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MAJORIZATION PROPERTIES FOR CERTAIN CLASSES OF MEROMORPHIC P-VALENT FUNCTIONS DEFINED BY INTEGRAL OPERATOR

E. E. ALI

Abstract. The object of the present paper is to investigate the majorization properties of certain classes of meromorphic p-valent functions defined by integral operator.

1. Introduction

Let $f(z)$ and $g(z)$ be analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. For analytic function $f(z)$ and $g(z)$ in U , we say that $f(z)$ is majorized by $g(z)$ in U (see [8]) and write

$$f(z) << g(z) \quad (z \in U), \tag{1}$$

if there exists a function $\varphi(z)$, analytic in U such that

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in U). \tag{2}$$

It may be noted that (1) is closely related to the concept of quasi-subordination between analytic functions.

If $f(z)$ and $g(z)$ are analytic functions in U, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as $f(z) \prec g(z)$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)),\ z \in U$. Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence, (see [9, p.4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\Sigma_{p,n}$ denote the class of meromorphic multivalent functions $f(z)$ of the form:

$$f(z) = z^{-p} + \sum_{k=n}^{\infty} a_k z^k, \qquad (n > -p;\ p, n \in \mathbb{N} = \{1, 2, ...\}) \tag{3}$$

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which are analytic in the open punctured unit disc $U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = U \backslash \{0\}$. Let $g(z) \in \Sigma_{p,n}$, be given by

$$g(z) = z^{-p} + \sum_{k=n}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = z^{-p} + \sum_{k=n}^{\infty} a_k b_k z^k = (g * f)(z). \tag{4}$$

For $p \in \mathbb{N},\ \alpha > 0,\ \lambda \geq 0$ and $f \in \Sigma_{p,n}$ given by (1), El-Ashwah and Aouf [5] defined the integral operator $J_{p,\alpha}^{\lambda}$ as follows:

$$J_{p,\alpha}^{\lambda} f(z) = z^{-p} + \sum_{k=n}^{\infty} \left(\frac{\alpha}{k+p+\alpha} \right)^{\lambda} a_k z^k \qquad (\alpha > 0;\ \lambda \geq 0;\ p, n \in \mathbb{N}) \tag{5}$$

From (5), it is easy to verify that (see [5]),

$$z(J_{p,\alpha}^{\lambda} f(z))' = \alpha J_{p,\alpha}^{\lambda-1} f(z) - (\alpha + p) J_{p,\alpha}^{\lambda} f(z) \qquad (\lambda \geq 1). \tag{6}$$

We note that
(i) For $n = 0$ and $\alpha = 1$, $J_{p,1}^{\lambda} f(z) = P_p^{\lambda} f(z)$ (Aqlan et al. [4]);
(ii) $J_{1,1}^{m} f(z) = J^m f(z)$ (Uralagaddi and Somanatha [11]);
(iii) $J_{1,\alpha}^{\lambda} f(z) = P_{\alpha}^{\lambda} f(z) (\alpha > 0,\ \lambda > 0)$ (Lashin [7]);
(iv) $J_{1,\alpha}^{1} f(z) = J_{\alpha} f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{\alpha}{k+1+\alpha} \right) a_k z^k \qquad (\alpha > 0).$

A function $f(z) \in \Sigma_{p,n}$ is said to be in the class $\Sigma_{p,n}^{\lambda,j}(\gamma)$ of meromorphic multivalent functions of complex order $\gamma \in \mathbb{C}^* = \mathbb{C} \backslash \{0\}$ in U, if and only if

$$Re \left\{ 1 - \frac{1}{\gamma} \left(\frac{z(J_{p,\alpha}^{\lambda} f(z))^{(j+1)}}{(J_{p,\alpha}^{\lambda} f(z))^{(j)}} + j + p \right) \right\} > 0$$

$$(p \in \mathbb{N};\ j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \alpha > 0;\ \lambda \geq 0; \gamma \in \mathbb{C}^*;\ z \in U). \tag{7}$$

Clearly, we have the following relationships:

$$(i) \Sigma_{p,n}^{0,0}(\gamma) = \Sigma_{p,n}(\gamma) \quad (\gamma \in \mathbb{C}^*),$$

$$(ii)\ \Sigma_{p,n}^{0,0}(p - \alpha) = \Sigma_{p,n}^{*}(\alpha) \qquad (0 \leq \alpha < p).$$

Also we note that

$$\Sigma_{p,n}^{*}(\alpha) \subseteq \Sigma_{p,n}^{*}(0) = \Sigma_{p,n}^{*} \quad (0 \leq \alpha < p).$$

The classes $\Sigma_{p,n}(\gamma)$ and $\Sigma_{p,n}^{*}(\alpha)$ are said to be classes of meromorphic starlike $p-$valent functions of complex order γ and meromorphic convex $p-$valent functions of order α $(0 \leq \alpha < p)$ in U^* see Aouf ([2] and [3]).

Definition 1. *Let* $-1 \leq B < A \leq 1, p \in \mathbb{N},\ j \in \mathbb{N}_0, \gamma \in \mathbb{C}^*, |\gamma(A - B) + (j + p)B| < (j + p),\ f \in \Sigma_{p,n}$ *. Then* $f \in \Sigma_{p,n}^{\lambda,j}(\gamma; A, B)$ *, the class of meromorphic multivalent* *functions of complex order* γ in U^* *if and only if*

$$\left\{1-\frac{1}{\gamma}\left(\frac{z(J_{p,\alpha}^{\lambda}f(z))^{(j+1)}}{(J_{p,\alpha}^{\lambda}f(z))^{(j)}}+j+p\right)\right\}\prec\frac{1+Az}{1+Bz}. \tag{8}$$

We note that $\Sigma_{1,1}^{\lambda,j}(\gamma;1,-1)=\Sigma^{\lambda,j}(\gamma)$ (see [6]).

A majorization problem for the subclasses of analytic function has recently been investigated by Altintas et al. [1] and MacGregor [8]. In this paper we investigate majorization problem for the class $\Sigma_{p,n}^{\lambda,j}(\gamma;A,B)$ and some related subclasses.

2. Main Results

Unless otherwise mentioned we shall assume throughout the paper that, $-1\leq B<A\leq 1,\gamma\in\mathbb{C}^*,\ \alpha>0,\ \lambda\geq 0,\ p\in\mathbb{N}$ and $j\in\mathbb{N}_0$.

Theorem 1. *Let the function* $f\in\Sigma_{p,n}$ and suppose that $g\in\Sigma_{p,n}^{\lambda,j}(\gamma;A,B)$. If $(J_{p,\alpha}^{\lambda}f(z))^{(j)}$ is majorized by $(J_{p,\alpha}^{\lambda}g(z))^{(j)}$ in U^*, then

$$\left|(J_{p,\alpha}^{\lambda}f(z))^{(j+1)}\right|\leq\left|(J_{p,\alpha}^{\lambda}g(z))^{(j+1)}\right|\qquad(|z|<r_0), \tag{9}$$

where $r_0=r_0(p,\gamma,j,A,B)$ is the smallest positive root of the equation

$$\begin{aligned}&|\gamma(A-B)+(j+p)B|\,r^3-[2\,|B|+(j+p)]\,r^2-\\&[2+|\gamma(A-B)+(j+p)B|]\,r+(j+p)=0.\end{aligned} \tag{10}$$

Proof. Since $g(z)\in\Sigma_{p,n}^{\lambda,j}(\gamma;A,B)$, we find from (8) that

$$1-\frac{1}{\gamma}\left(\frac{z(J_{p,\alpha}^{\lambda}g(z))^{(j+1)}}{(J_{p,\alpha}^{\lambda}g(z))^{(j)}}+j+p\right)=\frac{1+Aw(z)}{1+Bw(z)}, \tag{11}$$

where w is analytic in U with $w(0)=0$ and $|w(z)|<1$ $(z\in U)$. From (11), we have

$$\frac{z(J_{p,\alpha}^{\lambda}g(z))^{(j+1)}}{(J_{p,\alpha}^{\lambda}g(z))^{(j)}}=-\frac{(j+p)+[\gamma(A-B)+(j+p)B]w(z)}{1+Bw(z)}. \tag{12}$$

From (12), we have

$$\left|(J_{p,\alpha}^{\lambda}g(z))^{(j)}\right|\leq\frac{(1+|B|\,|z|)\,|z|}{(j+p)-|\gamma(A-B)+(j+p)B|\,|z|}\left|(J_{p,\alpha}^{\lambda}g(z))^{(j+1)}\right|. \tag{13}$$

Next, since $(J_{p,\alpha}^{\lambda}f(z))^{(j)}$ is majorized by $(J_{p,\alpha}^{\lambda}g(z))^{(j)}$ in U, from (2), we have

$$(J_{p,\alpha}^{\lambda}f(z))^{(j)}=\varphi(z)(J_{p,\alpha}^{\lambda}g(z))^{(j)}. \tag{14}$$

Differentiating (14) with respect to z, we have

$$(J_{p,\alpha}^{\lambda}f(z))^{(j+1)}=\varphi'(z)(J_{p,\alpha}^{\lambda}g(z))^{(j)}+\varphi(z)(J_{p,\alpha}^{\lambda}g(z))^{(j+1)}. \tag{15}$$

Thus, by noting that $\varphi(z)$ satisfies the inequality (see [10]),

$$\left|\varphi'(z)\right|\leq\frac{1-|\varphi(z)|^2}{1-|z|^2}\quad(z\in U), \tag{16}$$

using (13) and (16), in (15), we have

$$\left|(J_{p,\alpha}^{\lambda}f(z))^{(j+1)}\right| \leq$$
$$\left(|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2}.\frac{(1+|B|\,|z|)\,|z|}{(j+p)-|\gamma(A-B)+(j+p)B|\,|z|}\right)\left|(J_{p,\alpha}^{\lambda}g(z))^{(j+1)}\right|, \tag{17}$$

which upon setting

$$|z| = r \quad \text{and} \quad |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1),$$

leads us to the inequality

$$\left|(J_{p,\alpha}^{\lambda}f(z))^{(j+1)}\right| \leq$$
$$\frac{\Theta(\rho)}{(1-r^2)((j+p)-|\gamma(A-B)+(j+p)B|\,r)}\left|(J_{p,\alpha}^{\lambda}g(z))^{(j+1)}\right|,$$

where

$$\begin{aligned}\Theta(\rho) &= -r\,(1+|B|\,r)\,\rho^2 + (1-r^2)\left[(j+p)-|\gamma(A-B)+(j+p)B|\,r\right]\rho \\ &\quad +r\,(1+|B|\,r)\,,18\end{aligned} \tag{1}$$

takes its maximum value at $\rho = 1$, with $r_0 = r_0(p,\gamma,j,A,B)$, where $r_0(p,\gamma,j,A,B)$ is the smallest positive root of (10). Therefore the function $\Phi(\rho)$ defined by

$$\begin{aligned}\Phi(\rho) &= -\sigma\,(1+|B|\,\sigma)\,\rho^2 + (1-\sigma^2)\left[(j+p)-|\gamma(A-B)+(j+p)\,B|\,\sigma\right]\rho \\ &\quad +\sigma\,(1+|B|\,\sigma)\,19\end{aligned} \tag{2}$$

is an increasing function on the interval $0 \leq \rho \leq 1$, so that

$$\begin{aligned}\Phi(\rho) &\leq \Phi(1) = (1-\sigma^2)\left[(j+p)-|\gamma(A-B)+(j+p)B|\,\sigma\right] \\ &\quad (0 \leq \rho \leq 1;\ 0 \leq \sigma \leq r_0(p,\gamma,j,A,B))\,.20\end{aligned} \tag{3}$$

Hence upon setting $\rho = 1$ in (19), we conclude that (9) holds true for $|z| \leq r_0 = r_0(p,\gamma,j,A,B)$, where $r_0(p,\gamma,j,A,B)$, is the smallest positive root of (10). This completes the proof of Theorem 1.

Remark . Putting $p = 1, n = 0, A = 1$ and $B = -1$ in Theorem 1, we obtain the result obtained by Goyal and Goswami [6, Theorem 2.1].

Putting $A = 1$ and $B = -1$ in Theorem 1, we obtain the following result.

Corollary 1. *Let the function* $f \in \Sigma_{p,n}$ and suppose that $g \in \Sigma_{p,n}^{\lambda,j}(\gamma)$. If $(J_{p,\alpha}^{\lambda}f(z))^{(j)}$ is majorized by $(J_{p,\alpha}^{\lambda}g(z))^{(j)}$ in U^*, then

$$\left|(J_{p,\alpha}^{\lambda}f(z))^{(j+1)}\right| \leq \left|(J_{p,\alpha}^{\lambda}g(z))^{(j+1)}\right| \qquad (|z| < r_0)\,,$$

where $r_0 = r_0(p,\gamma,j)$ is given by

$$r_0 = r_0(p,\gamma,j) = \frac{k-\sqrt{k^2-4(j+p)\,|2\gamma-(j+p)|}}{2\,|2\gamma-(j+p)|}, \tag{21}$$

where $k = 2+(j+p)+|2\gamma-(j+p)|\,.$

Putting $\lambda = 0$ in Corollary 1, we obtain the following result
Corollary 2. *Let the function* $f \in \Sigma_{p,n}$ and suppose that $g \in \Sigma_{p,n}^{;j}(\gamma)$. If $f^{(j)}(z)$ is majorized by $g^{(j)}(z)$ in U^*, then

$$\left|f^{(j+1)}(z)\right| \leq \left|g^{(j+1)}(z)\right| \qquad (|z| < r_0)\,,$$

where $r_0 = r_0(p, \gamma, j)$ is given by (21).

Putting $\lambda = j = 0,\ A = 1$ and $B = -1$ in Theorem 1, we obtain the following result.
Corollary 3. *Let the function* $f \in \Sigma_{p,n}$ and suppose that $g \in \Sigma_{p,n}(\gamma)$. If $f(z)$ is majorized by $g(z)$ in U^*, then

$$|f'(z)| \leq |g'(z)| \qquad (|z| < r_0)\,,$$

where $r_0 = r_0(p, \gamma)$ is given by

$$r_0 = r_0(p; \gamma) = \frac{k - \sqrt{k^2 - 4p\,|2\gamma - p|}}{2\,|2\gamma - p|},$$

where $k = 2 + p + |2\gamma - p|\,.$

Putting $\gamma = p - \delta$ $(0 \leq \delta < p)$ in Corollary 3, we obtain the following result.
Corollary 4. *Let the function* $f \in \Sigma_{p,n}$ and suppose that $g \in \Sigma_{p,n}^{*}(\delta)$. If $f(z)$ is majorized by $g(z)$ in U^*, then

$$|f'(z)| \leq |g'(z)| \qquad (|z| < r_0)\,,$$

where $r_0 = r_0(p; \gamma)$ is given by

$$r_0 = r_0(p; \gamma) = \frac{k - \sqrt{k^2 - 4p\,|p - 2\delta|}}{2\,|p - 2\delta|},$$

where $k = 2 + p + |p - 2\delta|\,.$

Putting $\gamma = 1$ in Corollary 3, we obtain the following result.
Corollary 5. *Let the function* $f \in \Sigma_{p,n}$ and suppose that $g \in \Sigma_{p,n}^{*}$. If $f(z)$ is majorized by $g(z)$ in U^*, then

$$|f'(z)| \leq |g'(z)| \qquad (|z| < r_0)\,,$$

where r_0 is given by

$$r_0 = r_0(p) = \frac{k - \sqrt{k^2 - 4p\,|2 - p|}}{2\,|2 - p|},$$

where $k = 2 + p + |2 - p|\,.$

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Faculty of Science, Port Said University, Port Said , Egypt
E-mail address: ekram_008eg@yahoo.com