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THE EFFECT OF VACCINATION ON THE DYNAMICS OF CHILDHOOD DISEASES DESCRIBED BY A FRACTIONAL SIR EPIDEMIC MODEL

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Abstract. Childhood vaccination programs have had a dramatic impact on reducing child mortality worldwide. We introduce fractional-order into A SIR model that monitors the temporal dynamics of a childhood disease in the presence of preventive vaccine is presented in this paper. Generalized Euler method (GEM) is considered in this paper to obtain an analytic approximate solution of this model. The obtained results proved that the disease will persist within the population if the vaccination coverage level is below a certain threshold.

1. Introduction

Although the use of preventive vaccines has dramatically reduced the incidence of infectious diseases among children, childhood diseases remain an important public health problem. Decreased immunization coverage among children together with irregular supply of preventive vaccines are considered to be the major factors associated with the resurgence of numerous childhood diseases [1]. For some childhood diseases, such as Measles, Rubella and Chicken pox, preventive vaccines (administered to children who have not yet been infected) may induce a permanent immunity against the diseases. A universal effort to extend vaccination coverage to all children began in 1974, When the World Health Organization (WHO) founded the Expanded Program on Immunization (EPI). Mathematical models, of deterministic type, have often been used to provide deeper insights into the transmission dynamics of childhood diseases and to evaluate control strategies [4,5,15]. In the SIR model presented in this paper, the population that is involved in the spread of an infection is split into three epidemiological classes: a susceptible group (S), an infected group (I), and a removed group (R) denoting vaccinated as well as recovered people [10,13] with permanent immunity. This model assumes that the efficacy of the vaccine is 100% and the natural death rates μ in the classes remain unequal to births, so that the population size N is realistically not constant. Citizens are born into the population at a constant birth rate π with extremely very low childhood disease mortality rate. We denote the fraction of citizens vaccinated

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at birth each year as P (with $0 < P < 1$) and assume the rest are susceptible. A susceptible individual will move into the infected group through contact with an infected individual, approximated by an average contact rate β. An infected individual recovers at a rate γ, and enters removed group. The removed group also contains people who are vaccinated [12]. The differential equations for the SIR model are

$$\begin{aligned} \frac{dS}{dt} &= (1-P)\pi N - \beta\frac{SI}{N}\mu S, \\ \frac{dI}{dt} &= \beta\frac{SI}{N} - (\gamma+\mu)I, \\ \frac{dR}{dt} &= P\pi N + \gamma I - \mu R. \end{aligned} \tag{1}$$

Where $N = S + I + R$, and the parameters μ, π, β, γ are assumed to be positive. By adding the three equations of the above system (1), we obtain

$$\frac{dN}{dt} = (\pi - \mu)N,$$

so that we are now dealing with a varying total population [12]. The groups can be scaled by population N using the new variables, $s = S/N, i = I/N$, and $r = R/N$. The population is now normalised, meaning $s + i + r = 1$, and we have the new system:

$$\begin{aligned} \frac{ds}{dt} &= (1-P)\pi - \beta si - \pi s, \\ \frac{di}{dt} &= \beta si - (\gamma+\mu)i, \\ \frac{dr}{dt} &= P\pi + \gamma i - \mu r. \end{aligned} \tag{2}$$

The rest of the paper is organized as follows. Section 2 gives an idea about the fractional calculus. Generalized Taylor's formula is discussed in section 3. In section 4, a summary about the Generalized Euler method is presented. A fraction order is introduced to the presented model in section 5. Numerical results are reported in section 6.

2. Fractional calculus

Although fractional derivatives have a long mathematical history [8], for many years they were not used in biology or physics. One possible explanation of such unpopularity could be that there are multiple nonequivalent definitions of fractional derivatives [9]. Another difficult is that fractional derivatives have no evident geometrical interpretation because of their nonlocal character [10]. It was found that various; especially interdisciplinary applications can be elegantly modeled with the help of the fractional derivatives. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives, and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow. However, during the last ten years fractional calculus starts to attract much more attention of physicists and mathematicians [14]. In biology, it has been deduced that the membranes of cells of biological organism have fractional-order electrical conductance [6] and then are classified in groups of non-integer order models. FODE are naturally related to systems with

memory which exists in most biological systems. Also, they are closely related to fractals, which are abundant in biological systems [8,9]. We first give the definition of fractional-order integration and fractional-order differentiation [11]. For the concept of fractional derivative, we will adopt Caputo's definition, which is a modification of the Riemann-Liouville definition and has the advantage of dealing properly with initial value problems.

definition 1. The fractional integral of order $\alpha > 0$ of a function $f : R^+ \to R$ is given by

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, x > 0,$$
$$J^0 f(x) = f(x).$$

Hence we have

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}, \quad \alpha > 0, \gamma > -1, t > 0$$

Definition 2. Riemann-Liouville and Caputo fractional derivatives order α of a continuous function $f : R^+ \to R$ is given respectively by

$$D_*^\alpha f(x) = D^m(J^{m-\alpha} f(x)),$$
$$D^\alpha f(x) = J^{m-\alpha}(D^m f(x)), \text{ Where } m-1 < \alpha \leq m, m \in N.$$

The definition of fractional derivative involves an integration which is non-local operator (as it is defined on an interval) so fractional derivative is a non-local operator. In other word, calculating time-fractional derivative of a function $f(t)$ at some time $t = t_1$ requires all the previous history, i.e. all $f(t)$ from $t = 0$ to $t = t_1$.

3. Generalized Taylor's formula

In this section we introduce a generalization of Taylor's formula that involves Caputo fractional derivatives. This generalization is presented in [17]. Suppose that $D_*^{k\alpha} f(x) \in C(0, a]$, for $k = 0, 1, \cdots, n+1$ where $0 < \alpha \leq 1$. Then we have

$$f(x) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} (D_*^{i\alpha}(0+) + \frac{(D_*^{(n+1)\alpha})(\xi)}{\Gamma((n+1)\alpha+1)} x^{(n+1)\alpha} \tag{3}$$

With $0 \leq \xi \leq x, \forall\, x \in (0, a]$.

In case of $\alpha = 1$, the generalized Taylor's formula (5) reduces to the classical Taylor's formula.

4. Generalized Euler method (GEM)

Most nonlinear fractional differential equations do not have analytic solutions, so approximations and numerical techniques must be used [10]. The decomposition method (ADM) and the variational iteration method (VIM) are relatively new approaches to provide an analytical approximate solution to linear and nonlinear problems, and they are particularly valuable as tools for scientists and applied mathematicians, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations. In recent years, the application of the ADM, VIM, [11,13] in linear and nonlinear problems has been developed. On the other hand, these methods are effective for small time, i.e. $t << 1$, however the standard homotopy perturbation method (HPM) cannot solve the problem for larger time and in fact the solution of the chaotic system using HPM is an open problem

[13]. Nevertheless by chance, there are cases at which these methods give good approximation for a large range of t [4]. A few numerical methods for fractional differential equations have been presented in the literature [7,10]. However many of these methods are used for very specific types of differential equations, often just linear equations or even smaller classes. In [16], Odibat and Momani derived the generalized Euler's method that we have used for the numerical solution of initial value problems with Caputo derivatives. The method is a generalization of the classical Euler's method. Consider the initial value problem

$$D_*^{\alpha} y(t) = f(t, y(t)), y(0) = y_0, \;\; 0 < \alpha \leq 1, t > 0 \tag{4}$$

Let $[0, a]$ be the interval over which we want to find the solution of the problem (4). In actuality, we will not find a function $y(t)$ that satisfies the initial value problem (4). Instead, a set of points $\{(t_j, y(t_j))\}$is generated, and the points are used for our approximation. For convenience we subdivide the interval $[0, a]$into k subintervals $[t_j, t_{j+1}]$ of equal width $h = a/k$ by using the nodes $t_j = jh$, for $j = 0, 1, \cdots, k$. Assume that $y(t), D_*^{\alpha} y(t)$, and $D_*^{2\alpha} y(t)$ are continuous on $[0, a]$ and use the generalized Taylor's formula (3) to expand $y(t)$ about $t = t_0 = 0$. For each value t there is a value c_1 so that

$$y(t) = y(t_0) + (D_*^{\alpha}(t))(t_0)\frac{t^{\alpha}}{\Gamma(\alpha + 1)} + (D_*^{2\alpha} y(t))(c_1)\frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \tag{5}$$

When $(D_*^{\alpha} y(t))(t_0) = f(t_y, (t_0))$ and $h = t_1$ are substituted into equation (5), the result is an expression for $y(t_1)$:

$$y(t_1) = y(t_0) + f(t_0, y(t_0))\frac{h^{\alpha}}{\Gamma(\alpha + 1)} + (D_*^{2\alpha} y(t))(c_1)\frac{h^{2\alpha}}{\Gamma(2\alpha + 1)}$$

If the step size h is chosen small enough, then we may neglect the second-order term (involving $h^{2\alpha}$) and get

$$y(t_1) = y(t_0) + \frac{h^{\alpha}}{\Gamma(\alpha + 1)} f(t_0, y(t_0))$$

The process is repeated and generates a sequence of points that approximates the solution $y(t)$. The general formula for generalized Euler's method (GEM) when $t_{j+1} = t_j + h$ is

$$y(t_{j+1}) = y(t_j) + \frac{h^{\alpha}}{\Gamma(\alpha + 1)} f(t_j, y(t_j)) \tag{6}$$

for $j = 0, 1, \cdots, k-1$. It is clear that if $\alpha = 1$, then the generalized Euler's method (6) reduces to the classical Euler's method [2,3].

5. Fractional model derivation

Now we introduce fractional-order into the model (1). The new system is described by the following set of FDEs of order $\alpha_1, \alpha_2, \alpha_3 > 0$:

$$\begin{aligned} D^{\alpha_1(s)} &= (1 - P)\pi - \beta si - \pi s, \\ D^{\alpha_2}(i) &= \beta si - (\gamma + \pi)i, \\ D^{\alpha_3}(r) &= P\pi + \gamma i - \pi r. \end{aligned} \tag{7}$$

This paper attempts to find a numerical solution for a general class of fractional order model of childhood diseases (7). For this purpose the paper summarizes

specific techniques for generalized Euler method (GEM) [16,17], as well as the applications of Caputo fractional calculus.

The reason of using fractional order differential equations (FOD) is that FOD are naturally related to systems with memory which exists in most biological systems. Also they are closely related to fractals which are abundant in biological systems. The results derived of the fractional system (7) are of a more general nature. We would like to put your attention that time fractional derivatives change also the solutions we usually get in standard system (2). The concept of fractional or non-integer order derivation and integration can be traced back to the genesis of integer order calculus itself [7]. Most of the mathematical theory applicable to the study of non-integer order calculus was developed through the end of 19th century. However it is in the past hundred years that the most intriguing leaps in engineering and scientific application have been found. The calculation technique has in some cases had to change to meet the requirement of physical reality. The derivatives are understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses.

6. Numerical results and discussion

In this section, we will study the effect of vaccination on the dynamics of a childhood disease described by the SIR model (7) using GEM. If we consider that h=0.1 in (6), and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ in (7) the obtained results are shown in Figures 1,2,3.

Table. 1. Effect of vaccination coverage at various parameter values ($P_c = 0.4625$).

case	s_0	i_0	r_0	β	γ	π	P	R_v	Comments
1	1	0	0	0.8	0.03	0.4	0.9	0.18604	E_0 stable (disease eradication)
2	0.8	0.2	0	0.8	0.03	0.4	0.9	0.18604	E_0 stable (disease eradication)
3	0.8	0.2	0	0.8	0.03	0.4	0.3	1.30223	E_0 stable (no eradication)
4	0.8	0.2	0	0.8	0.03	0.4	0	1.86046	E_0 stable (no eradication)

7. Conclusion

In this paper, generalized Euler method (GEM) was implemented to describe the effect of vaccination on the dynamics of a childhood disease described by the fractional SIR model (7). The results show that the solution continuously depends on the time-fractional derivative and on the values of the parameters described in table 1. The figures1,2,3 describe the four presented cases in table 1. These figures show the effect of the high-vaccination coverage ($P > P_c$) on the disease free initial population groups. In case 1, the population of the susceptible group decreases with time while that of the removed group gradually increases due to inclusion of vaccinated susceptible group. The entire population generally remains disease free with all the time and the endemic equilibrium remains stable.

The figures that depicts Case 2, illustrates the impact of high vaccination coverage on the initial population groups with low level of infective group. The populations of the susceptible and infective groups decrease with time while that of the removed group increases due to inclusion of vaccinated and recovered people with permanent immunity and the disease outbreak ends. This clearly shows that a disease free equilibrium is achievable once the vaccination coverage level is greater than the threshold value (i.e.$P > P_c$).

Case 3 which is shown in the presented figures, illustrates the effect of low vaccination coverage on the initial population groups with low levels of infective group. The population of the susceptible group decreases with time. A small increase in the population of removed group is also noticed. However, it is noteworthy that the population of infective group will never disappear with time and the endemic situation persists.

Case 4 is shown in the presented Figures and the impact of initial low levels of infective group on the vaccination free population ($P < P_c$) is illustrated. As expected, the population of susceptible group decreases while that of infective group temporally increases. The disease rapidly spread to the entire population. The only contribution to removed group is the very small proportion of recovered people with permanent immunity. It is observed that the disease free equilibrium is stable provided the vaccination coverage level exceeds a certain threshold $P_c = (\beta - \gamma - \pi)/\beta = 0.4$.

As a definition of fractional calculus: $lim_{\alpha \to 1} D^{\alpha} f(t) = Df(t)$ has been provided. In the presented problem, the susceptible group $s(t)$, the infected group $i(t)$, and the removed group $r(t)$, have been obtained, the results obtained show that when $\alpha \to 1$ the solution of the fractional model (7) $s_{\alpha}(t), i_{\alpha}(t), r_{\alpha}(t)$, reduce to the standard solution $s(t), i(t), r(t)$.

In this paper, we modified the ODE model (1) into a system of fractional-order [7]. Our studies on the use of GEM for solving the presented SIR model shows that GEM is a good tool in solving the biological system. One of the advantages of GEM is its capability of presenting us with continuous solutions, thus giving us better understanding, insight as well as detail over the time interval.

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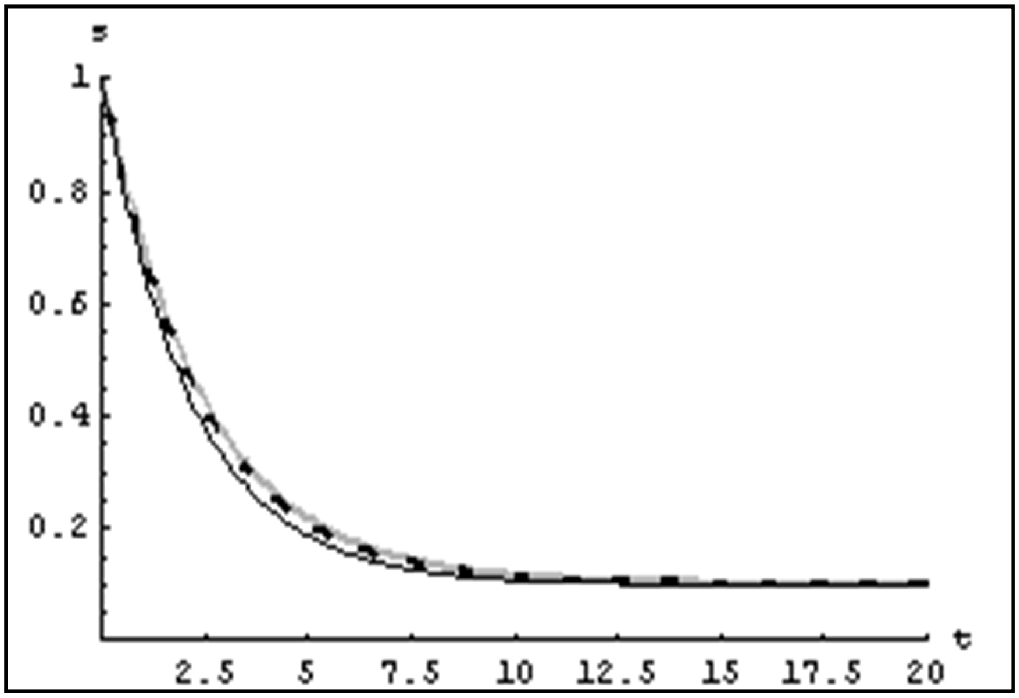
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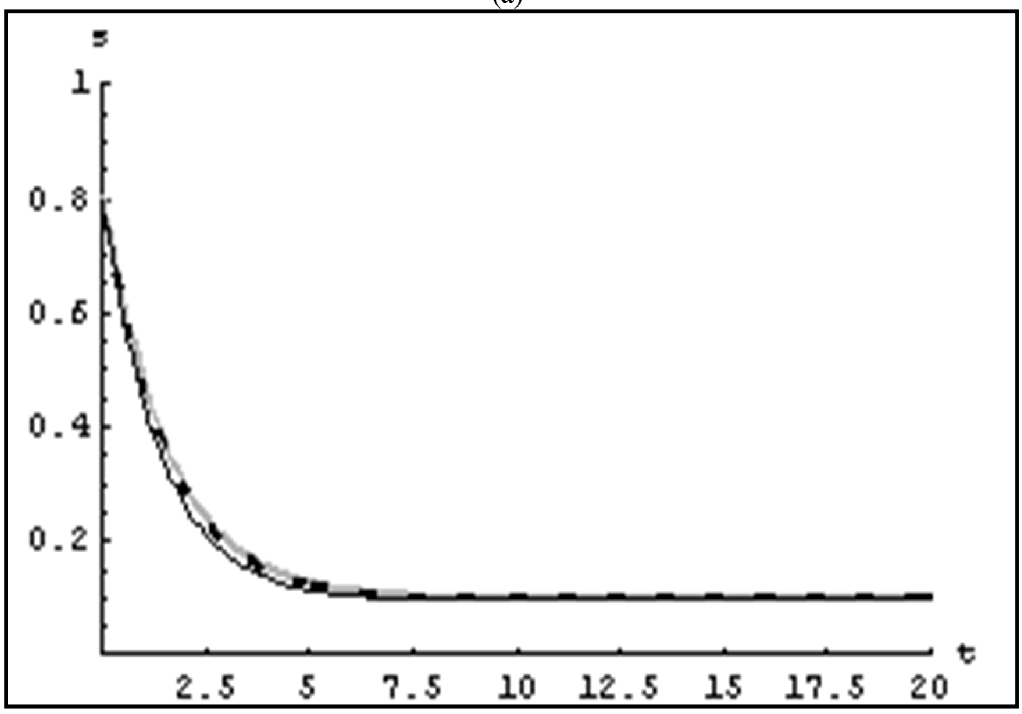
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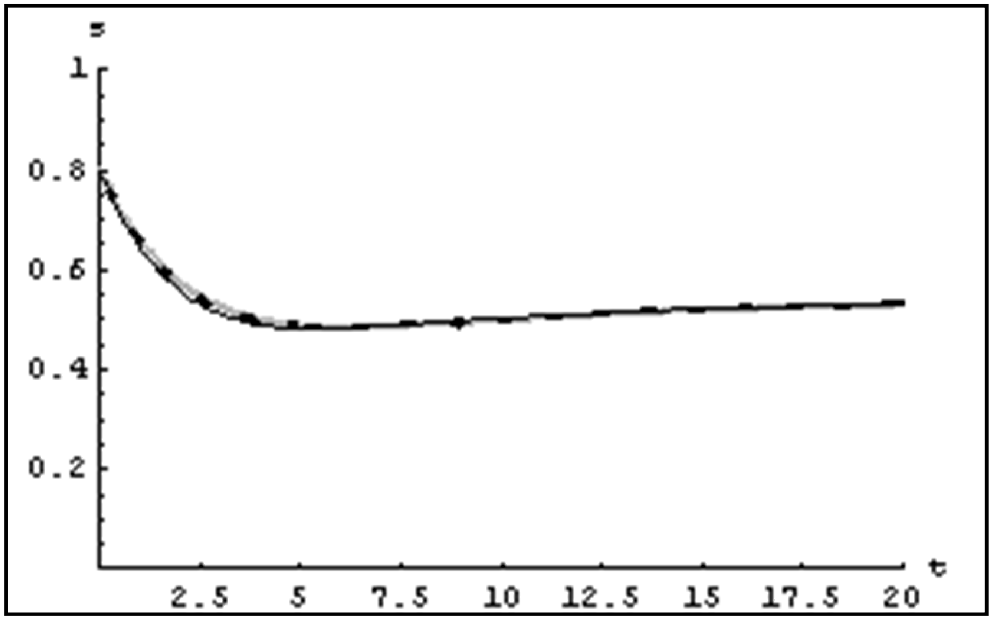

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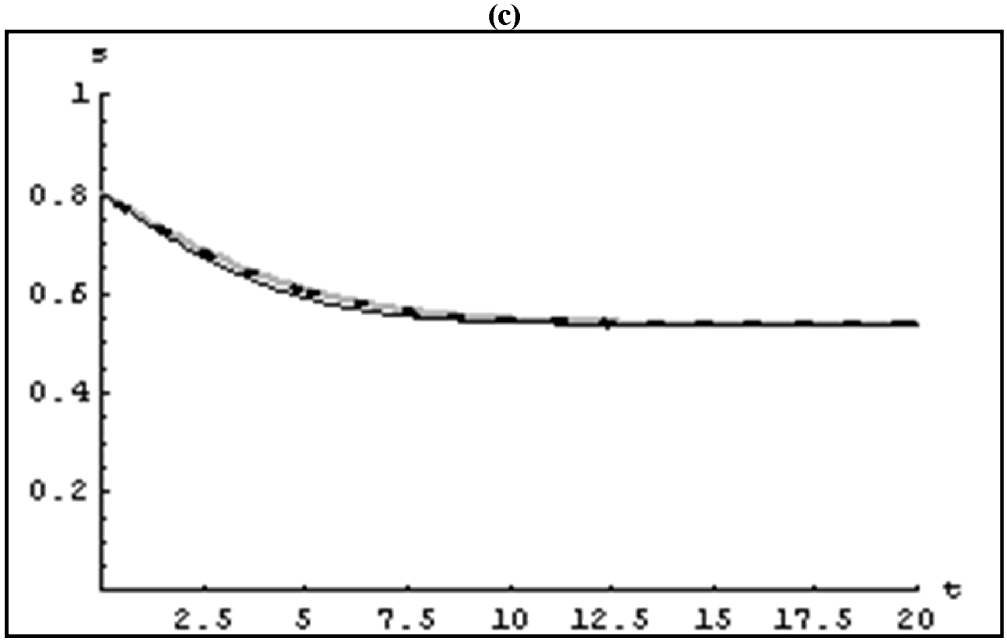
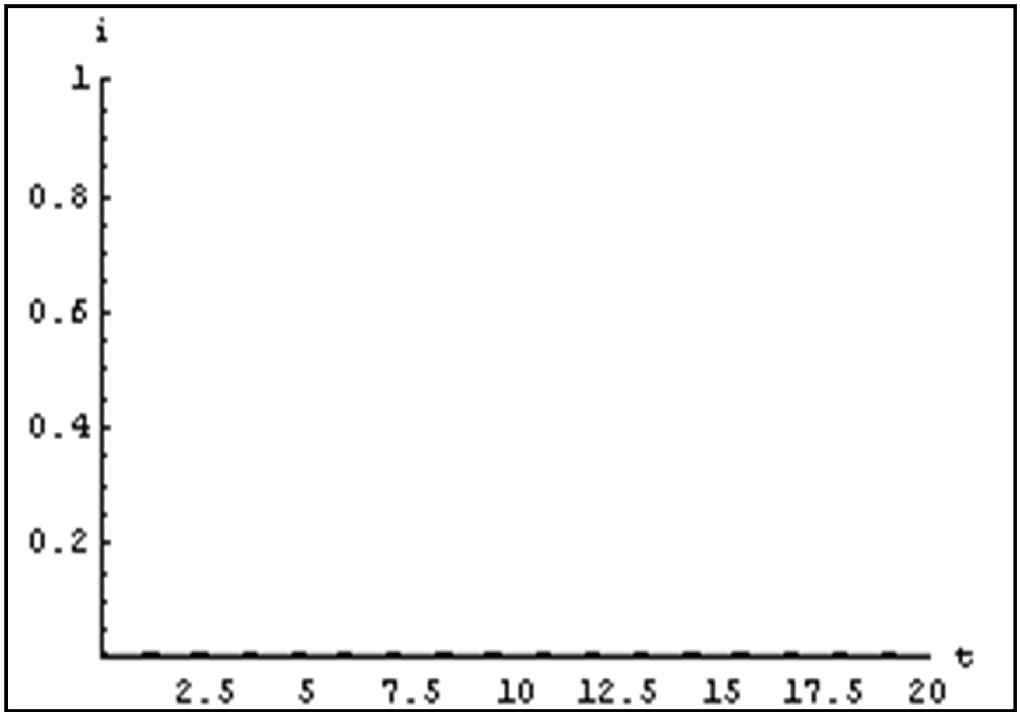


(d)

Fig. 1. susceptible fraction versus time for Case 1 (a), Case 2 (b), Case 3 (c), Case 4 (d): Gray solid line ($\alpha = 1$), Dotted line ($\alpha = 0.99$), Black solid line ($\alpha = 0.95$).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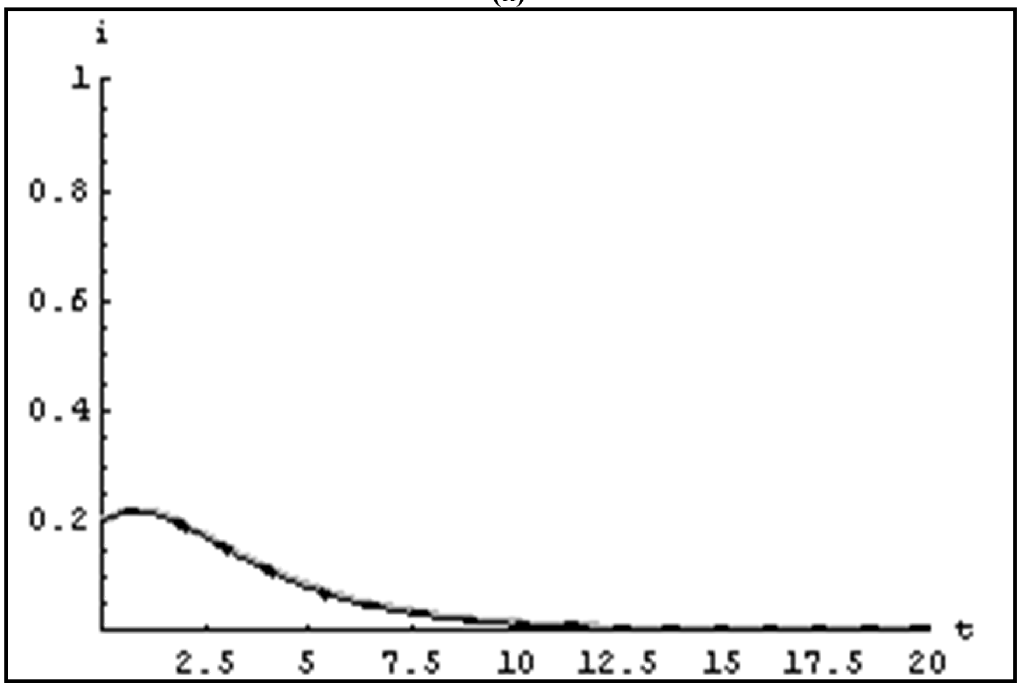
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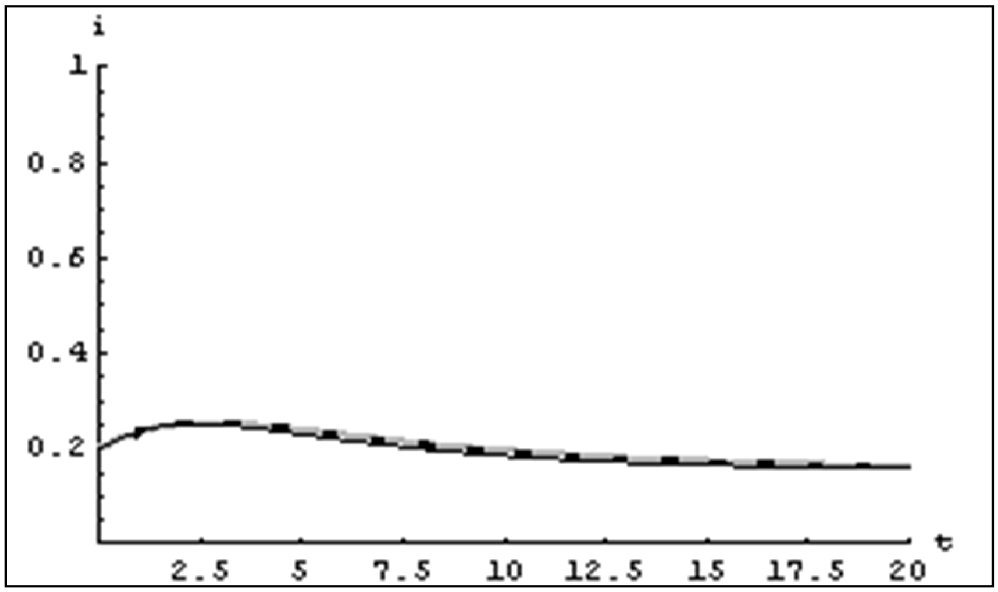

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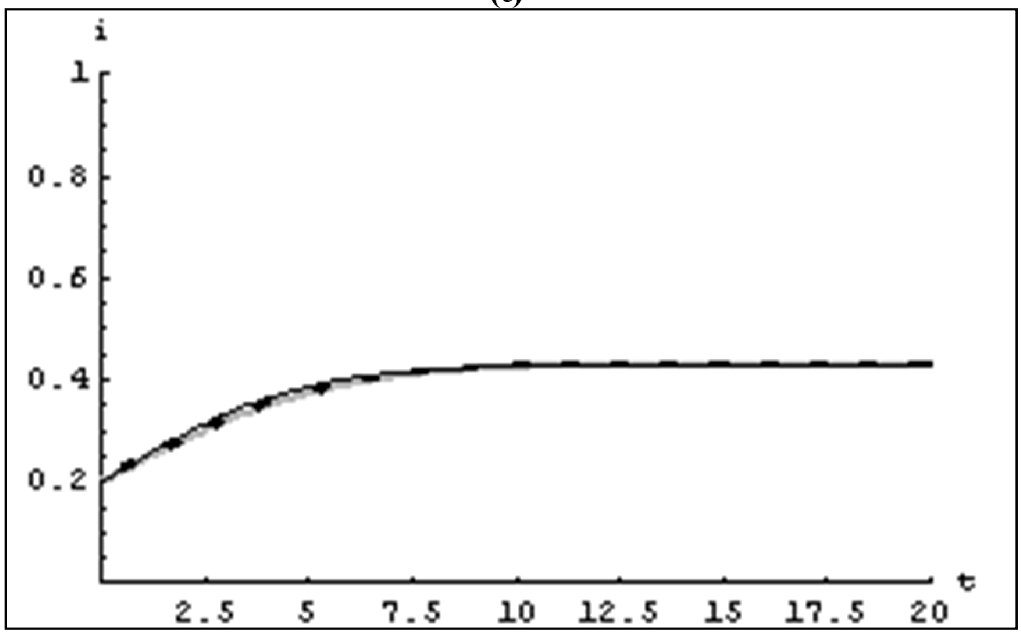

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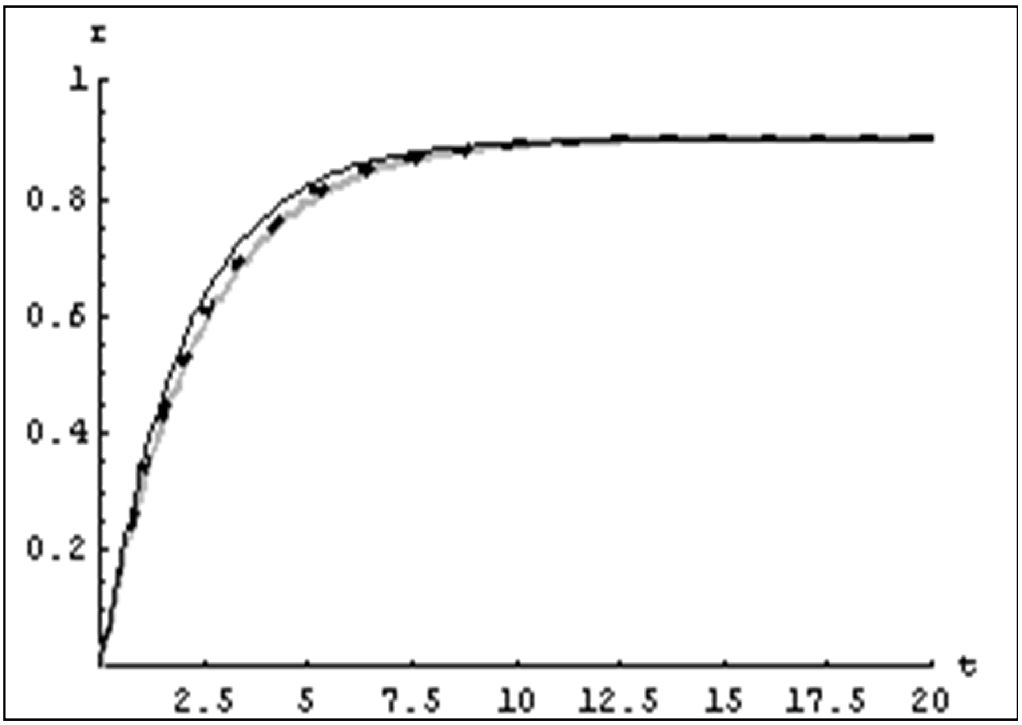
Fig. 2. Infectives fraction versus time for Case 1 (a), Case 2 (b), Case 3 (c), Case 4 (d): Gray solid line ($\alpha = 1$), Dotted line ($\alpha = 0.99$), Black solid line ($\alpha = 0.95$).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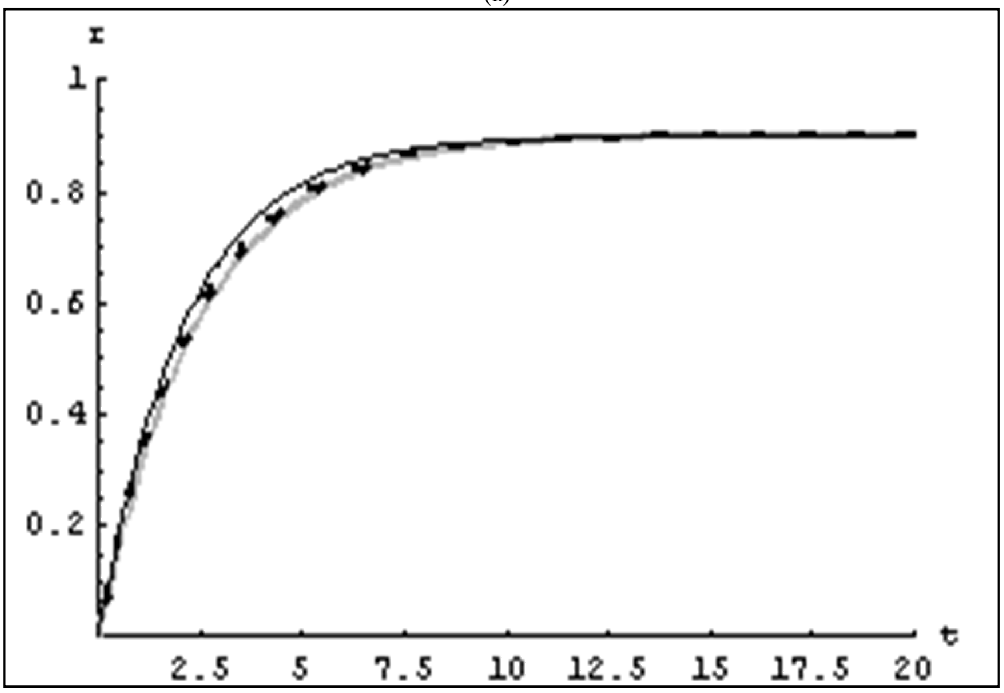
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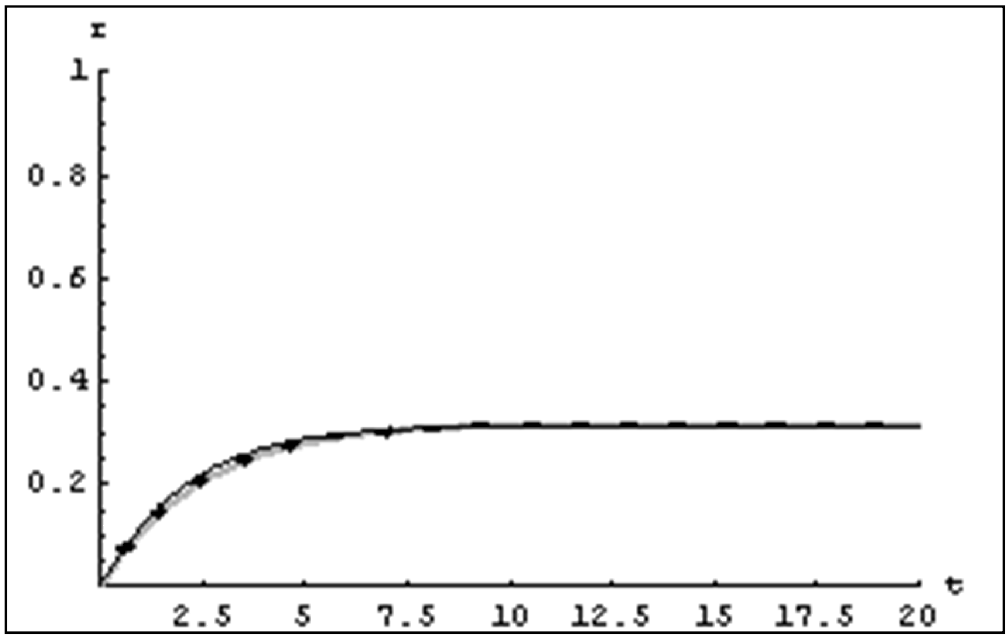

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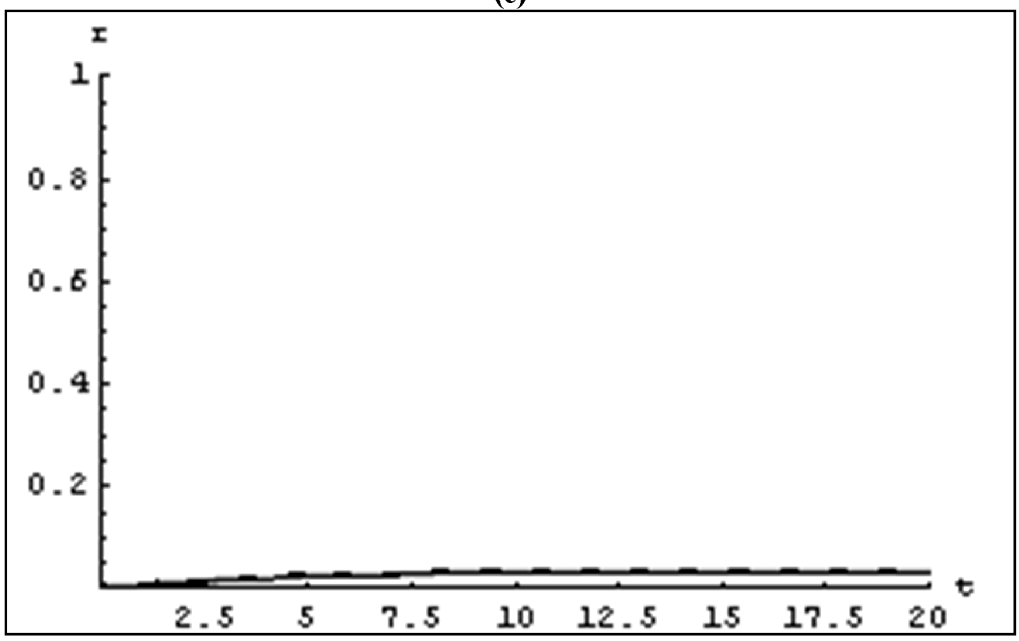

(d)

Fig. 3. Removed fraction versus time for Case 1 (a), Case 2 (b), Case 3 (c), Case 4 (d): Gray solid line ($\alpha = 1$), Dotted line ($\alpha = 0.99$), Black solid line ($\alpha = 0.95$).