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DIFFERENTIAL SUBORDINATION ASSOCIATED WITH AN EXTENDED FRACTIONAL DIFFERENTIAL OPERATOR

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ABSTRACT. In this paper we obtain certain sufficient conditions for multivalent functions defined by using an extended fractional differential operator.

1. Introduction

Let $A(p)$ denote the class of functions f of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in N = \{1, 2, ...\}), \tag{1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. Let also that $A = A(1)$. Recently, several authors ([1], [3] and [4]) obtained suffecient conditions associated with starlikeness in terms of the expression

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}.$$

In fact, Ravichandran [8] obtained the following results:
Theorem A [8, Theorem 3]. *Let q be convex univalent function and* $0 < \alpha \leq 1$,

$$Re\left\{\frac{1-\alpha}{\alpha} + 2q(z) + (1 + \frac{zq''(z)}{q'(z)})\right\} > 0.$$

If $f \in A$ *satisfies*

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec (1-\alpha)q(z) + \alpha q^2(z) + \alpha z q'(z),$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and q is the best dominant.

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Theorem B [8, Theorem 4]. *Let q be analytic in $U, q(0) = 1$ and $h(z) = zq'(z)/q(z)$ be starlike univalent in U. If $f \in A$ satisfies*

$$\frac{(zf(z))''}{f'(z)} - 2\frac{zf'(z)}{f(z)} \prec h(z),$$

then

$$\frac{z^2 f'(z)}{\{f(z)\}^2} \prec q(z)$$

and q is the best dominant.

For two functions f given by (1) and $g(z) = z^p + \sum\limits_{k=p+1}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} b_k a_k z^k = (g * f)(z).$$

Various operators of the fractional calculus (that is , fractional derivative and fractional integral) has been studied, we find it convenient to restrict ourselves to the following definitions used recently by Owa [5] and Owa and Srivastava [6].

Definition 1. *The fractional integral of order λ is defined for the function $f(z)$ by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int\limits_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt \quad (\lambda > 0), \tag{it2}$$

where $f(z)$ is analytic in a simply connected region containing the origin and multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring log $(z-t)$ to be real when $z-t > 0$.

Definition 2. *The fractional derivative of order λ is defined for the function $f(z)$ by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int\limits_0^z \frac{f(t)}{(z-t)^{\lambda}} dt, \tag{it3}$$

where $0 \leq \lambda < 1$, $f(z)$ is analytic in a simply connected region containing the origin and multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 1.

Definition 3. *Under the hypotheses of Definition 2, the fractional derivative of order $n + \lambda$ is defined for a function $f(z)$ by*

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \qquad (0 \leq \lambda < 1 \; ; \; n \in N_0 = N \cup \{0\}). \tag{it4}$$

For functions $f(z)$ in the form (1), Patel and Mishra [7] defined the extended fractional differintegral operator $\Omega_z^{(\lambda,p)} : A(p) \to A(p)$ by

$$\begin{aligned}\Omega_z^{(\lambda,p)} f(z) &= \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z) \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k z^k \\ &= z^p \; {}_2F_1(1, p+1; p+1-\lambda; z) * f(z) \;\; (z \in U; -\infty < \lambda < p+1), \end{aligned} \tag{5}$$

where ${}_2F_1$ is the Gaussian hypergeometric function defined by:

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k \quad (a,b,c \in C; c \notin Z_0 = \{0,-1,-2,...\}), $$

and $(d)_k$ is the Pochhammer symbol given in terms of the Gamma function by:

$$ (d)_k = \begin{cases} 1 & (k=0; d \in C\backslash\{0\}) \\ d(d+1)...(d+k-1) & (k \in N; d \in C). \end{cases} $$

We note that ${}_2F_1$ represents an analytic function in U (see for details [11, Ch. 14]).
It is easily seen from (5) that

$$ z(\Omega_z^{(\lambda,p)} f(z))' = (p-\lambda)\Omega_z^{(\lambda+1),p} f(z) + \lambda \Omega_z^{(\lambda,p)} f(z) \quad (z \in U; -\infty \leq \lambda < p). \quad (6) $$

We also note that:

$$ \Omega_z^{(0,p)} f(z) = f(z) \quad \text{and} \quad \Omega_z^{(1,p)} f(z) = \frac{zf'(z)}{p}. $$

The fractional differential operator $\Omega_z^{(\lambda,p)} f(z)$ with $0 \leq \lambda < 1$ was investigated by Srivastava and Aouf [10].
In this paper, we present extension of the results of Ravichandran [8] for functions defined through the extended fractional differential operator $\Omega_z^{(\lambda,p)} f(z)$.

2. Definitions and Preliminaries

In the present paper, we shall need the following Lemmas:
Lemma 1 [2]. *Let q be univalent in the unit disc U. Let φ be analytic in a domain containing $q(U)$. If $zq'(z)/\varphi(q(z))$ is starlike, then*

$$ z\psi'(z)\varphi(\psi(z)) \prec zq'(z)\varphi(q(z)) \quad (z \in U), $$

then$\psi(z) \prec q(z)$ and q is the best dominant.
Lemma 2 [9]. *If p and q are analytic in U, q is convex univalent , α, β and γ are complex and $\gamma \neq 0$. Further assume that*

$$ Re\left\{ \frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} q(z) + (1 + \frac{zq''(z)}{q'(z)}) \right\} > 0. $$

If $p(z) = 1 + cz + c_2 z^2 + ...$ is analytic in U and satisfies

$$ \alpha p(z) + \beta p^2(z) + \gamma z p'(z) \prec \alpha q(z) + \beta q^2(z) + \gamma z q'(z), $$

then $p(z) \prec q(z)$ and q is the best dominant.

3. Main Results

Theorem 1. *Let q be convex univalent, $\alpha \neq 0; 0 \leq \lambda < p-1; p > 1$. Further assume that*

$$Re\left\{\frac{(p-\lambda)(1-\alpha)-1}{\alpha} + 2(p-\lambda)q(z) + (1+\frac{zq''(z)}{q'(z)})\right\} > 0. \qquad \text{(it7)}$$

If $f \in A(p)$ satisfies

$$\frac{\Omega_z^{(\lambda+1,p)}f(z)}{\Omega_z^{(\lambda,p)}f(z)}\left\{1-\alpha+\frac{\Omega_z^{(\lambda+2,p)}f(z)}{\Omega_z^{(\lambda+1,p)}f(z)}\right\}$$

$$\prec \left[\frac{(p-\lambda)(1-\alpha)-1}{p-\lambda-1}\right]q(z) + \frac{\alpha(p-\lambda)}{p-\lambda-1}q^2(z) + \frac{\alpha}{p-\lambda-1}zq'(z), \qquad \text{(it8)}$$

then

$$\frac{\Omega_z^{(\lambda+1,p)}f(z)}{\Omega_z^{(\lambda,p)}f(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define a function Φ by

$$\Phi(z) = \frac{\Omega_z^{(\lambda+1,p)}f(z)}{\Omega_z^{(\lambda,p)}f(z)} \quad (z \in U). \qquad (9)$$

Then the function Φ is analytic in U and $\Phi(0) = 1$. Therefore, differentiating (9) logarithmically with respect to z and using the identity (6) in the resulting equation, we have

$$\frac{\Omega_z^{(\lambda+2,p)}f(z)}{\Omega_z^{\lambda+1,p}f(z)} = \frac{1}{p-1-\lambda}\left\{\frac{z\Phi'(z)}{\Phi(z)} + (p-\lambda)\Phi(z) - 1\right\}. \qquad (10)$$

Therefore from (10), we have

$$\frac{\Omega_z^{(\lambda+1,p)}f(z)}{\Omega_z^{(\lambda,p)}f(z)}\left\{1-\alpha+\frac{\Omega_z^{(\lambda+2,p)}f(z)}{\Omega_z^{(\lambda+1,p)}f(z)}\right\} = \left[\frac{(p-\lambda)(1-\alpha)-1}{p-\lambda-1}\right]\Phi(z) + \frac{\alpha(p-\lambda)}{p-\lambda-1}\Phi^2(z) + \frac{\alpha}{p-\lambda-1}z\Phi'(z). \qquad (11)$$

Using (11) in (8), we have

$$\left[\frac{(p-\lambda)(1-\alpha)-1}{p-\lambda-1}\right]\Phi(z) + \frac{\alpha(p-\lambda)}{p-\lambda-1}\Phi^2(z) + \frac{\alpha}{p-\lambda-1}z\Phi'(z)$$

$$\prec \left[\frac{(p-\lambda)(1-\alpha)-1}{p-\lambda-1}\right]q(z) + \frac{\alpha(p-\lambda)}{p-\lambda-1}q^2(z) + \frac{\alpha}{p-\lambda-1}zq'(z).$$

Hence the result now follows by using Lemma 2.

Theorem 2. *Let q be univalent in $U, q(0) = 1$. Let $\frac{zq'(z)}{q(z)}$ be starlike univalent in $U; 0 \leq \lambda < p-1, p > 1$. If $f(z) \in A(p)$ satisfies*

$$(p-1-\lambda)\frac{\Omega_z^{(\lambda+2,p)}f(z)}{\Omega_z^{(\lambda+1,p)}f(z)} - \alpha(p-\lambda)\frac{\Omega_z^{(\lambda+1,p)}f(z)}{\Omega_z^{(\lambda,p)}f(z)} \prec \frac{zq'(z)}{q(z)} + (1-\alpha)(1-\lambda) - 1, \qquad \text{(it12)}$$

then

$$\frac{z^{\alpha-1}\Omega_z^{(\lambda+1,p)}f(z)}{\left(\Omega_z^{(\lambda,p)}f(z)\right)^{\alpha}} \prec q(z)$$

and q is the best dominant.

Proof. Define a function ϕ by

$$\phi(z) = \frac{z^{\alpha-1}\Omega_z^{(\lambda+1,p)} f(z)}{\left(\Omega_z^{(\lambda,p)} f(z)\right)^{\alpha}} \quad (z \in U). \tag{13}$$

By a simple computation from (13), we have

$$(p-1-\lambda)\frac{\Omega_z^{(\lambda+2,p)} f(z)}{\Omega_z^{(\lambda+1,p)} f(z)} - \alpha(p-\lambda)\frac{\Omega_z^{(\lambda+1,p)} f(z)}{\Omega_z^{(\lambda,p)} f(z)} = \frac{z\phi'(z)}{\phi(z)} + (1-\alpha)(1-\lambda) - 1. \tag{14}$$

By using (14) in (12), we have

$$\frac{z\phi'(z)}{\phi(z)} \prec \frac{zq'(z)}{q(z)}$$

and the result follows by an application of Lemma 1.

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