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INCLUSION PROPERTIES FOR CERTAIN $k-$UNIFORMLY SUBCLASSES OF $p-$VALENT FUNCTIONS DEFINED BY CERTAIN INTEGRAL OPERATOR

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Abstract. We introduce several $k-$uniformly subclasses of $p-$valent functions defined by certain integral operator and investigate various inclusion relationships for these subclasses. Some interesting applications involving certain classes of integral operators are also considered.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \; (p \in \mathbb{N} = \{1, 2, 3, ...\}) \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(\omega(z))$ $(z \in \mathbb{U})$. In particular, if the function g is univalent in $\mathbb{U}$ the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (see [9] and [10]).

For $0 \leq \gamma, \eta < p, k \geq 0$ and $z \in \mathbb{U}$, we define $US_p^*(k;\gamma), UC_p(k;\gamma), UK_p(k;\gamma,\eta)$ and $UK_p^*(k;\gamma,\eta)$ the $k-$uniformly subclasses of $\mathcal{A}_p$ consisting of all analytic functions which are, respectively, $p-$valent starlike of order γ, $p-$valent convex of order γ, $p-$valent close-to-convex of order γ, and type η and $p-$valent quasi-convex of order γ, and type η as follows:

$$US_p^*(k;\gamma) = \left\{ f \in \mathcal{A}_p : \Re\left(\frac{zf'(z)}{f(z)} - \gamma\right) > k\left|\frac{zf'(z)}{f(z)} - p\right| \right\}, \tag{2}$$

$$UC_p(k;\gamma) = \left\{ f \in \mathcal{A}_p : \Re\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right) > k\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \right\}, \tag{3}$$

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$$UK_p(k;\gamma,\eta)=\left\{f\in\mathcal{A}_p:\exists\ g\in US_p^*(k;\eta),\Re\left(\frac{zf'(z)}{g(z)}-\gamma\right)>k\left|\frac{zf'(z)}{g(z)}-p\right|\right\},\tag{4}$$

$$UK_p^*(k;\gamma,\eta)=\left\{f\in\mathcal{A}_p:\exists\ g\in UC_p(k;\eta),\Re\left(\frac{\left(zf'(z)\right)'}{g'(z)}-\gamma\right)>k\left|\frac{\left(zf'(z)\right)'}{g'(z)}-p\right|\right\}.\tag{5}$$

These subclasses were introduced and studied by Al-Kharsani [1]. We note that

(i) $US_1^*(k;\gamma)=US^*(k;\gamma)$ and $UC_1(k;\gamma)=UC(k;\gamma)\,(0\leq\gamma<1)$ (see [7] and [14]);

(ii) $US_p^*(0;\gamma)=S_p^*(\gamma)$ $(0\leq\gamma<p)$ (see [12] and [13]);

(iii) $UC_p(0;\gamma)=C_p(\gamma)\,(0\leq\gamma<p)$ (see [12]);

(iv) $UK_p(0;\gamma,\eta)=K_p(\gamma,\eta)\,(0\leq\gamma,\eta<p)$ (see [2]);

(v) $UK_p^*(0;\gamma,\eta)=K_p^*(\gamma,\eta)\,(0\leq\gamma,\eta<p)$ (see [11]).

Corresponding to a conic domain $\Omega_{p,k,\gamma}$ defined by

$$\Omega_{p,k,\gamma}=\left\{u+iv:u>k\sqrt{(u-p)^2+v^2}+\gamma\right\},\tag{6}$$

we define the function $q_{p,k,\gamma}(z)$ which maps $\mathbb{U}$ onto the conic domain $\Omega_{p,k,\gamma}$ such that $1\in\Omega_{p,k,\gamma}$ as the following:

$$q_{k,\gamma}(z)=\begin{cases}\dfrac{p+(p-2\gamma)z}{1-z} & (k=0),\\[2ex] \frac{p-\gamma}{1-k^2}\cos\left\{\frac{2}{\pi}\left(\cos^{-1}k\right)i\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right\}-\frac{k^2p-\gamma}{1-k^2} & (0<k<1),\\[2ex] p+\dfrac{2(p-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 & (k=1),\\[2ex] \frac{p-\gamma}{k^2-1}\sin\left\{\dfrac{\pi}{2\zeta(k)}\int_0^{\frac{u(z)}{\sqrt{k}}}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}\right\}+\frac{k^2p-\gamma}{k^2-1} & (k>1).\end{cases}\tag{7}$$

where $u(z)=\frac{z-\sqrt{x}}{1-\sqrt{x}z},x\in(0,1)$ and $\zeta(k)$ is such that $k=\cosh\frac{\pi\zeta'(z)}{4\zeta(z)}$. By virture of the properties of the conic domain $\Omega_{p,k,\gamma}$, we have

$$\Re\{q_{p,k,\gamma}(z)\}>\frac{kp+\gamma}{k+1}.\tag{8}$$

Making use of the principal of subordination between analytic functions and the definition of $q_{p,k,\gamma}(z)$, we may rewrite the subclasses $US_p^*(k;\gamma)$, $UC_p(k;\gamma)$, $UK_p(k;\gamma,\beta)$ and $UK_p^*(k;\gamma,\beta)$ as the following:

$$US_p^*(k;\gamma)=\left\{f\in\mathcal{A}_p:\frac{zf'(z)}{f(z)}\prec q_{p,k,\gamma}(z)\right\},\tag{9}$$

$$UC_p(k;\gamma)=\left\{f\in\mathcal{A}_p:1+\frac{zf''(z)}{f'(z)}\prec q_{p,k,\gamma}(z)\right\},\tag{10}$$

$$UK_p(k;\gamma,\eta)=\left\{f\in\mathcal{A}_p:\exists\ g\in US_p^*(k;\eta),\ \frac{zf'(z)}{g(z)}\prec q_{p,k,\gamma}(z)\right\},\tag{11}$$

$$UK_p^*(k;\gamma,\eta)=\left\{f\in\mathcal{A}_p:\exists\ g\in UC_p(k;\eta),\ \frac{\left(zf'(z)\right)'}{g'(z)}\prec q_{p,k,\gamma}(z)\right\}.\tag{12}$$

Motivated essentially by Jung et al. [8], Shams et al. [15] introduced the integral operator $I_p^\alpha:\mathcal{A}_p\to\mathcal{A}_p$ as follows (see also Aouf et al. [3]):

$$I_p^\alpha f(z)=\begin{cases}\dfrac{(p+1)^\alpha}{z\Gamma(\alpha)}\int_0^z\left(\log\dfrac{z}{t}\right)^{\alpha-1}f(t)\,dt & (\alpha>0;p\in\mathbb{N}),\\ f(z) & (\alpha=0;p\in\mathbb{N}).\end{cases}\tag{13}$$

For $f\in\mathcal{A}_p$ given by (1), then from (13), we deduce that

$$I_p^\alpha f(z)=z^p+\sum_{n=p+1}^{\infty}\left(\frac{p+1}{n+1}\right)^\alpha a_nz^n,\quad(\alpha\geq 0;p\in\mathbb{N}).\tag{14}$$

Using the above relation, it is easy to verify the identity:

$$z\left(I_p^{\alpha+1}f(z)\right)'=(p+1)\,I_p^\alpha f(z)-I_p^{\alpha+1}f(z).\tag{15}$$

We note that the one-parameter family of integral operator $I_1^\alpha=I^\alpha$ was defined by Jung et al. [8].

Next, using the operator I_p^α, we introduce the following $k-$uniformly subclasses of $p-$valent functions for $\alpha\geq 0, p\in\mathbb{N}, k\geq 0$ and $0\leq\gamma,\eta<p$:

$$US_p^*(\alpha;k;\gamma)=\left\{f\in\mathcal{A}_p:I_p^\alpha f(z)\in US_p^*(k;\gamma)\,(z\in\mathbb{U})\right\},\tag{16}$$

$$UC_p(\alpha;k;\gamma)=\left\{f\in\mathcal{A}_p:I_p^\alpha f(z)\in UC_p(k;\gamma)\,(z\in\mathbb{U})\right\},\tag{17}$$

$$UK_p(\alpha;k;\gamma,\eta)=\left\{f\in\mathcal{A}_p:I_p^\alpha f(z)\in UK_p(k;\gamma,\eta)\,(z\in\mathbb{U})\right\},\tag{18}$$

$$UK_p^*(\alpha;k;\gamma,\eta)=\left\{f\in\mathcal{A}_p:I_p^\alpha f(z)\in UK_p^*(k;\gamma,\eta)\,(z\in\mathbb{U})\right\}.\tag{19}$$

We also note that

$$f\in US_p^*(\alpha;k;\gamma)\Leftrightarrow\frac{zf'}{p}\in UC_p(\alpha;k;\gamma),\tag{20}$$

and

$$f\in UK_p(\alpha;k;\gamma,\eta)\Leftrightarrow\frac{zf'}{p}\in UK_p^*(\alpha;k;\gamma,\eta).\tag{21}$$

In this paper, we investgate several inclusion properties of the classes $US_p^*(\alpha;k;\gamma)$, $UC_p(\alpha;k;\gamma)$, $UK_p(\alpha;k;\gamma,\eta)$, and $UK_p^*(\alpha;k;\gamma,\eta)$ associated with the operator I_p^α. Some applications involving integral operators are also considered.

2. Inclusion properties involving the operator I_p^α

In order to prove the main results, we shall need The following lemmas.

Lemma 1 [6] Let $h(z)$ be convex univalent in $\mathbb{U}$ with $\Re\{\eta h(z)+\gamma\}>0(\eta,\gamma\in\mathbb{C})$. If $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=h(0)$, then

$$p(z)+\frac{zp'(z)}{\eta p(z)+\gamma}\prec h(z) \tag{22}$$

implies

$$p(z)\prec h(z). \tag{23}$$

Lemma 2 [9] Let $h(z)$ be convex univalent in $\mathbb{U}$ and let w be analytic in $\mathbb{U}$ with $\Re\{w(z)\}\geq 0$. If $p(z)$ is analytic in $\mathbb{U}$ and $p(0)=h(0)$, then

$$p(z)+w(z)zp'(z)\prec h(z) \tag{24}$$

implies

$$p(z)\prec h(z). \tag{25}$$

Theorem 1 Let $k\geq 0$ and $0\leq\gamma<p$. Then,

$$US_p^*(\alpha;k;\gamma)\subset US_p^*(\alpha+1;k;\gamma). \tag{26}$$

Proof. Let $f\in US_p^*(\alpha;k;\gamma)$ and set

$$p(z)=\frac{z\left(I_p^{\alpha+1}f(z)\right)'}{I_p^{\alpha+1}f(z)}\quad(z\in\mathbb{U}), \tag{27}$$

where the function $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=p$. Using (15), (26) and (27), we have

$$\frac{z\left(I_p^{\alpha}f(z)\right)'}{I_p^{\alpha}f(z)}=p(z)+\frac{zp'(z)}{p(z)+1}\prec q_{p,k,\gamma}(z). \tag{28}$$

Since $k\geq 0$ and $0\leq\gamma<p$, we see that

$$\Re\{q_{p,k,\gamma}(z)+1\}>0\quad(z\in\mathbb{U}). \tag{29}$$

Applying Lemma 1 to (28), it follows that $p(z)\prec q_{p,k,\gamma}(z)$, that is, $f\in US_p^*(\alpha+1;k;\gamma)$. Therefore, we complete the proof of Theorem 1. □

Theorem 2 Let $k\geq 0$ and $0\leq\gamma<p$. Then,

$$UC_p(\alpha;k;\gamma)\subset UC_p(\alpha+1;k;\gamma). \tag{30}$$

Proof. Applying (21) and Theorem 1, we observe that

$$\begin{aligned} f\in UC_p(\alpha;k;\gamma) &\Longleftrightarrow \tfrac{zf'}{p}\in US_p^*(\alpha;k;\gamma)\\ &\Longrightarrow \tfrac{zf'}{p}\in US_p^*(\alpha+1;k;\gamma)\quad\text{(by Theorem 1)},\\ &\Longleftrightarrow f\in UC_p(\alpha+1;k;\gamma), \end{aligned}$$

which evidently proves Theorem 2. □

Next, by using Lemma 2, we obtain the following inclusion relation for the class $UK_p(\alpha;k;\gamma,\eta)$.

Theorem 3 Let $k\geq 0$ and $0\leq\gamma,\eta<p$. Then,

$$UK_p(\alpha;k;\gamma,\eta)\subset UK_p(\alpha+1;k;\gamma,\eta). \tag{31}$$

Proof. Let $f \in UK_p(\alpha;k;\gamma,\eta)$. Then, from the definition of $UK_p(\alpha;k;\gamma,\eta)$, there exists a function $r(z) \in US_p^*(k;\eta)$ such that

$$\frac{z\left(I_p^\alpha f(z)\right)'}{r(z)} \prec q_{p,k,\gamma}(z). \tag{32}$$

Choose the function g such that $I_p^\alpha g(z) = r(z)$. Then, $g \in US_p^*(\alpha;k;\eta)$ and

$$\frac{z\left(I_p^\alpha f(z)\right)'}{I_p^\alpha g(z)} \prec q_{p,k,\gamma}(z). \tag{33}$$

Now let

$$p(z) = \frac{z\left(I_p^{\alpha+1} f(z)\right)'}{I_p^{\alpha+1} g(z)} \quad (z \in \mathbb{U}), \tag{34}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = p$. Since $g \in US_p^*(\alpha;k;\eta)$, by Theorem 1, we know that $g \in US_p^*(\alpha+1;k;\eta)$. Let

$$t(z) = \frac{z\left(I_p^{\alpha+1} g(z)\right)'}{I_p^{\alpha+1} g(z)} \quad (z \in \mathbb{U}), \tag{35}$$

where $t(z)$ is analytic in $\mathbb{U}$ with $\Re\{t(z)\} > \frac{kp+\eta}{k+1}$. Also, from (34), we note that

$$I_p^{\alpha+1} zf'(z) = I_p^{\alpha+1} g(z)\; p(z). \tag{36}$$

Differentiating both sides of (36) with respect to z, we obtain

$$\begin{aligned} \frac{z\left(I_p^{\alpha+1} zf'(z)\right)'}{I_p^{\alpha+1} g(z)} &= \frac{z\left(I_p^{\alpha+1} g(z)\right)'}{I_p^{\alpha+1} g(z)} p(z) + zp'(z) \\ &= t(z)\, p(z) + zp'(z). \end{aligned} \tag{37}$$

Now using the identity (15) and (35), we obtain

$$\begin{aligned} \frac{z\left(I_p^\alpha f(z)\right)'}{I_p^\alpha g(z)} &= \frac{I_p^\alpha zf'(z)}{I_p^\alpha g(z)} = \frac{z\left(I_p^{\alpha+1} zf'(z)\right)' + I_p^{\alpha+1} zf'(z)}{z\left(I_p^{\alpha+1} g(z)\right)' + I_p^{\alpha+1} g(z)} \\ &= \frac{\dfrac{z\left(I_p^{\alpha+1} zf'(z)\right)'}{I_p^{\alpha+1} g(z)} + \dfrac{z\left(I_p^{\alpha+1} f(z)\right)'}{I_p^{\alpha+1} g(z)}}{\dfrac{z\left(I_p^{\alpha+1} g(z)\right)'}{I_p^{\alpha+1} g(z)} + 1} \\ &= \frac{t(z)\, p(z) + zp'(z) + p(z)}{t(z)+1} \\ &= p(z) + \frac{zp'(z)}{t(z)+1}. \end{aligned} \tag{38}$$

Since $k \geq 0, 0 \leq \eta < p$.and $\Re\{t(z)\} > \frac{kp+\eta}{k+1}$, we see that

$$\Re\{t(z)+1\} > 0 \quad (z \in \mathbb{U}).$$

Hence, applying Lemma 2, we can show that $p(z) \prec q_{p,k,\gamma}(z)$ so that $f \in UK_p(\alpha; k; \gamma, \eta)$. Therefore, we complete the proof of Theorem 3. □

Theorem 4 Let $k \geq 0$ and $0 \leq \gamma, \eta < p$. Then,

$$UK_p^*(\alpha; k; \gamma, \eta) \subset UK_p^*(\alpha + 1; k; \gamma, \eta). \tag{2.18}$$

Proof. Just as we derived Theorem 2 as consequence of Theorem 1 by using the equivalence (21), we can also prove Theorem 4 by using Theorem 3 and the equivalence (**??**). □

3. Inclusion properties involving the integral operator $F_{c,p}$

In this section, we present several integral-preserving properties of the $p-$valent function classes introduced here. We consider the generalized Libera integral operator $F_{c,p}(f)$ (see [5] and [4]) defined by

$$F_{c,p}(f)(z) = \frac{c+p}{z^c} \int t^{c-1} f(z)\, dt \;\; (c > -p). \tag{39}$$

Theorem 5 Let $kp + \gamma + c(k+1) \geq 0$. If $f \in US_p^*(\alpha; k; \gamma)$, then $F_{c,p}(f) \in US_p^*(\alpha; k; \gamma)$.

Proof. Let $f \in US_p^*(\alpha; k; \gamma)$ and set

$$p(z) = \frac{z\left(I_p^\alpha F_{c,p}(f)(z)\right)'}{I_p^\alpha F_{c,p}(f)(z)} \quad (z \in \mathbb{U}), \tag{40}$$

where $p(z)$ is analytic in U with $p(0) = p$. From (39), we have

$$z\left(I_p^\alpha F_{c,p}(f)(z)\right)' = (c+p) I_p^\alpha f(z) - cI_p^\alpha F_{c,p}(f)(z). \tag{41}$$

Then, by using (40) and (41), we obtain

$$(c+p)\frac{I_p^\alpha f(z)}{I_p^\alpha F_{c,p}(f)(z)} = p(z) + c. \tag{42}$$

Taking the logarithmic differentiation on both sides of (42) and multiplying by z, we have

$$\frac{z\left(I_p^\alpha f(z)\right)'}{I_p^\alpha f(z)} = p(z) + \frac{zp'(z)}{p(z)+c} \prec q_{k,\gamma}(z). \tag{43}$$

Hence, by virtue of Lemma 1, we conclude that $p(z) \prec q_{k,\gamma}(z)$ in $\mathbb{U}$, which implies that $F_{c,p}(f) \in US_p^*(\alpha; k; \gamma)$. □

Next, we derive an inclusion property involving $F_{c,p}(f)$, which is given by the following.

Theorem 6 Let $kp + \gamma + c(k+1) \geq 0$. If $f \in UC_p(\alpha; k; \gamma)$, then $F_{c,p}(f) \in UC_p(\alpha; k; \gamma)$.

Proof. By applying Theorem 5, it follows that

$$\begin{aligned} f\in UC_p(\alpha;k;\gamma) &\iff \frac{zf'}{p}\in US_p^*(\alpha;k;\gamma)\\ &\Longrightarrow F_{c,p}\left(\frac{zf'}{p}\right)\in US_p^*(\alpha;k;\gamma)\\ &\iff \frac{z\left(F_{c,p}(f)\right)'}{p}\in US_p^*(\alpha;k;\gamma)\\ &\iff F_{c,p}(f)\in UC_p(\alpha;k;\gamma), \end{aligned}$$

which proves Theorem 6. □

Theorem 7 Let $kp+\eta+c(k+1)\geq 0$. If $f\in UK_p(\alpha;k;\gamma,\eta)$, then $F_{c,p}(f)\in UK_p(\alpha;k;\gamma,\eta)$.

Proof. Let $f\in UK_p(\alpha;k;\gamma,\eta)$. Then, in view of the definition of the class $UK_p(\alpha;k;\gamma,\eta)$, there exists a function $g\in US_p^*(\alpha;k;\eta)$ such that

$$\frac{z\left(I_p^\alpha f(z)\right)'}{I_p^\alpha g(z)}\prec q_{k,\gamma}(z). \tag{44}$$

Thus, we set

$$p(z)=\frac{z\left(I_p^\alpha F_{c,p}(f)(z)\right)'}{I_p^\alpha F_{c,p}(g)(z)}\quad (z\in\mathbb{U}), \tag{45}$$

where $p(z)$ is analytic in $\mathbb{U}$ with $p(0)=p$. Since $g\in US_p^*(\alpha;k;\gamma)$, we see from Theorem 5 that $F_{c,p}(f)\in US_p^*(\alpha;k;\gamma)$. Let

$$t(z)=\frac{z\left(I_p^\alpha F_{c,p}(g)(z)\right)'}{I_p^\alpha F_{c,p}(g)(z)}\quad (z\in\mathbb{U}), \tag{46}$$

where $t(z)$ is analytic in $\mathbb{U}$ with $\Re\{t(z)\}>\dfrac{kp+\eta}{k+1}$. Also, from (45), we note that

$$I_p^\alpha zF_{c,p}'(f)(z)=I_p^\alpha F_{c,p}(g)(z)\ .\ p(z). \tag{47}$$

Differentiating both sides of (47) with respect to z, we obtain

$$\begin{aligned} \frac{z\left(I_p^\alpha zF_{c,p}'(f)(z)\right)'}{I_p^\alpha F_{c,p}(g)(z)} &= \frac{z\left(I_p^\alpha F_{c,p}(g)(z)\right)'}{I_p^\alpha F_{c,p}(g)(z)}p(z)+zp'(z)\\ &= t(z)p(z)+zp'(z). \end{aligned} \tag{48}$$

Now using the identity (41) and (48), we obtain

$$\begin{aligned}\frac{z\left(I_p^{\alpha}f\left(z\right)\right)^{'}}{I_p^{\alpha}g\left(z\right)} &= \frac{z\left(I_p^{\alpha}zF'_{c,p}\left(f\right)\left(z\right)\right)^{'}+cI_p^{\alpha}zF'_{c,p}\left(f\right)\left(z\right)}{z\left(I_p^{\alpha}F_{c,p}\left(g\right)\left(z\right)\right)^{'}+cI_p^{\alpha}F_{c,p}\left(g\right)\left(z\right)} \\ &= \frac{\dfrac{z\left(I_p^{\alpha}zF'_{c,p}\left(f\right)\left(z\right)\right)^{'}}{I_p^{\alpha}F_{c,p}\left(g\right)\left(z\right)}+c\dfrac{z\left(I_p^{\alpha}F_{c,p}\left(f\right)\left(z\right)\right)^{'}}{I_p^{\alpha}F_{c,p}\left(g\right)\left(z\right)}}{\dfrac{z\left(I_p^{\alpha}F_{c,p}\left(g\right)\left(z\right)\right)^{'}}{I_p^{\alpha}F_{c,p}\left(g\right)\left(z\right)}+c} \\ &= \frac{t\left(z\right)p\left(z\right)+zp^{'}\left(z\right)+cp\left(z\right)}{t\left(z\right)+c} \\ &= p\left(z\right)+\frac{zp^{'}\left(z\right)}{t\left(z\right)+c}. \qquad (49)\end{aligned}$$

Since $kp+\eta+c\left(k+1\right)\geq 0$ and $\Re\left\{t\left(z\right)\right\}>\dfrac{kp+\eta}{k+1}$, we see that

$$\Re\left\{t\left(z\right)+c\right\}>0 \quad \left(z\in\mathbb{U}\right). \qquad (50)$$

Hence, applying Lemma 2 to (49), we can show that $p\left(z\right)\prec q_{p,k,\gamma}\left(z\right)$ so that $f\in UK_p\left(\alpha;k;\gamma,\eta\right)$. □

Theorem 8 Let $kp+\eta+c\left(k+1\right)\geq 0$. If $f\in UK_p^{*}\left(\alpha;k;\gamma,\eta\right)$, then $F_{c,p}\left(f\right)\in UK_p^{*}\left(\alpha;k;\gamma,\eta\right)$

Proof. Just as we derived Theorem 6 as consequence of Theorem 5, we easily deduce the integral-preserving property asserted by Theorem 8 by using Theorem 7. □

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