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# EXISTENCE OF A UNIQUE CONTINUOUS SOLUTION FOR A QUADRATIC INTEGRAL EQUATIONS

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Abstract. In this paper, we are concerned with the existence of a unique continuous solution for the quadratic integral equation .

$$x(t) \ = \ a(t) \ + \lambda \int_0^t k_1(t,s) f_1(s,x(s)) ds \int_0^t k_2(t,s) f_2(s,x(s)) ds, \ t \in [0,T]$$

where $f_1$ and $f_2$ are two measurable lipschitz functions.

## 1. Introduction

Quadratic integral equations(QIES) are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory.
The quadratic integral equations can be very often encountered in many applications(see[1]-[16]).
In some papers (see[3]-[4]) the authors studied the existence of unique solutions.

Here we are concerned with the nonlinear quadratic integral equation

$$x(t) \ = \ a(t) \ + \lambda \int_0^t k_1(t,s) f_1(s,x(s)) ds \int_0^t k_2(t,s) f_2(s,x(s)) ds, \ t \in [0,T]. \quad (1)$$

The existence of a unique continuous solution $x \in C[0,T]$ well be proved. Some initial value problems will be concerned as an application.

## 2. Main results

Consider the following assumptions
(i) $a$ :$I = [0,T] \to R$ is continuous, $a = \sup |a(t)|, \ t \in [0,T]$ .
(ii) $f_i \ : \ I \times R \to R$ are measurable in $t$ for all $x \ \in \ R$ and satisfy the Lipschitz condition with respect to the second argument $x$ for almost all $t \ \in \ [0,T]$, $| \ f_i(t,x) - f_i(t,y) \ | \ \leq \ L \ | \ x - y \ |$.

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for each $(t,x),(t,y) \in I \times R$.
(iii) There exist two functions $m_i \in L^1[0,T]$ such that.
$| \, f_i(t,x) \, | \leq m_i(t)$ , $i = 1,2$
(iv) $k_i$ : $[0,T] \times R \to R$ are continuous in $t \in [0,T]$ for every $s \in [0,T]$ and measurable in $s \in [0,T]$ for all $t \in [0,T]$ such that.
$\int_0^t |k_i(t,s)| m_i(s) ds \leq K$ , $i = 1,2$ .

Now for the existence of a unique continuous solution of the quadratic integral equation (1) we have the following theorem.
**Theorem (2.1)** Let the assumptions (i)-(iv) are satisfied. If $2 \, |\lambda| \, L \, K^2 \, T < 1$, then the quadratic integral equation (1) has a unique continuous solution $x \in C[0,T]$.

**Proof.** Define the operator $F$ associated with the quadratic integral equation (1) by
$Fx(t) = a(t) + \lambda \int_0^t k_1(t,s) \, f_1(s,x(s)) \, ds \cdot \int_0^t k_2(t,s) \, f_2(s,x(s)) \, ds.$
The operator $F$ maps $C[0,T]$ into itself.
Let $x \in C[0,T]$, let $t_1, \; t_2 \in [0,T]$, $t_1 < t_2$ and $|t_2 - t_1| \leq \delta$, then

$$
\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= |a(t_2) - a(t_1) \\
&+ \lambda \int_0^{t_2} k_1(t_2,s) f_1(s,x(s)) ds \int_0^{t_2} k_2(t_2,s) f_2(s,x(s)) ds \\
&- \lambda \int_0^{t_1} k_1(t_1,s) f_1(s,x(s)) ds \int_0^{t_1} k_2(t_1,s) f_2(s,x(s)) ds| \\
&\leq |a(t_2) - a(t_1)| \\
&+ |\lambda \int_0^{t_2} k_1(t_2,s) f_1(s,x(s)) ds \int_0^{t_2} k_2(t_2,s) f_2(s,x(s)) ds \\
&- \lambda \int_0^{t_1} k_1(t_1,s) f_1(s,x(s)) ds \int_0^{t_1} k_2(t_1,s) f_2(s,x(s)) ds \\
&+ \lambda \int_0^{t_2} k_1(t_2,s) f_1(s,x(s)) ds \int_0^{t_1} k_2(t_1,s) f_2(s,x(s)) ds \\
&- \lambda \int_0^{t_2} k_1(t_2,s) f_1(s,x(s)) ds \int_0^{t_1} k_2(t_1,s) f_2(s,x(s)) ds| \\
&\leq |a(t_2) - a(t_1)| \\
&+ |\lambda \int_0^{t_1} k_2(t_1,s) f_2(s,x(s)) ds [\int_0^{t_2} k_1(t_2,s) f_1(s,x(s)) ds \\
&- \int_0^{t_1} k_1(t_1,s) f_1(s,x(s)) ds] \\
&+ \lambda \int_0^{t_2} k_1(t_2,s) f_1(s,x(s)) ds [\int_0^{t_2} k_2(t_2,s) f_2(s,x(s)) ds \\
&- \int_0^{t_1} k_2(t_1,s) f_2(s,x(s)) ds]|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & |\ a(t_2) - a(t_1)| \\
+\ & |\lambda| \int_0^{t_1} |k_2(t_1,s)||f_2(s,x(s))|ds \int_0^{t_1} |k_1(t_2,s) - k_1(t_1,s)||f_1(s,x(s))|ds \\
+\ & |\lambda| \int_0^{t_1} |k_2(t_1,s)||f_2(s,x(s))|ds \int_{t_1}^{t_2} |k_1(t_2,s)||f_1(s,x(s))|ds \\
+\ & |\lambda| \int_0^{t_2} |k_1(t_2,s)||f_1(s,x(s))|ds \int_0^{t_1} |k_2(t_2,s) - k_1(t_1,s)|f_2(s,x(s))ds \\
+\ & |\lambda| \int_0^{t_2} |k_1(t_2,s)||f_1(s,x(s))|ds \int_{t_1}^{t_2} |k_2(t_2,s)||f_2(s,x(s))|ds \\
\leq\ & |a(t_2) - a(t_1)| \\
+\ & |\lambda| \int_0^{t_1} |k_2(t_1,s)|m_2(s)ds. \int_0^{t_1} |k_1(t_2,s) - k_1(t_1,s)|m_1(s)ds \\
+\ & |\lambda| \int_0^{t_1} |k_2(t_1,s)|m_2(s)ds. \int_{t_1}^{t_2} |k_1(t_2,s)|m_1(s)ds \\
+\ & |\lambda| \int_0^{t_2} |k_1(t_2,s)|m_1(s)ds. \int_0^{t_1} |k_2(t_2,s) - k_2(t_1,s)|m_2(s)ds \\
+\ & |\lambda| \int_0^{t_2} |k_1(t_2,s)|m_1(s)ds. \int_{t_1}^{t_2} |k_2(t_2,s)|m_2(s)ds \\
\leq\ & |a(t_2) - a(t_1)| \\
+\ & |\lambda|\ K. \int_0^{t_1} |k_1(t_2,s) - k_1(t_1,s)|m_1(s)ds \\
+\ & |\lambda|\ K. \int_{t_1}^{t_2} |k_1(t_2,s)|m_1(s)ds \\
+\ & |\lambda|\ K. \int_0^{t_1} |k_2(t_2,s) - k_2(t_1,s)|m_2(s)ds \\
+\ & |\lambda|\ K. \int_{t_1}^{t_2} |k_2(t_2,s)|m_2(s)ds.
\end{aligned}
$$

This proves that $F : C[0,T] \to C[0,T]$.
Now to prove that $F$ is contraction, we have the following.
Let $x,\ y \in\ C[0,T]$, then

$$
\begin{aligned}
|Fx(t) - Fy(t)| \ =\ & |\lambda \int_0^t\ k_1\ (t,s)\ f_1(s,x(s))\ ds \int_0^t\ k_2\ (t,s)\ f_2(s,x(s))\ ds \\
-\ & \lambda \int_0^t\ k_1\ (t,s)\ f_1(s,y(s))\ ds \int_0^t\ k_2\ (t,s)\ f_2(s,y(s))\ ds|
\end{aligned}
$$

$$
\begin{aligned}
&= |\lambda \int_0^t k_1(t,s)\ f_1(s,x(s))\ ds \int_0^t k_2(t,s)\ f_2(s,x(s))\ ds \\
&- \lambda \int_0^t k_1(t,s)\ f_1(s,y(s))\ ds \int_0^t k_2(t,s)\ f_2(s,y(s))\ ds \\
&+ \lambda \int_0^t k_1(t,s)\ f_1(s,x(s))\ ds \int_0^t k_2(t,s)\ f_2(s,y(s))\ ds \\
&- \lambda \int_0^t k_1(t,s)\ f_1(s,x(s))\ ds \int_0^t k_2(t,s)\ f_2(s,y(s))\ ds| \\
&= |\lambda \int_0^t k_1(t,s)\ f_1(s,x(s))\ ds \int_0^t k_2\ (t,s)\ [f_2(s,x(s)) - f_2(s,y(s))]\ ds \\
&+ \lambda \int_0^t k_2(t,s)\ f_2(s,y(s))\ ds \int_0^t k_1\ (t,s)\ [f_1(s,x(s)) - f_1(s,y(s))]\ ds| \\
&\leq |\lambda| \int_0^t |k_1(t,s)|\ |f_1(s,x(s))|\ ds \int_0^t |k_2(t,s)|\ |f_2(s,x(s)) - f_2(s,y(s))|\ ds \\
&+ |\lambda| \int_0^t |k_2(t,s)|\ f_2(s,y(s))|\ ds \int_0^t |k_1(t,s)|\ |f_1(s,x(s)) - f_1(s,y(s))|\ ds \\
&\leq |\lambda|\ K \int_0^t k\ L\ |x(s) - y(s)|\ ds \\
&+ |\lambda|\ K \int_0^t k\ L\ |x(s) - y(s)|\ ds \\
&\leq |\lambda|\ K^2\ L \int_0^t |x(s) - y(s)|\ ds \\
&+ |\lambda|\ K^2\ L \int_0^t |x(s) - y(s)|\ ds \\
&\leq 2\ |\lambda|\ L\ K^2\ T\ ||x - y||.
\end{aligned}
$$

This means that

$$||Fx - Fy|| \leq\ 2\ |\lambda|\ L\ K^2\ T\ ||x - y||.$$

Then by using Banach fixed point Theorem (see[15]), the operator $F$ has a unique fixed point $x \in\ C[0,T]$ .
This completes the proof of existence a unique solution $x \in C[0,T]$ of the quadratic integral equation (1).■

When $k_1 = k_2 = k$ and $f_1 = f_2 = f$, we have the following corollary.
**Corollary(2.2)** Let the assumpations of theorem (2.1) are satisfied .
If $2\ |\lambda|\ L\ K^2\ T <\ 1$ , then the quadratic integral equation

$$x(t)\ =\ a(t)\ + \lambda(\int_0^t k(t,s) f(s,x(s)) ds)^2,\ t \in [0,T]. \tag{2}$$

has a unique continuous solution $x \in C[0,T]$.
Letting $k=1$, $a(t)=0$ and $\lambda=1$ in (2),then we have the following corollary.
**corollary(2.3)** Let the assumpations (ii)-(iii) of theorem (2.1) are satisfied .
If $2\ L\ T<\ 1$ , then the quadratic integral equation

$$x(t)\ =\ (\int_0^t f(s,x(s))ds)^2,\ t\in[0,T]. \tag{3}$$

has a unique continuous solution $x \in C[0,T]$.
Now let $k_1(t)=k_2(t)=1$ in equation (1) then we have the following corollary,
**Corollary(2.4)** Let the assumptions (i)-(iii) of Theorem (2.1) are satisfied, then the quadratic integral equation

$$x(t)\ =\ a(t)\ +\lambda\int_0^t f_1(s,x(s))\ ds\int_0^t f_2(s,x(s))\ ds.$$

has the existence of a unique continuous solution $x \in C[0,T]$.

## 3. Initial value problems

Consider now the initial value problem

$$\frac{d\sqrt{x(t)}}{dt}=f(t,x(t)),\ t>0 \tag{4}$$

with the initial condition

$$x(t)=0 \tag{5}$$

We have the following corollary.
**Corollary(3.1)** The initial value problem (4)and (5) is equivalent to the quadratic integral equation (3)
**Proof** Let $x(t)$ be a solution of the problem (4)-(5), then

$$\frac{d\sqrt{x(t)}}{dt}=f(t,x(t)).$$

Integrating, we get

$$\sqrt{x(t)}-\sqrt{x(0)}=\int_0^t f(s,x(s))ds$$

but $x(t)$= 0, then

$$\sqrt{x(t)}=\int_0^t f(s,x(s))ds$$

and

$$x(t)=(\int_0^t f(s,x(s))ds)^2.$$

Let $x(t)$ be a solution of (3), then

$$\sqrt{x(t)}=\int_0^t f(s,x(s))ds$$

Differentiating, we get

$$\frac{d\sqrt{x(t)}}{dt} = f(t, x(t)).$$

Also at $t = 0$ , we find that

$$x(0) = (\int_0^t f(t, x(t))|_{t=0})^2 = 0 \ .\blacksquare$$

Now we have the following corollary.

**Corollary(3.2)** The initial value problem (4)and (5) has a unique continuous solution $x \in C[0, T]$.

**Proof.**

From the equivalent of the initial value problem (4)-(5) and the quadratic integral equation (3)(Corollary 3.1),we deduce that from corollary (2.3) that the initial value problem (4)-(5) has a unique continuous solution $x \in C[0, T]$.$\blacksquare$

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