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# TIME DOMAIN CIRCUIT RESPONSE USING FRACTIONAL DIFFERENTIAL TRANSFORM METHOD (FDTM).

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Abstract. This parer introduces the use of fractional differential transform method to obtain the transient response in circuit applications. The existance and uniqueness of the solution and continuous dependence on initial conditions are studied. The mathematical model is solved by fractional differential transform method and the numerical results are compared with the results obtained by the circuit state eqyations.

## 1. Introduction

In recent years fractional differential equations have gained a considerable amount of interest due to their many applications in several fields: signal processing, fluid mechanics, viscoelasticity, mathematical biology, and bioengineering [1, 2, 3, 4, 5]. Accurate modeling of engineering and scientific systems have become imperative these days due to their extensive usage in safety-critical domains, such as, medicine and transportation. This fact has led to the widespread usage of fractional calculus. The influences of fractional-order modeling are discussed in many established fields and generate new concepts such as the fractional-order circuit theories [6, 7, 8], electromagnetic and Smith chart [9, 10], chaotic systems [11], stability analysis [12, 13, 14]. In control engineering, the concept of fractional operations is mostly used in fractional system identification [15], biomimetic control [16],$PI^{\lambda}D^{\mu}$ controllers [17], fractional $PI^{\alpha}$ [18], $PD^{\lambda}$ controller [19], in signal processing, fractional operators are used in the design of fractional order differentiators and integrators [20] and for modeling the speech signals [21].

The electric system may be modeled more accurately by fractional differential equation. For instance, in integer model, the current flow in ideal capacitor was modeled by the relation $i_c(t) = C\frac{dv_c(t)}{dt}$ where $i_c(t)$ is the capacitor current, $C$ is the capacitance of the capacitor. It has been shown that for $0<\beta<1$, the relationship $i_c(t) = CD^{\beta}v_c(t)$ where $D^{\beta}$ is the Caputo derivative of order $\beta$ , provides more accurate representations of real capacitor [22, 23, 24].

In general, the differential transform method is applied to solve the electric circuit problems [25]. This method is an iterative procedure for obtaining analytic Taylor series solution of ordinary or partial differential equations.

Recently, the application of differential transform method is successfully extended to obtain analytical approximate solutions to linear and nonlinear ordinary differential equations of fractional order [26]. Consequently, Erturk and Momani developed a new application of this method to provide approximate solutions for the system of fractional differential equations [27].
In this paper, some definitions are introduced in section 2. The basic fractional-order differential equations for a circuit with active elements is discussed in section 3.the circuit state equation is illustrated in section 4.The existence and uniqueness of the solution and continuous dependence on initial conditions are studied in section 5 and 6. The mathematical model is solved by fractional differential transform method in section 7. Numerical example is studied in section 8.

## 2. Preliminaries and notation

In this section, some definitions and properties of the fractional calculus [29] and differential transform method [25, 26, 27, 28], are introduced.

### 2.1. Fractional calculus.

Let $L^1 = L^1[a,b], 0 \leq a, b < \infty$ be a class of Lebesgue integralable functions on $[a,b]$.

**Definition 1.** *The Riemann-Liouville integral operator $J^\alpha$ of order $\alpha \geq 0$ for a function $f(t) \in L^1$, is defined by*

$$J^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1} f(s)}{\Gamma(\alpha)} ds \tag{1}$$

$$J^0 f(t) = f(t) \tag{2}$$

where $\Gamma(\alpha)$ is the gamma function, $\alpha > 0$.

**Definition 2.** *The Riemann-Liouville fractional derivative of $f(t)$ of order $\alpha$, $m-1<\alpha<m$ , is defined by*

$$^R_aD^\alpha_t f(t) = \frac{d^m}{dt^m} J_a^{m-\alpha} f(t) \tag{3}$$

Where the subscripts $a$ and $t$ denote the two limits related to the operation of fractional differentiation.
The Caputo representation for fractional order derivative satisfies these requirements. In the Caputo case, the derivative of a constant is zero, therefore we can define, properly, the initial conditions for the fractional differential equations which can be handled by using an analogy with the classical integer case. As a result, in this manuscript we use the Caputo fractional derivative for a function of time, $f(t)$ defined as

$$^c_aD^\alpha_t f(t) = J_a^{n-\alpha} \frac{d^n}{dt^n} f(t) = \int_a^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(\tau) d\tau \tag{4}$$

Where $\Gamma(.)$ is the Euler Gamma function, $a$ is the integration initial condition, $n = 1, 2, 3, ... \in N$ and $n-1 < \alpha \leq n$.

### 2.2. The fractional differential transform method (FDTM). .

In this section, the fractional differential transform method is used to obtain approximate analytical solutions for the system of fractional differential equations

[25, 26, 27, 28].
Firstly, expand the analytical function in terms of a fractional power series

$$f(t) = \sum_{k=0}^{\infty} F(k)(t - t_0)^{\frac{k}{\theta}} \tag{5}$$

Where $\theta$ is the order of the fraction to be selected such that $\alpha\theta$ is a positive integer, $\alpha$ is the order of the fractional differential equation (FDE) being considered and $F(k)$ is the fractional differential transform of $f(t)$.

The transformation of the initial conditions is defined as follows

$$F(k) = \begin{cases} \frac{1}{(k/\theta)!}[\frac{d^{k/\theta} f(t)}{dt^{k/\theta}}]_{t=t_0} & for \quad \frac{k}{\theta} \in Z^+ \\ 0, & for \quad \frac{k}{\theta} \notin Z^+ \end{cases} \tag{6}$$

where $k = 0, 1, 2, ..., (\alpha\theta - 1)$.

The basic properties of the differential transformation

(1) If $f(t) = g(t) \pm h(t)$, then $F(k) = G(k) \pm H(k)$.

(2) If $f(t) = g(t)h(t)$, then $F(k) = \sum_{l=0}^{k} G(l)H(k - l)$.

(3) If $f(t) = (t - t_0)^p$, then $F(k) = \delta(k - \theta p)$

where, $$\delta(k) = \begin{cases} 1, & if \quad k = 0, \\ 0, & if \quad k \neq 0. \end{cases}$$

(4) If $f(t) = D^\alpha[g(t)]$, then $F(k) = \frac{\Gamma(\alpha+1+\frac{k}{\theta})}{\Gamma(1+\frac{k}{\theta})} G(k + \alpha\theta)$.

where $\theta$ is the order of the fraction to be selected and $\alpha$ is the order of the fractional derivative.

## 3. **Fractional order circuit example**

. Figure 1 shows a circuit example which consists of three linear resistors and two fractional order capacitors [14].

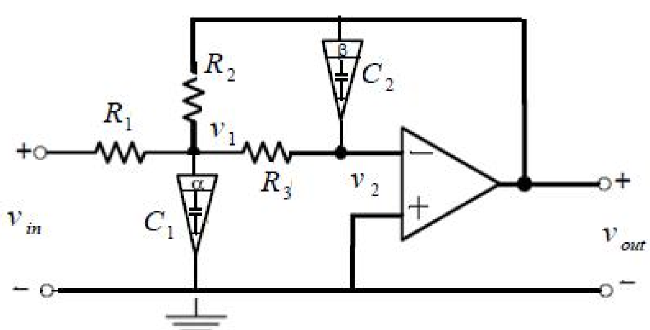


FIGURE 1. Fractional-order circuit example

This circuit can be described by the following system of fractional order differential equations

$$\begin{aligned} {}_0^cD_t^\alpha v_1(t) &= -k_1 v_1(t) + k_2 v_{out}(t) + k_3 v_{in}(t) \\ {}_0^cD_t^\beta v_{out}(t) &= -k_4 v_1(t) \end{aligned} \tag{7}$$

Subject to the initial conditions

$$v_{c_1}(0) = v_1(0) = c_1, v_{c_2}(0) = v_{out}(0) = c_2 \tag{8}$$

where $k_1 = (\frac{1}{R_1C_1} + \frac{1}{R_2C_1} + \frac{1}{R_3C_1}), k_2 = (\frac{1}{R_2C_1}, k_3 = \frac{1}{R_1C_1}$ and $k_4 = \frac{1}{R_3C_2}$

## 4. The circuit state equation

. The state equation of the circuit is described by the form:

$$\begin{bmatrix} {}_0^cD_t^\alpha & x_1(t) \\ {}_0^cD_t^\beta & x_2(t) \end{bmatrix} = \begin{bmatrix} -k_1 & k_2 \\ -k_4 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} k_3 \\ 0 \end{bmatrix} \begin{bmatrix} v_{in}(t) \end{bmatrix} \tag{9}$$

where $x_1(t) = v_1(t)$ and $x_2(t) = v_{out}(t)$,
Assuming the same fractional order i.e. $\alpha = \beta = q$, one obtain

$${}_0^cD_t^qX(t) = AX(t) + Be(t) \qquad\qquad 0 < q < 1. \tag{10}$$

With initial condition given by

$$X(0) = X_0 \tag{11}$$

where
$A = \begin{bmatrix} -k_1 & k_2 \\ -k_4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} k_3 \\ 0 \end{bmatrix}$, $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $X_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$
and $e(t) = \begin{bmatrix} v_{in}(t) \end{bmatrix}$

The solution of equation (10)is given by:

$$x(t) = \Phi_0(t)x_0 + \int_0^\infty \Phi(t-s)Bv_{in}(s)ds, \qquad x(0) = x_0 \tag{12}$$

where

$$\Phi_0(t) = E_q(At^q) = \sum_{m=0}^{\infty} \frac{A^m t^{mq}}{\Gamma(mq+1)} \tag{13}$$

$$\Phi(t) = \sum_{m=0}^{\infty} \frac{A^m t^{(m+1)q-1}}{\Gamma((m+1)q)} \tag{14}$$

and $E_q(At^q)$ denotes the Mittag-Leffler matrix function [30].

## 5. Existence and uniqueness of the solution

. Let $C^*[0,T]$ be a class of continuous column vector $X(t)$ and $x_1, x_2 \in C[0,T]$, the class of continuous functions on the interval $[0,T]$ . The norm of $X \in C^*[0,T]$ is given by [31]

$$\|X\| = \sum_{i,j} \sup_{t\in(0,t]} |x_{ij}(t)|$$

The solution of (10) and (11) is given by:

$$X = X_0 + J^q(AX + Be(t)) \tag{15}$$

Let $F(X) = X$, then

$$\begin{aligned} F(X) - F(Y) = J^q(AX) - J^q(AY) = \\ \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}AXd\tau - \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}AYd\tau \end{aligned} \tag{16}$$

$$|F(X)-F(Y)| \leq \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}\,|AX-AY|\,d\tau \leq \frac{T^q}{q\Gamma(q)}\,|AX-AY| \quad (17)$$

After some calculations, we get the following inequality

$$\|F(X)-F(Y)\| \leq \frac{T^q}{q\Gamma(q)}\max\{(|k_1|+|k_4|),(|k_2|)\}\,\|X-Y\| \quad (18)$$

Therefore the mapping $F$ is contraction if the following condition is satisfied

$$\frac{T^q}{q\Gamma(q)}\max\{(|k_1|+|k_4|),(|k_2|)\} < 1 \quad (19)$$

**Theorem 1.**The sufficient condition for existence and uniqueness of the solution of system (7) with initial conditions (8) and $t \in (0,T]$ is:

$$\frac{T^q}{q\Gamma(q)}\max\{(|k_1|+|k_4|),(|k_2|)\} < 1$$

## 6. **Continuous dependence on initial conditions**

Assume that there are two sets of initial conditions to system (7), $X_0$ and $Y_0$ , which satisfy

$$\|X_0-Y_0\| \leq \delta \quad (20)$$

And assume that the condition in theorem (1) is satisfied. Then

$$X = X_0 + J^q(AX+Be(t)) \quad (21)$$

$$Y = Y_0 + J^q(AY+Be(t)) \quad (22)$$

We get the following inequality

$$\|X-Y\| \leq \|X_0-Y_0\| + g\,\|X-Y\| \quad (23)$$

where

$$g = \frac{T^q}{q\Gamma(q)}\max\{(|k_1|+|k_4|),(|k_2|)\}, \qquad g<1$$

then

$$(1-g)\,\|X-Y\| \leq \|X_0-Y_0\|$$

$$\|X-Y\| \leq \frac{\delta}{(1-g)} \quad (24)$$

Let $\varepsilon = \frac{\delta}{(1-g)}$ then the following relation hold

$$\|X-Y\| \leq \varepsilon \quad (25)$$

**Theorem 2.** For system (7) satisfying the condition of Theorem (1). Then, $\forall\varepsilon > 0, \exists\delta(\varepsilon) = (1-g)\varepsilon > 0$ such that $\|X_0-Y_0\| \leq \delta \Rightarrow \|X-Y\| \leq \varepsilon$ i.e. the solution has continuous dependence on initial conditions

## 7. The solution of the system by using differential transform method

System (7) is transformed by using properties (1) and (4) as follows:

$$\begin{aligned} V_1(k+\alpha\theta) &= \frac{\Gamma(1+\frac{k}{\theta})}{\Gamma(\alpha+1+\frac{k}{\theta})}[-k_1V_1(k)+k_2V_{out}(k)+k_3V_{in}(k)] \\ V_{out}(k+\beta\theta) &= \frac{\Gamma(1+\frac{k}{\theta})}{\Gamma(\beta+1+\frac{k}{\theta})}[-k_4V_1(k)] \end{aligned} \tag{26}$$

where $\theta$ is selected such that $\alpha\theta$ and $\beta\theta$ are positive integer, respectively.
The conditions in equation (8) can be transformed by using equation (6) as follows:

$$\begin{cases} V_1(0)=c_1, V_1(k)=0 & for \quad k=1,2,3,...,\alpha\theta-1. \\ V_{out}(0)=c_2, V_{out}(k)=0 & for \quad k=1,2,3,...,\beta\theta-1. \end{cases} \tag{27}$$

Using equations (26) and (27), $V_1(k)$ for $k=\alpha\theta,\alpha\theta+1,...,n$ and $V_{out}(k)$ for $k=\beta\theta,\beta\theta+1,...,n$ are calculated and using the inverse transformation rule (5), $V_{out}(k)$ is calculated for different values of $\alpha$ and $\beta$.

## 8. Numerical example

In this example, assume the following data [32]: $R_1=200K\Omega, R_2=40K\Omega, R_3=50K\Omega, C_1=25nF, C_2=10nF, v_{in}(t)=6.25\cos(6280t)u(t)$ and $v_1(0)=v_{out}(0)=0$
By substituting the above data in the recurrence relations (26),(27), then, applying the inverse transformation in equation (5) to get the solution

Figure (2), (3) and (4) shows the output voltage at different cases when $(\alpha,\beta)$ is equal to $\{(0.95,1.0),(0.90,1.0),(0.85,1.0),(1.0,0.98),(1.0,0.95),(1.0,0.90),(0.98,0.98),(0.95,0.95),(0.92,0.92),(0.90,0.90)\}$

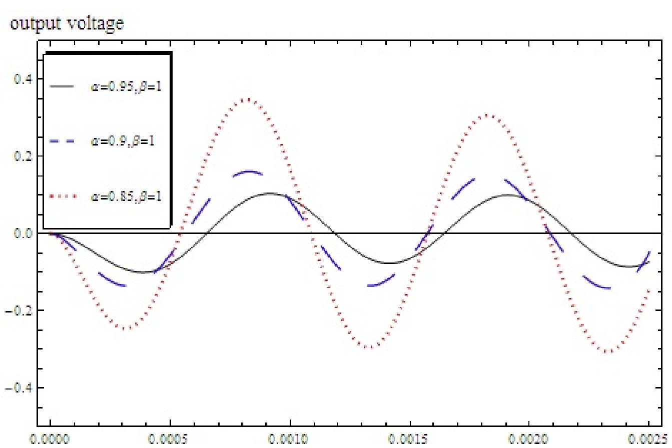


FIGURE 2. the output voltage for different $\alpha$ and fixed $\beta=1$

Table (1) shows the absolute errors between the results obtained by FDTM and the exact solution in equation (12), when $\alpha=\beta=q$ for different values of $q$.

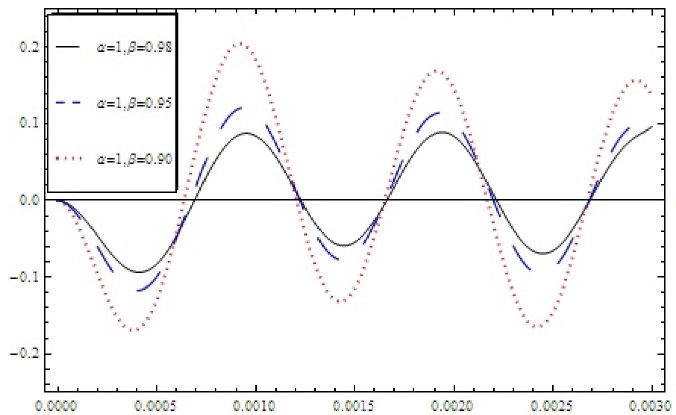


FIGURE 3. the output voltage for different $\beta$ and fixed $\alpha = 1$

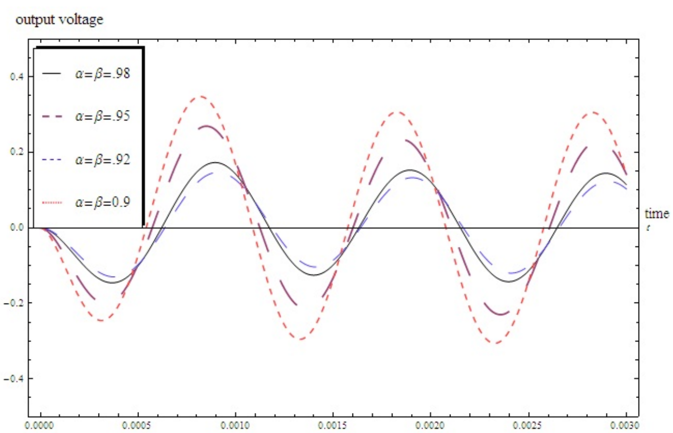


FIGURE 4. the output voltage by FDTM for $\alpha = \beta$ at different values

Table (1) Absolute error of the output voltage for $\alpha = \beta$ at different values

| $t$ | $\alpha = \beta = 0.9$ | $\alpha = \beta = 0.92$ | $\alpha = \beta = 0.95$ | $\alpha = \beta = 0.96$ | $\alpha = \beta = 1$ |
|---|---|---|---|---|---|
| 0.0005 | $6.52256 * 10^{-16}$ | $1.38778 * 10^{-17}$ | $7.63278 * 10^{-16}$ | $1.94289 * 10^{-16}$ | $2.77556 * 10^{-17}$ |
| 0.001 | $1.54876 * 10^{-14}$ | $6.05072 * 10^{-15}$ | $2.36478 * 10^{-14}$ | $1.72085 * 10^{-15}$ | $5.55112 * 10^{-16}$ |
| 0.0015 | $9.41164 * 10^{-13}$ | $3.1336 * 10^{-13}$ | $2.99011 * 10^{-13}$ | $2.55754 * 10^{-13}$ | $7.16094 * 10^{-15}$ |
| 0.002 | $1.85043 * 10^{-11}$ | $7.93148 * 10^{-11}$ | $1.01961 * 10^{-11}$ | $5.34392 * 10^{-13}$ | $2.13607 * 10^{-13}$ |
| 0.0025 | $5.30888 * 10^{-10}$ | $3.9708 * 10^{-8}$ | $6.91724 * 10^{-10}$ | $3.2284 * 10^{-12}$ | $7.56319 * 10^{-13}$ |
| 0.003 | $3.90383 * 10^{-8}$ | $7.5677 * 10^{-6}$ | $1.62107 * 10^{-7}$ | $1.18532 * 10^{-7}$ | $1.90664 * 10^{-10}$ |

## 9. Conclusion

This paper demonstrates generalized concepts from the very narrow integer-order scope systems to fractional-order systems. The fractional modeling introduces new parameters which provide more accurate representations of real capacitor. Fractional differential transform method is suitable for solving many circuits in the fractional-order domain.

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