_m(s_{n,i}, s_{n,r}) \\ + \lambda_2 \sum_{r=0}^{n} w_{n,r} k_2(s_{n,i}, s_{n,r}) B_m(s_{n,r}, s_{n,j}),\end{aligned} \tag{9}$$

$m \geq 0$, $s_{n,i} = a + ih$, $i = 0, 1, ..., n$, $s_{n,j} = j\frac{T}{n}$, $j = 0, 1, ..., n$ and $w_{n,j}$ are the weight functions of any numerical integration scheme. The approximate solution of equation (1) using DADM can be computed as

$$\widetilde{u}(s_{n,i}, s_{n,j}) = \sum_{m=0}^{\infty} \widetilde{u_m}(s_{n,i}, s_{n,j}). \tag{10}$$

Rewriting equations (8), (9) and (10) in matrix form

$$U_0 = F, \tag{11}$$

$$U_{m+1} = C\; G_m + D\; H_m\;, \quad m \geq 0, \quad \text{and} \tag{12}$$

$$U = \sum_{m=0}^{\infty} U_m \tag{13}$$

where U_0, U_m, U, F, C, D, G_m and H_m are all matrices with dimension $(n+1)\times(n+1)$ such that

$$\begin{aligned}
U_m &= \left[\widetilde{u}_m(s_{n,i}, s_{n,j})\right], \; i=0,1,...,n, \; j=0,1,...,n, \; m\geq 0,\\
F &= \left[f(s_{n,i}, s_{n,j})\right], \; i=0,1,...,n, \; j=0,1,...,n,\\
U &= \left[\widetilde{u}(s_{n,i}, s_{n,j})\right], \; i=0,1,...,n, \; j=0,1,...,n,\\
G_m &= \left[A_m(s_{n,i}, s_{n,r})\right], \; i=0,1,...,n, \; r=0,1,...,n,\\
H_m &= \left[B_m(s_{n,r}, s_{n,j})\right], \; r=0,1,...,n, \; j=0,1,...,n,\\
C &= \left[c_{r,j}\right], \; r=0,1,...,n, \; j=0,1,...,n,\\
c_{r,j} &= \{.\lambda_1 w_{n,r} \; k_1(s_{n,j}, s_{n,r}), \quad j\geq 1 \text{ and } r\leq j0 \qquad , \qquad j=0 \text{ or } r>j,\\
D &= \left[\lambda_2 w_{n,r} k_2(s_{n,i}, s_{n,r})\right], \; i=0,1,...,n, \; r=0,1,...,n.
\end{aligned}$$

The main advantage of DADM is that the computation of the solution need not to solve nonlinear algebraic system of equations like Nystrom method and projection methods. Another advantage of DADM is that the matrices C and D are unchanged during the computation of components U_m, $m\geq 1$ in equation (12). Also, DADM can be used for solving equation (1) with degenerate or non-degenerate kernels $k_1(t,\tau)$ and $k_2(x,s)$.

3. Convergence Analysis

Convergence of the Adomian series solution was studied for different problems and by many authors. In ([32] -[33]), convergence was investigated when the method applied to a general functional equations and to specific type of equations in ([34] -[35]). In convergence analysis, Adomian polynomials play a very important role however, these polynomials cannot utilize all the information concerning the obtained successive terms of the series solution, which could affect directly the accuracy as well as the convergence region and the convergence rate.

3.1. **Uniqueness Theorem.** *Problem* (1) *has a unique solution whenever* $0<\alpha<1$, $\alpha=\alpha_1+\alpha_2$, $\alpha_1=|\lambda_1|\,L_1M_1T$, $\alpha_2=|\lambda_2|\,L_2M_2(b-a)$.

Proof. see [36]. □

3.2. **Convergence Theorem.** The series solution (4) of problem (1) using ADM converges if: $0<\alpha<1$ and $f(x,t)$ bounded on the interval J.

Proof. see [36]. □

3.3. **Error Estimate.** The maximum absolute truncation error of the series solution (4) to problem (1) is estimated to be: $\max\limits_{\forall x,t\in J}\left|u(x,t)-\sum_{i=0}^{m}u_i(x,t)\right| \leq \frac{\alpha^m}{1-\alpha}\left(\frac{\phi_1\alpha_1}{L_1}+\frac{\phi_2\alpha_2}{L_2}\right)$ where $\phi_1=\max\limits_{\forall x,t\in J}|f_1(u_0)|$ and $\phi_2=\max\limits_{\forall x,t\in J}|f_2(u_0)|$.

Proof. see [36]. □

3.4. **Equivalence between DADM and ADM.** Let $D=[a,b]$ and $D^*=[0,T]$ are closed bounded sets in $\mathbf{R}^2$, $J=D\times D^*$. Define operators κ and κ^* such that

$$\kappa^* f_1(u) = \int_0^t k_1(t,\tau)\ f_1(u(x,\tau))\,d\tau, \quad t\in D^*, x\in D,\ u\in C(J)\,, \tag{14a}$$

$$\kappa f_2(u) = \int_D k_2(x,s)\ f_2(u(s,t))\,ds, \quad t\in D^*, x\in D,\ u\in C(J)\,, \tag{14b}$$

where κ and κ^* are compact bounded operators on $C(J)$ to $C(J)$ since

$$\|\kappa^* f_1(u)\| \le \|\kappa^*\|\cdot\|f_1(u)\| \quad \text{and}\ \|\kappa^*\| = \max_{t\in D^*}\int_0^t |k_1(t,\tau)|\,d\tau,$$

$$\|\kappa f_2(u)\| \le \|\kappa\|\cdot\|f_2(u)\| \quad \text{and}\ \|\kappa\| = \max_{x\in D}\int_D |k_2(x,s)|\,ds.$$

Now, equation (1) can be written as

$$u = f + \lambda_1\kappa^* f_1(u) + \lambda_2\kappa f_2(u)\,,$$

let u be the solution obtained by using ADM, where $u=\sum_{m=0}^{\infty}u_m$ and $u_0=f$.

Define numerical integral operators κ_n and κ_n^* as

$$\kappa_n^*\ f_1\left(\tilde{u}(x,t)\right) = \sum_{r=0}^{j} w_{n,r}\ k_1(t,s_{n,r})\ f_1\left(\tilde{u}(x,s_{n,r})\right), \tag{15a}$$

$$\kappa_n\ f_2\left(\tilde{u}(x,t)\right) = \sum_{r=0}^{n} w_{n,r}\ k_2(x,s_{n,r})\ f_2\left(\tilde{u}(s_{n,r},t)\right), \tag{15b}$$

where κ_n and κ_n^* are linear finite rank bounded operator on $C(J)$ to $C(J)$ since

$$\|\kappa_n^*\| = \max_{t\in D^*}\sum_{r=0}^{j}|w_{n,r}\ k_1(t,s_{n,r})| \ \ \textit{and}\ \ \|\kappa_n\| = \max_{x\in D}\sum_{j=0}^{n}|w_{n,r}\ k_2(x,s_{n,r})|$$

With the operators κ_n and κ_n^* , equation (1) can be written as

$$\tilde{u} = f + \lambda_1\kappa_n^* f_1\left(\tilde{u}\right) + \lambda_2\kappa_n f_2\left(\tilde{u}\right),$$

where $\tilde{u}$ here is the solution obtained by using *DADM,* and $\tilde{u}=\sum_{m=0}^{\infty}\tilde{u}_m$ and $\tilde{u}_0=f$.

Since $\|\kappa_n\ g-\kappa g\|\to 0$ and $\|\kappa_n^*\ g-\kappa^* g\|\to 0$ as $n\to\infty$ where $g\in C(J)$ [23]. Then, the solution of equation (1), using DADM converges to the solution of the same equation when using ADM, i.e.

$$\tilde{u}\ \to u \text{ as } n\to\infty,$$

Proof. Since $u=\sum_{m=0}^{\infty}u_m$, $u_0=f$, $\tilde{u}=\sum_{m=0}^{\infty}\tilde{u}_m$, and $\tilde{u}_0=f$. Starting with

$$\left\|\tilde{u}-u\right\| = \left\|\sum_{m=0}^{\infty}\left(\tilde{u}_m-u_m\right)\right\| \le \sum_{m=0}^{\infty}\left\|\tilde{u}_m-u_m\right\|. \tag{16}$$

Since

$$\begin{aligned} \left\|\widetilde{u}_0 - u_0\right\| &= \|f - f\| = 0, \text{ and} & (17) \\ \left\|\widetilde{u}_m - u_m\right\| &= \|(\lambda_1 \kappa_n^* A_{m-1} + \lambda_2 \kappa_n B_{m-1}) - (\lambda_1 \kappa^* A_{m-1} + \lambda_2 \kappa B_{m-1})\| \\ &\leq \lambda_1 \|\kappa_n^* A_{m-1} - \kappa^* A_{m-1}\| + \lambda_2 \|\kappa_n B_{m-1} - \kappa B_{m-1}\| & (18) \end{aligned}$$

thus $\left\|\widetilde{u}_m - u_m\right\| \to 0$ as $n \to \infty$. Then, by induction and substituting from equation (17) and equation (18) into inequality (16), this completes the proof. □

4. Numerical Experiment

Example (1) Consider the following nonlinear Fredholm Volterra integral equation

$$u(x,t) = f(x,t) + \frac{1}{10}\int_0^t \exp\left(t^2 + \tau^3\right) u^2(x,\tau)\, d\tau + \frac{1}{10}\int_0^1 \exp\left(x^2 + s^4\right) u^3(s,t)\, ds,$$

$f(x,t) = xt + \frac{1}{30}x^2 \exp\left(t^2\right)\left[1 - \exp\left(t^3\right)\right] + \frac{1}{40}t^3 \exp\left(x^2\right)\left[1 - \exp(1)\right]$ with exact solution $u(x,t) = xt$.In this example the ADM can not be applied because the integral $\int_0^1 \exp\left(s^4\right) ds$ has no analytical solution. Here, DADM is the suitable method to obtain solution using equations (8) and (9). Table (1) shows the absolute error $|e_m^n(x,t)| = \left|u(x,t) - \widetilde{u}(x,t)\right|$ at nodes of the quadrature rule, $n+1$ is number of nodes and m is number of components computed form equation (9).

Table (1) the absolute error of example (1)

t	x	$\left\|e_7^4(x,t)\right\|$	t	x	$\left\|e_7^4(x,t)\right\|$
0.25	0.25	$8.6521e-9$	0.75	0.25	$5.1963e-8$
	0.50	$2.0034e-8$		0.50	$9.8064e-8$
	0.75	$6.0231e-8$		0.75	$1.9803e-7$
	1.00	$2.6562e-7$		1.00	$8.6432e-7$
0.50	0.25	$3.2846e-8$	1.00	0.25	$9.7771e-8$
	0.50	$5.6684e-8$		0.50	$2.6187e-7$
	0.75	$9.1503e-8$		0.75	$6.3509e-7$
	1.00	$4.3159e-7$		1.00	$1.2695e-6$

Example (2) consider the following nonlinear Fredholm Volterra integral equation

$$u(x,t) = f(x,t) + \frac{1}{10}\int_0^t t\tau u^2(x,\tau)\, d\tau + \frac{1}{10}\int_0^1 \cos(\exp(s) + x)\exp(u(s,t))\, ds,$$

$f(x,t) = x + t^2 + \frac{1}{10}\exp\left(t^2\right)\left[\sin(1+x) - \sin(\exp(1) + x)\right] + \frac{1}{20}t\left[x^3 - \left(x + t^2\right)^3\right]$ whose exact solution is $u(x,t) = x + t^2$. In this example the ADM can not be applied because the integral $\int_0^1 \cos[\exp(s) + s]\ \exp[\sin(s)]\, ds$ has no analytical solution. Here, DADM is the suitable method to obtain solution using equations (8) and (9). Table (2) shows the absolute error at nodes of the quadrature rule.

Table (2) the absolute error of example (2)

t	x	$\left\|e_8^4(x,t)\right\|$	t	x	$\left\|e_8^4(x,t)\right\|$
0.25	0.25	$7.4738e-8$	0.75	0.25	$4.0951e-7$
	0.50	$1.8469e-7$		0.50	$8.2409e-7$
	0.75	$5.7642e-7$		0.75	$2.1163e-6$
	1.00	$1.6942e-6$		1.00	$9.6853e-6$
0.50	0.25	$1.5231e-7$	1.00	0.25	$8.8996e-7$
	0.50	$4.3761e-7$		0.50	$2.3301e-6$
	0.75	$8.9357e-7$		0.75	$7.6582e-6$
	1.00	$5.6554e-6$		1.00	$2.3214e-5$

Example (3) Consider the following nonlinear Fredholm Volterra integral equation

$$u(x,t) = f(x,t) + \frac{1}{10}\int_0^t (t-\tau)\, u^2(x,\tau)\, d\tau + \frac{1}{2}\int_0^1 \left(x^2 - s\right) u^2(s,t)\, ds,$$

$f(x,t) = \frac{t}{120}[t\left(1-2x^2\right)+120x(1-x)-t^3x^2(1-x)^2]$ with exact solution $u(x,t) = xt(1-x)$.In this example the ADM can be applied. Also, DADM can be used to obtain solution using equations (8) and (9). Table (3) shows the absolute error at nodesof the quadrature rule.

Table (3) the absolute error of example (3)

t	x	$\left\|e_7^4(x,t)\right\|$	t	x	$\left\|e_7^4(x,t)\right\|$
0.25	0.25	$9.3245e-8$	0.75	0.25	$6.5621e-7$
	0.50	$1.9654e-7$		0.50	$9.2604e-7$
	0.75	$2.8536e-7$		0.75	$2.4397e-6$
	1.00	$5.6472e-7$		1.00	$6.5371e-6$
0.50	0.25	$2.6210e-7$	1.00	0.25	$9.9768e-7$
	0.50	$4.0821e-7$		0.50	$3.1943e-6$
	0.75	$8.6458e-7$		0.75	$6.3905e-6$
	1.00	$1.9301e-6$		1.00	$1.7654e-5$

5. Conclusion

DADM is cricullay when the integrals through the classical ADM can not performed analytically. The main advantage of DADM is that the computation of the solution need not to solve nonlinear algebraic system of equations like Nystrom method or projection methods. Another advantage of DADM is that the matrices C and D are unchanged during the computation of components U_m, $m \geq 1$ in equation (12). Also, DADM can be used for solving equation (1) with degenerate or non-degenerate kernels $k_1(t,\tau)$ and $k_2(x,s)$. Convergence of the technique is discussed and the error is estimated.

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