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A NEW CLASS OF HARMONIC FUNCTIONS OF COMPLEX ORDER DEFINED BY DUAL CONVOLUTION

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Abstract. In this paper, we investigate several properties of the harmonic classes $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \beta, b, t, \sigma)$ and $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, b, t)$. We obtain distortion theorem, extreme points, convolution condition, convex combinations and closure property under integral operator for functions in these two classes.

1. Introduction

A continuous complex valued functions $f = u + iv$ which is defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write

$$f(z) = h(z) + \overline{g(z)}\,, \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h^{'}(z)| > |g^{'}(z)|$ in D (see [6]).

Denote by S_H, the class of functions f of the form (1.1) that are harmonic univalent and sense preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$.

For $f = h + \overline{g} \in S_H$, we may express the analytic functions h and g are of the form:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ g(z) = \sum_{k=1}^{\infty} b_k z^k,\ |b_1| < 1. \tag{1.2}$$

Also let $S_{\overline{H}}$ denote the subclass of S_H consisting of functions $f = h + \overline{g}$ such that the functions h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k|\, z^k,\ g(z) = \sum_{k=1}^{\infty} |b_k|\, z^k\ ,\ |b_1| < 1. \tag{1.3}$$

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In 1984 Clunie and Sheil-Small [6] investigated the class S_H as well as its geometric subclasses and its properties. Since then, there have been several studies related to the class S_H and its subclasses. Following Clunie and Sheil-Small [6], Frasin [12], Frasin and Murugusundaramoorthy [13], Jahangiri [14, 15], Silverman [23], Silverman and Silvia [24], Dixit and porwal [8] and others have investigated various subclasses of S_H and its properties.

The Hadamard product (or convolution) of two power series

$$\Phi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k \ (\lambda_k \geq 0) \text{ and } \Psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k \ (\mu_k \geq 0), \tag{1.4}$$

be defined by

$$(\Phi * \Psi)(z) = z + \sum_{k=2}^{\infty} \lambda_k \mu_k z^k, \tag{1.5}$$

and the integral convolution is defined by

$$(\Phi \Diamond \Psi)(z) = z + \sum_{k=2}^{\infty} \frac{\lambda_k \mu_k}{k} z^k, \tag{1.6}$$

note that by (1.5) and (1.6), we have

$$(\Phi \Diamond \Psi)(z) =_0^z \frac{(\Phi * \Psi)(t)}{t} dt\ . \tag{1.7}$$

Motivated by the work of Dixit et al. [9], and Frasin and Murugusundaramoorthy [13].

We consider the class $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \beta, b, t, \sigma)$ consisting of functions $f = h + \overline{g}$, where h and g of the form (1.2) and satisfying the condition

$$\left| \frac{1}{b} \left[\frac{h(z) * \Phi(z) - \sigma \overline{g(z) * \Psi(z)}}{h_t(z) \Diamond \Phi(z) + \sigma \overline{g_t(z) \Diamond \Psi(z)}} - 1 \right] \right| < \beta, \tag{1.8}$$

where $0 < \beta \leq 1,\ b \in \mathbb{C}^* = \mathbb{C} \backslash \{0\},\ |\sigma| = 1,\ h_t(z) = (1-t)z + th(z),\ g_t(z) = tg(z),\ 0 \leq t \leq 1,\ \Phi(z)$ and $\Psi(z)$ are given by (1.4).

Further, let for $\sigma = 1, \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, b, t)$ be the subclass of $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \beta, b, t, \sigma)$ consisting of functions of the form (1.3).

Specializing the functions $\Phi(z)$ and $\Psi(z)$ and the parameters $\beta,\ b$, t and σ we obtain the following subclasses studied by various authors:
(i) $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; 1, 1-\gamma, t) = \mathcal{G}_{\overline{\mathcal{H}}}(\Phi, \Psi; \gamma, t)$ $(0 \leq \gamma < 1,\ 0 \leq t \leq 1)$ (see Magesh and Porwal [20, with $\beta = 0$]);
(ii) $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; 1, 1-\alpha, 1) = \overline{HS}(\Phi, \Psi, \alpha)$ $(0 \leq \alpha < 1)$ (see Dixit et al. [9]);
(iii) $\mathcal{S}_{\overline{\mathcal{H}}}\left(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 1, b, 1\right) = \overline{S_H}(b, 1, \beta)$ (see Aouf et al. [4, with $p = 1$]);
(iv) $\mathcal{S}_{\overline{\mathcal{H}}}\left(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 1, 1, 1\right) = T_H^*$ (see Silverman [23]);
(v) $\mathcal{S}_{\mathcal{H}}\left(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; 1, 1-\alpha, 1, 1\right) = S_H^*(\alpha)$ $(0 \leq \alpha < 1)$ (see Jahangiri [15]);

(vi) $\mathcal{S}_{\mathcal{H}}\left(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \beta, b, 1, 1\right) = \mathcal{HS}^*(b,\beta)$ (see Janteng [17]).

Also we note that:
(i) $\mathcal{S}_{\mathcal{H}}(z+\sum\limits_{k=2}^{\infty}k^{n+1}z^k, z+\sum\limits_{k=2}^{\infty}k^{n+1}z^k; \beta, b, 1, (-1)^n) = \mathcal{S}_{\mathcal{H}}(n;\beta,b)$
$= \left\{ f\in S_H : \left|\frac{1}{b}\left[\frac{D^{n+1}h(z)-(-1)^{n+1}\overline{(D^{n+1}g(z))}}{D^n h(z)+(-1)^n\overline{D^n g(z)}}\right]\right| < \beta \ (n\in\mathbb{N}_0=\mathbb{N}\cup\{0\};\right.$
$\left.\mathbb{N}=\{1,2,...\})\right\}$, where D^n is the modefied Salagean operator (see [16], [22] and [25]);
(ii) $\mathcal{S}_{\mathcal{H}}(z+\sum\limits_{k=2}^{\infty}k^{-n}z^k, z+\sum\limits_{k=2}^{\infty}k^{-n}z^k; \beta, b, 1, (-1)^{n+1}) = E_H(n;\beta,b)$
$= \left\{ f\in S_H : \left|\frac{1}{b}\left[\frac{I^n h(z)-(-1)^n\overline{(I^n g(z))}}{I^{n+1}h(z)+(-1)^{n+1}\overline{I^{n+1}g(z)}}\right]\right| < \beta \ (n\in\mathbb{N}_0)\right\}$, where I^n is the modefied Salagean integral operator (see [7], with $p=1$, also see [22]);
(iii) $\mathcal{S}_{\mathcal{H}}(z+\sum\limits_{k=2}^{\infty}k\left[1+\lambda(k-1)\right]^n z^k, z+\sum\limits_{k=2}^{\infty}k\left[1+\lambda(k-1)\right]^n z^k; \beta, b, 1, (-1)^n)$
$= \mathcal{S}_{\mathcal{H}}(n;\beta,,b,\lambda) = \left\{\left|\frac{1}{b}\left[\frac{z(D_\lambda^n h(z))'-(-1)^n\overline{z(D_\lambda^n g(z))'}}{D_\lambda^n h(z)+(-1)^n\overline{D_\lambda^n g(z)}}\right]\right| < \beta \ (\lambda\geq 0; n\in\mathbb{N}_0)\right\}$,
where D_λ^n is the modefied Al-Oboudi operator (see [1, 26], also see [2], with $p=1$);
(iv) $\mathcal{S}_{\mathcal{H}}(z+\sum\limits_{k=2}^{\infty}k\left[1+\lambda(k-1)\right]^{-n} z^k, z+\sum\limits_{k=2}^{\infty}k\left[1+\lambda(k-1)\right]^{-n} z^k; \beta, b, 1, (-1)^n)$
$= \mathcal{L}_{\mathcal{H}}(n;\beta,,b,\lambda) = \left\{\left|\frac{1}{b}\left[\frac{z(I_\lambda^n h(z))'-(-1)^n\overline{z(I_\lambda^n g(z))'}}{I_\lambda^n h(z)+(-1)^n\overline{I_\lambda^n g(z)}}\right]\right| < \beta \ (\lambda\geq 0; n\in\mathbb{N}_0)\right\}$ where I_λ^n is modefied integral operator see ([3], with $p=1$, also see [11], with $\ell=0$);
(v) $\mathcal{S}_{\mathcal{H}}\left(z+\sum\limits_{k=2}^{\infty}k\left(\frac{1+\ell+\lambda(k-1)}{1+\ell}\right)^m z^k, \sum\limits_{k=2}^{\infty}k\left(\frac{1+\ell+\lambda(k-1)}{1+\ell}\right)^m z^k; \beta,,b,1,(-1)^m\right)$
$= \mathcal{S}_{\mathcal{H}}(m,\ell;\beta,b,\lambda) = \left\{\left|\frac{1}{b}\left[\frac{z(J^m(\lambda,\ell)h(z))'-(-1)^m\overline{z(J^m(\lambda,\ell)g(z))'}}{J^m(\lambda,\ell)h(z)+(-1)^m\overline{J^m(\lambda,\ell)g(z)}}\right]\right| < \beta \ (\lambda\geq 0;\right.$
$\left.\ell>-1; m\in\mathbb{Z}=\{\pm 1,...\})\right\}$, where $J^m(\lambda,\ell)$ is the modefied Prajapat operator (see [21, 10], with $p=1$);
(vi) $\mathcal{S}_{\mathcal{H}}(\Phi,\Psi;\beta,(1-\alpha)e^{-i\lambda}\cos\lambda,t,1) = \mathcal{S}_{\mathcal{H}}(\Phi,\Psi,\alpha;\beta,\lambda,t)$
$= \left\{ f\in S_H : \left|\frac{h(z)*\Phi(z)-\overline{g(z)*\Psi(z)}}{h_t(z)\lozenge\Phi(z)+\overline{g_t(z)\lozenge\Psi(z)}}-1\right| < \beta(1-\alpha)\cos\lambda \ (|\lambda|<\frac{\pi}{2}; 0\leq\alpha<1)\right\}$.

In this paper, we have obtained the coefficient bounds for the classes $\mathcal{S}_{\mathcal{H}}(\Phi,\Psi;\beta,b,t,\sigma)$ and $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$. Further distortion bounds, extreme points, convolution and convex combination properties, closure property under integral operator for functions in the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$ are also obtained.

2. Coefficient bounds

Unless otherwise mentioned, we assume throughout this paper that $0 < \beta \leq 1$, $b \in \mathbb{C}^*$, $|\sigma| = 1$, $h_t(z) = (1-t)z + th(z)$, $g_t(z) = tg(z)$, $0 \leq t \leq 1$ and $\Phi(z)$, $\Psi(z)$ are given by (1.4). We begin with a sufficient condition for functions in $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \beta, b, t, \sigma)$.

Theorem 1. *Let* $f = h + \overline{g}$ be given by (1.1). Furthermore, let

$$\sum_{k=2}^{\infty} \frac{\lambda_k}{k} \left(k - (1-\beta\,|b|)\,t\right)|a_k| + \sum_{k=1}^{\infty} \frac{\mu_k}{k} \left(k + (1-\beta\,|b|)\,t\right)_k |b_k| \leq \beta\,|b|\,, \tag{2.1}$$

where,

$$k^2\beta\,|b| \leq \lambda_k(k - (1-\beta\,|b|)\,t) \text{ and } k^2\beta\,|b| \leq \mu_k(k + (1-\beta\,|b|)\,t) \text{ for } k \geq 2. \tag{2.2}$$

Then $f(z)$ is sense-preserving, harmonic univalent in U and $f(z) \in \mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \beta, b, t, \sigma)$.
Proof. If $z_1 \neq z_2$, then by using (2.2), we have

$$\begin{aligned}
\left|\frac{f(z_2)-f(z_1)}{h(z_2)-h(z_1)}\right| &\geq 1 - \left|\frac{g(z_2)-g(z_1)}{h(z_2)-h(z_1)}\right| = 1 - \left|\frac{\sum_{k=1}^{\infty} b_k\left(z_2^k - z_1^k\right)}{(z_2 - z_1) + \sum_{k=2}^{\infty} a_k\left(z_1^k - z_2^k\right)}\right| \\
&\geq 1 - \frac{\sum_{k=1}^{\infty} k\,|b_k|}{1 - \sum_{k=2}^{\infty} k\,|a_k|} \geq 1 - \frac{\sum_{k=1}^{\infty} \frac{\mu_k}{k}\left(\frac{(k+(1-\beta\,|b|)\,t}{\beta\,|b|}\right)_k |b_k|}{1 - \sum_{k=2}^{\infty} \frac{\lambda_k}{k}\left(\frac{(k-(1-\beta\,|b|)\,t}{\beta\,|b|}\right)|a_k|} \geq 0,
\end{aligned}$$

which proves the univalence. Also f is sense-preserving in U since

$$\begin{aligned}
|h'(z)| &\geq 1 - \sum_{k=2}^{\infty} k\,|a_k|\,|z|^{k-1} > 1 - \sum_{k=2}^{\infty} k\,|a_k| \\
&\geq 1 - \sum_{k=2}^{\infty} \frac{\lambda_k}{k}\left(\frac{(k-(1-\beta\,|b|)\,t}{\beta\,|b|}\right)|a_k| \\
&\geq \sum_{k=1}^{\infty} \frac{\mu_k}{k}\left(\frac{(k+(1-\beta\,|b|)\,t}{\beta\,|b|}\right)|a_k| \\
&\geq \sum_{k=1}^{\infty} k\,|b_k|\,|z|^{k-1} \geq |g'(z)|.
\end{aligned}$$

Now we show that $f(z) \in \mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \beta, b, t, \sigma)$. We only need to show that if (2.1) holds then the condition (1.7) is satisfied, then we want to prove that

$$Re\left\{\frac{h(z)*\Phi(z) - \sigma\overline{g(z)*\Psi(z)}}{h_t(z)\Diamond\Phi(z) + \sigma\overline{g_t(z)\Diamond\Psi(z)}} - (1-\beta\,|b|)\right\} = Re\frac{A(z)}{B(z)} > 0. \tag{2.3}$$

Using the fact that $Re\{w\} \geq 0$ if and only if $|1+w| \geq |1-w|$, it suffices to show that

$$|A(z) + B(z)| - |A(z) - B(z)| \geq 0, \tag{2.4}$$

where $A(z) = \left[h(z)*\Phi(z) - \sigma\overline{g(z)*\Psi(z)} - (1-\beta\,|b|)(h_t(z)\Diamond\Phi(z) + \sigma\overline{g_t(z)\Diamond\Psi(z)})\right]$ and $B(z) = \left[h_t(z)\Diamond\Phi(z) + \sigma\overline{g_t(z)\Diamond\Psi(z)}\right]$. Substituting for $A(z)$ and $B(z)$ in the left side of (2.4) we obtain

$$\begin{aligned}
& \left|A(z)+B(z)\right|-\left|A(z)-B(z)\right| \\
= & \left|(1+\beta|b|)z+\sum_{k=2}^{\infty}\left(1+\frac{t\beta|b|}{k}\right)\lambda_k a_k z^k-\sigma\sum_{k=1}^{\infty}\left(1-\frac{t\beta|b|}{k}\right)\mu_k\overline{b_k z^k}\right| \\
& -\left|(1-\beta|b|)z+\sum_{k=2}^{\infty}\left(1-(2-\beta|b|)\frac{t}{k}\right)\lambda_k a_k z^k+\sigma\sum_{k=1}^{\infty}\left(1+(2-\beta|b|)\frac{t}{k}\right)\mu_k\overline{b_k z^k}\right| \\
\geq & (1+\beta|b|)|z|-\sum_{k=2}^{\infty}\left(1+\frac{t\beta|b|}{k}\right)\lambda_k|a_k||z|^k-\sum_{k=1}^{\infty}\left(1-\frac{t\beta|b|}{k}\right)\mu_k|b_k||z|^k \\
& -(1-\beta|b|)|z|-\sum_{k=2}^{\infty}\left(1-(2-\beta|b|)\frac{t}{k}\right)\lambda_k|a_k||z|^k-\sum_{k=1}^{\infty}\left(1+(2-\beta|b|)\frac{t}{k}\right)\mu_k|b_k||z|^k \\
\geq & 2\left\{\beta|b|-\sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left(k-(1-\beta|b|)t\right)|a_k|-\sum_{k=1}^{\infty}\frac{\mu_k}{k}\left(k+(1-\beta|b|)t\right)|b_k|\right\} \\
\geq & 0,\ \text{this by using (2.1).}
\end{aligned}$$

The harmonic univalent functions

$$f(z)=z+\sum_{k=2}^{\infty}\frac{k\beta|b|}{\left[k-(1-\beta|b|)t\right]\lambda_k}x_k z^k+\sum_{k=1}^{\infty}\frac{k\beta|b|}{\left[k+(1-\beta|b|)t\right]\mu_k}\overline{y_k z^k}, \tag{2.5}$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, shows that the coefficient bound given by (2.1) is sharp. This completes the proof of Theorem 1.

In the following theorem, it is shown that the condition (2.1) is also necessary for function $f=h+\overline{g}$, where h and g are of the form (1.3) and belongs to the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$.

Theorem 2. Let $f=h+\overline{g}$, where h and g are given by (1.3). Then $f(z)\in\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$, if and only if the coefficient bound (2.1) holds.

Proof. Since $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)\subseteq\mathcal{S}_{\mathcal{H}}(\Phi,\Psi;\beta,b,t,\sigma)$, we only need to prove the "only if" part of the theorem. For functions $f=h+\overline{g}$, where h and g are given by (1.3), the inequality (2.2) is equivalent to

$$Re\left\{\frac{z-\sum_{k=2}^{\infty}\lambda_k|a_k|z^k-\sum_{k=1}^{\infty}\mu_k|b_k|\overline{z^k}}{z-\sum_{k=2}^{\infty}\frac{t\lambda_k}{k}|a_k|z^k+\sum_{k=1}^{\infty}\frac{t\mu_k}{k}|b_k|\overline{z^k}}\right\}>1-\beta|b|.$$

Upon choosing the values of z on the positive real axis, where $0\leq z=r<1$, we must have

$$\frac{E}{1-\sum_{k=2}^{\infty}\frac{t\lambda_k}{k}|a_k|r^{k-1}+\sum_{k=1}^{\infty}\frac{t\mu_k}{k}|b_k|r^{k-1}}\geq 0, \tag{2.6}$$

where

$$E=\beta|b|-\sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left(k-(1-\beta|b|)t\right)|a_k|r^{k-1}-\sum_{k=1}^{\infty}\frac{\mu_k}{k}\left(k+(1-\beta|b|)t\right)|b_k|r^{k-1}.$$

If the inequality (2.1) does not hold, then E is negative for r sufficiently close to 1. Thus there exists $z_0=r_0$ in $(0,1)$ for which the quotient in (2.6) is negative. But this is a contradiction, then the proof of Theorem 2 is completed.

3. Distortion bounds and Extreme points

Theorem 3. Let $f = h + \overline{g}$, where h and g are given by (1.3) be in the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, b, t)$ and $A_k \leq \frac{\lambda_k}{k}\left(k - (1 - \beta\,|b|)\,t\right)$, $B_k \leq \frac{\mu_k}{k}\left(k + (1 - \beta\,|b|)\,t\right)$ for $k \geq 2$, $C = \min\{A_2, B_2\}$. Then for $|z| = r < 1$, we have

$$|f(z)| \geq (1 - |b_1|)r - \left[\frac{\beta\,|b|}{C} - \frac{[1 + (1 - \beta\,|b|)\,t]}{C}\,|b_1|\right] r^2, \tag{3.1}$$

and

$$|f(z)| \leq (1 + |b_1|)r + \left[\frac{\beta\,|b|}{C} - \frac{[1 + (1 - \beta\,|b|)\,t]}{C}\,|b_1|\right] r^2. \tag{3.2}$$

The bounds in (3.1) and (3.2) are attained for the functions f given by

$$f(z) = (1 + |b_1|)\overline{z} + \left[\frac{\beta\,|b|}{C} - \frac{[1 + (1 - \beta\,|b|)\,t]}{C}\,|b_1|\right] \overline{z}^2, \tag{3.3}$$

and

$$f(z) = (1 - |b_1|)\overline{z} - \left[\frac{\beta\,|b|}{C} - \frac{[1 + (1 - \beta\,|b|)\,t]}{C}\,|b_1|\right] \overline{z}^2, \tag{3.4}$$

for $|b_1| \leq \frac{\beta\,|b|}{[1 + (1 - \beta\,|b|)\,t]}$.

Proof. Let $f(z) \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, b, t)$, then we have

$$\begin{aligned} |f(z)| &\geq (1 - |b_1|)r - \sum_{k=2}^{\infty}(|a_k| + |b_k|)r^k \\ &\geq (1 - |b_1|)r - \frac{\beta\,|b|}{C}\sum_{k=2}^{\infty}\left(\frac{C}{\beta\,|b|}\right)(|a_k| + |b_k|)r^2 \\ &\geq (1 - |b_1|)r - \frac{\beta\,|b|}{C}\,\cdot \\ &\quad \cdot\sum_{k=2}^{\infty}\left(\frac{\lambda_k}{k}\left(k - (1 - \beta\,|b|)\,t\right)|a_k| + \frac{\mu_k}{k}\left(k - (1 - \beta\,|b|)\,t\right)|b_k|\right) r^2 \\ &\geq (1 - |b_1|)r - \frac{\beta\,|b|}{C}\left[1 - \frac{[1 + (1 - \beta\,|b|)\,t]}{\beta\,|b|}\,|b_1|\right] r^2 \\ &= (1 - |b_1|)r - \left[\frac{\beta\,|b|}{C} - \frac{[1 + (1 - \beta\,|b|)\,t]}{C}\,|b_1|\right] r^2, \end{aligned}$$

which proves the asserion (3.1) of Theorem 3. The proof of the assertion (3.2) is similar, thus, we omit it.

The following covering result follows from the left hand inequality of Theorem 3.

Corollary 1. Let $f = h + \overline{g}$, where h and g are given by (1.3) be in the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi, \Psi; \beta, b, t)$, where $|b_1| < \frac{C - \beta\,|b|}{C - [1 + (1 - \beta\,|b|)\,t]}$ and $A_k \leq \frac{\lambda_k}{k}\left(k - (1 - \beta\,|b|)\,t\right)$, $B_k \leq \frac{\mu_k}{k}\left(k + (1 - \beta\,|b|)\,t\right)$ for $k \geq 2$, $C = \min\{A_2, B_2\}$. Then for $|z| = r < 1$, we have

$$\left\{w : |w| < \frac{C - \beta\,|b|}{C} - \frac{C - [1 + (1 - \beta\,|b|)\,t]}{C}\,|b_1|\right\} \subset f(U).$$

Now we determine the extreme points of the closed convex hull of the class

$\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$ denoted by *clco* $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$.

Theorem 4. Let $f = h + \overline{g}$, where h and g are given by (1.3), Then $f(z) \in$ *clco* $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$ if and only if

$$f(z) = \sum_{k=1}^{\infty} \left[X_k h_k(z) + Y_k g_k(z)\right], \tag{3.5}$$

where

$$\begin{aligned} h_1(z) &= z, \\ h_k(z) &= z - \frac{k\beta\,|b|}{[k-(1-\beta\,|b|)\,t]\,\lambda_k} z^k \ (k = 2,3,...), 3.6 \end{aligned} \tag{1}$$

and

$$g_k(z) = z + \frac{k\beta\,|b|}{[k+(1-\beta\,|b|)\,t]\,\mu_k} \overline{z}^k \ (k = 1,2,...), \tag{3.7}$$

where $\sum_{k=1}^{\infty}(X_k + Y_k) = 1,\ X_k \geq 0$ and $Y_k \geq 0$. In particular, the extreme points of the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$ are $\{h_k\}$ and $\{g_k\}$, respectively.

Proof. For a function $f(z)$ of the form (3.5), we have

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}\left[X_k h_k(z) + Y_k g_k(z)\right] \\ &= \sum_{k=1}^{\infty}\left[X_k\left(z - \frac{k\beta\,|b|}{[k-(1-\beta\,|b|)\,t]\,\lambda_k} z^k\right) + Y_k\left(z + \frac{k\beta\,|b|}{[k+(1-\beta\,|b|)\,t]\,\mu_k}\overline{z}^k\right)\right] \\ &= z - \sum_{k=2}^{\infty}\frac{k\beta\,|b|}{[k-(1-\beta\,|b|)\,t]\,\lambda_k} X_k z^k + \sum_{k=1}^{\infty}\frac{k\beta\,|b|}{[k+(1-\beta\,|b|)\,t]\,\mu_k} Y_k \overline{z}^k. \end{aligned}$$

But

$$\begin{aligned} &\sum_{k=2}^{\infty}\left(\frac{[k-(1-\beta\,|b|)\,t]\,\lambda_k}{k\beta\,|b|}\frac{k\beta\,|b|}{[k-(1-\beta\,|b|)\,t]\,\lambda_k}X_k\right) \\ &\quad + \sum_{k=1}^{\infty}\left(\frac{[k+(1-\beta\,|b|)\,t]\,\mu_k}{k\beta\,|b|}\frac{k\beta\,|b|}{[k+(1-\beta\,|b|)\,t]\,\mu_k}Y_k\right) \\ &= \sum_{k=2}^{\infty}X_k + \sum_{k=1}^{\infty}Y_k = 1 - X_1 \leq 1. \end{aligned}$$

Thus $f(z) \in$ *clco* $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$.

Conversely, assume that $f(z) \in$ *clco* $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$. set

$$\begin{aligned} X_k &= \frac{[k-(1-\beta\,|b|)\,t]}{k\beta\,|b|}\lambda_k\,|a_k| \ \ (k = 2,3,...), \\ Y_k &= \frac{[k+(1-\beta\,|b|)\,t]}{k\beta\,|b|}\mu_k\,|b_k| \ \ (k = 1,2,...). \end{aligned}$$

Then by using (2.1), we have $0 \leq X_k \leq 1$ $(k = 2,3,...)$ and $0 \leq Y_k \leq 1$ $(k = 2,3,...)$. Define $X_1 = 1 - \sum_{k=2}^{\infty}X_k - \sum_{k=1}^{\infty}Y_k$. Thus we obtain $f(z) = \sum_{k=1}^{\infty}\left[X_k h_k(z) + Y_k g_k(z)\right]$. This completes the proof of Theorem 4.

4. Convolution and convex combination

Let the functions $f_m(z)$ define by

$$f_m(z) = z - \sum_{k=2}^{\infty}|a_{k,m}|\,z^k + \sum_{k=1}^{\infty}|b_{k,m}|\,\overline{z}^k \ (m = 1,2), \tag{4.1}$$

are in the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$, the convolution of $f_1(z)$ and $f_2(z)$ is defined as follows

$$(f_1 * f_2)(z) = z - \sum_{k=2}^{\infty} |a_{k,1}|\,|a_{k,2}|\,z^k + \sum_{k=1}^{\infty} |b_{k,1}|\,|b_{k,2}|\,\overline{z}^k, \tag{4.2}$$

while the integral convolution is defined by

$$(f_1 \Diamond f_2)(z) = z - \sum_{k=2}^{\infty} \frac{|a_{k,1}|\,|a_{k,2}|}{k} z^k + \sum_{k=1}^{\infty} \frac{|b_{k,1}|\,|b_{k,2}|}{k}\overline{z}^k. \tag{4.3}$$

Theorem 5. For $0 < \delta \leq \beta \leq 1$, let the functions $f_1 \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$ and $f_2 \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\delta,b,t)$. Then

$$\begin{aligned} (f_1 * f_2)(z) &\in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t) \subset \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\delta,b,t), 4.4 \qquad (2)\\ (f_1 \Diamond f_2)(z) &\in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t) \subset \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\delta,b,t).4.5 \qquad (3)\end{aligned}$$

Proof. Let $f_m(z)$ $(m = 1,2)$ are given by (4.1), where $f_1(z)$ be in the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$ and $f_2(z)$ be in the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\delta,b,t)$. We wish to show that the coefficients of $(f_1 * f_2)(z)$ satisfy the required condition given in (2.1). For $f_2 \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\delta,b,t)$, we note that $|a_{k,2}| < 1$ and $|b_{k,2}| < 1$. Now for the convolution functions $(f_1 * f_2)(z)$, we obtain

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{\lambda_k}{k}\left(\frac{k-(1-\delta|b|)\,t}{\delta|b|}\right)|a_{k,1}|\,|a_{k,2}| + \sum_{k=1}^{\infty} \frac{\mu_k}{k}\left(\frac{k+(1-\delta|b|)\,t}{\delta|b|}\right)|b_{k,1}|\,|b_{k,2}| \\ \leq\ & \sum_{k=2}^{\infty} \frac{\lambda_k}{k}\left(\frac{k-(1-\delta|b|)\,t}{\delta|b|}\right)|a_{k,1}| + \sum_{k=1}^{\infty} \frac{\mu_k}{k}\left(\frac{k+(1-\delta|b|)\,t}{\delta|b|}\right)|b_{k,1}| \\ \leq\ & \sum_{k=2}^{\infty} \frac{\lambda_k}{k}\left(\frac{k-(1-\beta|b|)\,t}{\beta|b|}\right)|a_{k,1}| + \sum_{k=1}^{\infty} \frac{\mu_k}{k}\left(\frac{k+(1-\beta|b|)\,t}{\beta|b|}\right)|b_{k,1}| \\ \leq\ & 1, \end{aligned}$$

since $0 < \delta \leq \beta \leq 1$ and $f_1 \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$. Thus $(f_1 * f_2)(z) \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t) \subset \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\delta,b,t)$. The proof of the assertion (4.5) is similar, thus, we omit it. This completes the proof of Theorem 5.

Next we show that $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$ is closed under convex combinations of its members.

Theorem 6. The class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$ is closed under convex combination.
Proof. For $i = 1,2,...,$ let $f_i \in \mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$, where

$$f_i(z) = z - \sum_{k=2}^{\infty} |a_{k,i}|\,z^k + \sum_{k=1}^{\infty} |b_{k,i}|\,\overline{z}^k, \tag{4.6}$$

then from (2.1), for $\sum\limits_{i=1}^{\infty} t_i = 1,\ 0 \leq t_i < 1$, the convex combination of f_i can be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i\,|a_{k,i}|\right) z^k + \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i\,|b_{k,i}|\right)\overline{z}^k. \tag{4.7}$$

Then by (2.1), we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left(k-(1-\beta|b|)t\right)\left(\sum_{i=1}^{\infty}t_i|a_{k,i}|\right)+_{k=1}^{\infty}\frac{\mu_k}{k}\left(k+(1-\beta|b|)t\right)\left(\sum_{i=1}^{\infty}t_i|b_{k,i}|\right)\\
&= \sum_{i=1}^{\infty}t_i\left[\sum_{k=2}^{\infty}\frac{\lambda_k}{k}\left(k-(1-\beta|b|)t\right)|a_{k,i}|+\sum_{k=1}^{\infty}\frac{\mu_k}{k}\left(k+(1-\beta|b|)t\right)|b_{k,i}|\right]\\
&\leq \beta|b|.
\end{aligned}$$

This completes the proof of Theorem 6.

5. Integral operator

Now, we examine a closure property of the class $\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$ under the generalized Bernardi-Libera-Livingston integral operator (see [5, 18, 19]) $L_c(f)$ which is defined by

$$L_c(f)=\frac{c+1}{z^c}{}_0^z t^{c-1}f(t)dt\ (c>-1). \tag{5.1}$$

Theorem 7. Let $f(z)\in\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$. Then $L_c(f(z))\in\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$.
Proof. From the representation of $L_c(f(z))$, it follows that

$$\begin{aligned}
L_c(f(z)) &= \frac{c+1}{z^c}{}_0^z t^{c-1}\left[h(t)+\overline{g(t)}\right]dt\\
&= \frac{c+1}{z^c}{}_0^z t^{c-1}\left[z-\sum_{k=2}^{\infty}|a_k|t^k+\sum_{k=1}^{\infty}|b_k|\overline{t^k}\right]dt\\
&= z-\sum_{k=2}^{\infty}\frac{c+1}{c+k}|a_k|z^k+\sum_{k=1}^{\infty}\frac{c+1}{c+k}|b_k|\overline{z^k}.5.2
\end{aligned} \tag{4}$$

Therefore

$$\begin{aligned}
&{}_{k=2}^{\infty}\frac{\lambda_k}{k}\left(k-(1-\beta|b|)t\right)\frac{(c+1)}{(c+k)}|a_k|+_{k=1}^{\infty}\frac{\mu_k}{k}\left(k+(1-\beta|b|)t\right)\frac{(c+1)}{(c+k)}\mu_k|b_k|\\
&{}_{k=2}^{\infty}\frac{\lambda_k}{k}\left(k-(1-\beta|b|)t\right)|a_k|+_{k=1}^{\infty}\frac{\mu_k}{k}\left(k+(1-\beta|b|)t\right)|b_k|\\
\leq\ &\beta|b|.
\end{aligned}$$

Since $f(z)\in\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$, by using Theorem 2, then $L_c(f(z))\in\mathcal{S}_{\overline{\mathcal{H}}}(\Phi,\Psi;\beta,b,t)$. This completes the proof of Theorem 7.

Remarks. (i) Putting $\beta=\sigma=1,\ b=1-\gamma\ (0\leq\gamma<1)$ in the above results, we obtain some analogous results obtaind by Magesh and Porwal [20, with $\beta=0$];
(ii) Putting $\beta=t=\sigma=1$ and $b=1-\alpha\ (0\leq\alpha<1)$ in the above results, we obtain some analogous obtaind by Dixit et al. [9];
(iii) Putting $\Phi(z)=\Psi(z)=\dfrac{z}{(1-z)^2}$ and $t=\sigma=1$ in the above results, we obtain some analogous results obtaind by Aouf et al. [4, with $p=1$];
(iv) Putting $\Phi(z)=\Psi(z)=\dfrac{z}{(1-z)^2}$ and $t=\sigma=1$ in the above results, we improve the results obtained by Janteng [17];

(v) Putting $\Phi(z)=\Psi(z)=\dfrac{z}{(1-z)^2}$, $\beta=t=\sigma=1$ and $b=1-\alpha$ $(0\leq\alpha<1)$ in the above results, we obtain some analogous results btaind by Jahangiri [15];
(vi) Putting $\Phi(z)=\Psi(z)=\dfrac{z}{(1-z)^2}$, $\beta=b=t=1$ in the above results, we obtain some analogous results obtaind by Silverman [23].
(vii) Putting $\varphi=\psi=z+\sum\limits_{k=2}^{\infty}k^{n+1}z^k$, $t=1$, $n\in\mathbb{N}_0$ and $\sigma=(-1)^n$ in the above results, we obtain new results for the class $\mathcal{S}_{\mathcal{H}}(n;\beta,b)$;
(iix) Putting $\varphi=\psi=z+\sum\limits_{k=2}^{\infty}k^{-n}z^k$, $t=1$, $n\in\mathbb{N}_0$ and $\sigma=(-1)^{n+1}$ in the above results, we obtain new results for the class $E_H(n;\beta,b)$;
(ix) Putting $\varphi=\psi=z+\sum\limits_{k=2}^{\infty}k[1+\lambda(k-1)]^n z^k$, $t=1$, $\lambda\geq 0$, $n\in\mathbb{N}_0$ and $\sigma=(-1)^n$ in the above results, we obtain new results for the class $\mathcal{S}_{\mathcal{H}}(n;\beta,b,\lambda)$;
(x) Putting $\varphi=\psi=z+\sum\limits_{k=2}^{\infty}k[1+\lambda(k-1)]^{-n}z^k$, $t=1$, $\lambda\geq 0$, $n\in\mathbb{N}_0$ and $\sigma=(-1)^n$ in the above results, we obtain new results for the class $\mathcal{L}_{\mathcal{H}}(n;\beta,b,\lambda)$;
(xi) Putting $\varphi=\psi=z+\sum\limits_{k=2}^{\infty}k\left(\dfrac{1+\ell+\lambda(k-1)}{1+\ell}\right)^m z^k$, $t=1$, $\ell,\lambda\geq 0$, $m\in\mathbb{N}_0$ and $\sigma=(-1)^m$ in the above results, we obtain new results for the class $\mathcal{S}_{\mathcal{H}}(m,\ell;\beta,b,\lambda)$;
(xii) Putting $b=(1-\alpha)e^{-i\lambda}\cos\lambda(|\lambda|<\frac{\pi}{2},\ 0\leq\alpha<1)$ and $\sigma=1$ in the above results, we obtain new results for the class $\mathcal{S}_{\mathcal{H}}(\Phi,\Psi,\alpha;\beta,\lambda,t)$.

References

[1] F. M. Al-Aoboudi, On univalent functions defined by a generalized operator, Internat. J. Math. Math. Sci., 27 (2004), 1429-1436.

[2] M. K. Aouf, Certain subclasses of multivalent prestarlike functions with negative coefficients, Demonstratio Math., 40 (2007), no. 4, 799-814.

[3] M. K. Aouf and T. M. Seoudy, On differential sandwich theorems of p-valent analytic functions defined by the integral operator, Arab. J. Math., (2013), 147-158.

[4] M. K. Aouf, A. A. Shamandy, A. O. Mostafa and A. K. Wagdy, A new class of harmonic p-valent functions of complex order, Proc. Pakistan Acad. Sci., 49(2012), no. 3, 219-225.

[5] S. D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc., 135 (1969), 429-446.

[6] J. Clunie and T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A. I. Math., 9, 3-25 (1984).

[7] L. I. Cotirla, Harmonic univalent functions defined by an integral operator, Acta Univ. Apulensis, 17 (2009), 95-105.

[8] K. K. Dixit and S. Porwal, Some properties of harmonic functions defined by convolution, Kyungpook Math. J., 49 (2009), no. 4, 751-761.

[9] K. K. Dixit, A. L. Pathak, S. Porwal and R. Agarwal, On a subclass of harmonic univalent functions defined by convolution and integral convolution, Internat. J. Pure Appl. Math., 69 (2011), no. 3, 255-264.

[10] R. M. El-Ashwah and M. K. Aouf, Some properties of new integral operator, Acta Univ. Apulensis 24 (2010), 51-61.

[11] R. M. El-Ashwah and M. K. Aouf, Differential subordination and superordination for certain subclasses of analytic functions involving an extended integral operator, Acta Univ. Apulensis, 28 (2011), 341-350.

[12] B. A. Frasin, Comprehensive family of harmonic univalent functions, SUT J. Math., 42 (2006), no. 1, 145-155.

[13] B. A. Frasin and G. Murugusundaramoorthy, Comprehensive family of k-uniformly harmonic starlike functions, Mat. Vesnik, 63 (2011), no. 3, 171-180.

[14] J. M. Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients, Ann. Univ. Mariae Curie-Sk lodowska Sect. A., 52 (1998), no. 2, 57-66.
[15] J. M. Jahangiri, Harmonic functions starlike in the unit disk, J. Math. Anal. Appl., 235 (1999), no. 2, 470-477.
[16] J. M. Jahangiri, G. Murugusundaramoorthy and K. Vijya, Salagean-type harmonic univalent functions, Southwest J. Pure Appl. Math., 2 (2002), 77-82.
[17] A. Janteng, Starlike functions of complex order, Appl. Math. Sci., 3(2009), no. 12, 557-564.
[18] R. J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc. 16 (1965), 755–758.
[19] A. E. Livingston, On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc., 17 (1966), 352-357.
[20] N. Magesh and S. Porwal, Harmonic Uniformly $\beta-$Starlike Functions defined by convolution and integral convolution, Acta Univ. Apulensis, 32 (2012), 129-141.
[21] J. K. Prajapat, Subordination and superordination preserving properties for generalized multiplier transformation operator, Math. Comput. Modelling, (2011), 1-10.
[22] G. S. Salagean, Subclasses of univalent function, Lecture Notes in Math. (Springer-Verlag) 1013 (1983), 368-372.
[23] H. Silverman, Harmonic univalent functions with negative coe cients, J. Math. Anal. Appl. 220 (1998), no. 1, 283-289.
[24] H. Silverman and E. M. Silvia, Subclasses of harmonic univalent functions, New Zealand J. Math. 28 (1999), no. 2, 275-284.
[25] S. Yalçin, M. Ö. Ztürk and M. Yamankaradeniz, On the subclass of Salagean-type harmonic univalent functions, J. Inequal. Pure Appl. Math., 8 (2007), no. 2, Art. 54, 1–9.
[26] E. Yaşar and S. Yalçin, Generalized Salagean-type harmonic univalent functions, Stud. Univ. Babes-Bolyai Math., 57 (2012), no. 3, 395-403.

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