

**EXISTENCE RESULTS FOR A COUPLED SYSTEM OF
NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH
THREE-POINT BOUNDARY CONDITIONS.**

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ABSTRACT. This paper studies a coupled system of nonlinear fractional differential equation with three-point boundary conditions. Applying the Schauder fixed point theorem, an existence result is proved for the following system

$$\begin{aligned} D^\alpha u(t) &= f(t, v(t), D^m v(t)), & t \in (0, 1), \\ D^\beta v(t) &= g(t, u(t), D^n u(t)), & t \in (0, 1), \\ u(0) &= 0, D^\theta u(1) = \delta D^\theta u(\eta), & v(0) = 0, D^\theta v(1) = \delta D^\theta v(\eta), \end{aligned}$$

where $\alpha, \beta, m, n, \eta, \delta, \theta$ satisfy certain condition.

1. INTRODUCTION

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer-rheology, etc. involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see [2, 4, 6, 8, 9, 10, 11, 13] and the references therein.

On the other hand, the study of coupled systems involving fractional differential equations is also important as such systems occur in various problems of applied nature, for instance, see [5, 7]. Recently, in [12], the existence of nontrivial solutions was investigated for a coupled system of nonlinear fractional differential equations with two-point boundary conditions by using Schauder's fixed point theorem. Ref. [3] established the existence of a positive solution to a singular coupled system of fractional order. The existence of nontrivial solutions for a coupled system of nonlinear fractional differential equations with three-point boundary conditions was investigated in [1] by using Schauder's fixed point theorem.

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In this paper, we consider a three-point boundary value problem for a coupled system of nonlinear fractional differential equation given by

$$\begin{aligned} D^\alpha u(t) &= f(t, v(t), D^m v(t)), & t \in (0, 1), \\ D^\beta v(t) &= g(t, u(t), D^n u(t)), & t \in (0, 1), \\ u(0) &= 0, D^\theta u(1) = \delta D^\theta u(\eta), & v(0) = 0, D^\theta v(1) = \delta D^\theta v(\eta), \end{aligned} \quad (1)$$

where $1 \leq \alpha, \beta < 2$, $m, n, \delta > 0$, $0 < \eta < 1$, $\alpha - n \geq 1$, $\beta - m \geq 1$, $\delta \eta^{\alpha-1} < 1$, $\delta \eta^{\beta-1} < 1$, $0 \leq \theta < 1$, $\alpha - \theta - 1 \geq 0$, $\beta - \theta - 1 \geq 0$, D is the standard Riemann-Liouville fractional derivative and $f, g: [0, 1] \times R \times R \rightarrow R$ are given continuous function.

The organization of this paper is as follows. In Section 2, we present some necessary definition and preliminary results that will be used to prove our main results. The proofs of our main results are given in Section 3. In Section 4, we will give an example to ensure our main result.

2. PRELIMINARIES

For the convenience of the reader, we present here the necessary definition from fractional calculus theory and preliminary results.

([9]) The Riemann-Liouville fractional integral of order $q > 0$ of function $f: (0, \infty) \rightarrow R$ is given by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds,$$

provided that the integral exists.

([9]) The Riemann-Liouville fractional derivative of order $q > 0$ of function $f: (0, \infty) \rightarrow R$ is given by

$$D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{f(s)}{(t-s)^{q-n+1}} ds,$$

where $n = [q] + 1$ and $[q]$ denotes the integral part of number q , provided that the right side is pointwise defined on $(0, \infty)$.

([9]) Let $n - 1 < \alpha \leq n$, $D^\alpha u(t)$ exists for $t \in (0, 1)$. Then

$$I^\alpha D^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_n t^{\alpha-n}$$

The following properties are useful for our discussion: $I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t)$, $D^\alpha I^\alpha f(t) = f(t)$, $\alpha > 0$, $\beta > 0$, $f \in L_1(0, 1)$; $I^\alpha D^\alpha f(t) = f(t)$, $0 < \alpha < 1$, $f(t) \in C[0, 1]$ and $D^\alpha f(t) \in C(0, 1) \cap L_1(0, 1)$; $I^\alpha: C[0, 1] \rightarrow C[0, 1]$, $\alpha > 0$.

Let $C(J)$ Denote the space of all continuous functions defined on $J = [0, 1]$. Let $X =$

$\{u(t) : u \in C(J) \text{ and } D^n u \in C(J)\}$ be a Banach space endowed with the norm $\|u\|_X = \max_{t \in J} |u(t)| + \max_{t \in J} |D^n u(t)|$, where $1 < \alpha < 2$, $0 < n \leq \alpha - 1$, (see [12] Lemma 3.2), and $Y = \{v(t) : v \in C(J) \text{ and } D^m v \in C(J)\}$ be a Banach space equipped with the norm $\|v\|_Y = \max_{t \in J} |v(t)| + \max_{t \in J} |D^m v(t)|$, where $1 < \beta < 2$, $0 < m \leq \beta - 1$. Thus, $(X \times Y, \|\cdot\|_{X \times Y})$ is a Banach with the norm defined by $\|(u, v)\|_{X \times Y} = \max\{\|u\|_X, \|v\|_Y\}$ for $(u, v) \in X \times Y$.

Let $y \in C(J)$ be a given function and $1 < \alpha < 2$. Then the unique solution of

$$D^\alpha u(t) = y(t), \quad t \in (0, 1), \quad (2)$$

$$u(0) = 0, \quad D^\theta u(1) = \delta D^\theta u(\eta), \quad (3)$$

is given by

$$u(t) = \int_0^1 K_1(t, s) y(s) ds,$$

where $K_1(t, s)$ is the Green's function given by

$$K_1(t, s) = \frac{1}{(1 - \delta\eta^{\alpha-\theta-1})\Gamma(\alpha)} \begin{cases} K_{11}(t, s), & 0 \leq t \leq \eta \\ K_{12}(t, s), & \eta < t \leq 1 \end{cases} \quad (4)$$

$$K_{11}(t, s) = \begin{cases} \left[(t-s)^{\alpha-1} (1 - \delta\eta^{\alpha-\theta-1}) + \delta t^{\alpha-1} (\eta-s)^{\alpha-\theta-1} - t^{\alpha-1} (1-s)^{\alpha-\theta-1} \right], & 0 \leq s \leq t \\ \left[\delta t^{\alpha-1} (\eta-s)^{\alpha-\theta-1} - t^{\alpha-1} (1-s)^{\alpha-\theta-1} \right], & t < s \leq \eta \\ -t^{\alpha-1} (1-s)^{\alpha-\theta-1}, & \eta < s \leq 1 \end{cases}$$

$$K_{12}(t, s) = \begin{cases} \left[(t-s)^{\alpha-1} (1 - \delta\eta^{\alpha-\theta-1}) + \delta t^{\alpha-1} (\eta-s)^{\alpha-\theta-1} - t^{\alpha-1} (1-s)^{\alpha-\theta-1} \right], & 0 \leq s \leq \eta \\ \left[(t-s)^{\alpha-1} (1 - \delta\eta^{\alpha-\theta-1}) - t^{\alpha-1} (1-s)^{\alpha-\theta-1} \right], & \eta < s \leq t \\ -t^{\alpha-1} (1-s)^{\alpha-\theta-1}, & t < s \leq 1 \end{cases}$$

Proof. For $C_1, C_2 \in R$, the general solution of (2) can be written as

$$\begin{aligned} u(t) &= I^\alpha y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} \\ &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} \end{aligned} \quad (5)$$

The boundary condition $u(0) = 0$ implies that $C_2 = 0$. Thus

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds + C_1 t^{\alpha-1}$$

By lemma 2, we have

$$D^\theta u(t) = \frac{1}{\Gamma(\alpha-\theta)} \int_0^t (t-s)^{\alpha-\theta-1} y(s) ds + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\theta)} t^{\alpha-\theta-1}$$

By the boundary condition

$$D^\theta u(1) = \delta D^\theta u(\eta)$$

$$\begin{aligned} \frac{1}{\Gamma(\alpha-\theta)} \int_0^1 (1-s)^{\alpha-\theta-1} y(s) ds + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\theta)} &= \frac{\delta}{\Gamma(\alpha-\theta)} \int_0^\eta (\eta-s)^{\alpha-\theta-1} y(s) ds \\ &+ C_1 \frac{\delta \Gamma(\alpha)}{\Gamma(\alpha-\theta)} \eta^{\alpha-\theta-1} \end{aligned}$$

$$C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\theta)} (1 - \delta\eta^{\alpha-\theta-1}) = \frac{\delta}{\Gamma(\alpha-\theta)} \int_0^\eta (\eta-s)^{\alpha-\theta-1} y(s) ds - \frac{1}{\Gamma(\alpha-\theta)} \int_0^1 (1-s)^{\alpha-\theta-1} y(s) ds$$

$$C_1 = \frac{\delta}{(1 - \delta\eta^{\alpha-\theta-1})} \int_0^\eta \frac{(\eta-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} y(s) ds - \frac{1}{(1 - \delta\eta^{\alpha-\theta-1})} \int_0^1 \frac{(1-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} y(s) ds$$

Thus, the unique solution of (2) and (3) is

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ &+ \frac{\delta t^{\alpha-1}}{(1 - \delta\eta^{\alpha-\theta-1})} \int_0^\eta \frac{(\eta-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} y(s) ds \\ &- \frac{t^{\alpha-1}}{(1 - \delta\eta^{\alpha-\theta-1})} \int_0^1 \frac{(1-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} y(s) ds \\ &= \int_0^1 K_1(t, s) y(s) ds \end{aligned}$$

where $K_1(t, s)$ is given by (4). □

Similarly, the general solution of

$$\begin{aligned} D^\beta v(t) &= y(t), \quad t \in (0, 1), \\ v(0) &= 0, \quad D^\theta v(1) = \delta D^\theta v(\eta), \end{aligned}$$

is

$$v(t) = \int_0^1 K_2(t, s) y(s) ds$$

where $K_2(t, s)$ can be obtained from $K_1(t, s)$ by replacing α with β . Let K_1, K_2 denotes the Green's function for the boundary value problem (1).

Consider the coupled system of integral equation

$$\begin{cases} u(t) = \int_0^1 K_1(t, s) f(s, v(s), D^p v(s)) ds, \\ v(t) = \int_0^1 K_2(t, s) g(s, u(s), D^q u(s)) ds. \end{cases} \quad (6)$$

3. MAIN RESULTS

Assume that $f, g: J \times R \times R \rightarrow R$ are continuous function. Then $(u, v) \in X \times Y$ is a solution of (1) if and only if $(u, v) \in X \times Y$ is a solution of (6).

Proof. The proof is immediate from lemma 2, so we omit it. □

Let us define an operator $F: X \times Y \rightarrow X \times Y$ as

$$F(u, v)(t) = (F_1 v(t), F_2 u(t)),$$

where

$$F_1 v(t) = \int_0^1 K_1(t, s) f(s, v(s), D^p v(s)) ds, \quad F_2 u(t) = \int_0^1 K_2(t, s) g(s, u(s), D^q u(s)) ds.$$

In view of the continuity of K_1, K_2, f, g , it follows that F is continuous. Moreover, by lemma 3, the fixed point of the operator F coincides with the solution of (1).

For the forthcoming analysis, we introduce the growth condition on f and g as (A_1) there exists a nonnegative function $a(t) \in L_1(0, 1)$ such that

$$|f(t, x, y)| \leq a(t) + \epsilon_1 |x|^{\rho_1} + \epsilon_2 |y|^{\rho_2}, \quad \epsilon_1, \epsilon_2 > 0, \quad 0 < \rho_1, \rho_2 < 1,$$

(A₂) there exists a nonnegative function $b(t) \in L_1(0, 1)$ such that

$$|g(t, x, y)| \leq b(t) + \delta_1 |x|^{\sigma_1} + \delta_2 |y|^{\sigma_2}, \quad \delta_1, \delta_2 > 0, \quad 0 < \sigma_1, \sigma_2 < 1.$$

Let us set the following notations for convenience:

$$A = \frac{(1 - \delta\eta^{\alpha-\theta}) \Gamma(\alpha - q + 1) + [(\alpha - \theta)(1 - \delta\eta^{\alpha-\theta-1}) + (\alpha - q)(1 + \delta\eta^{\alpha-\theta})] \Gamma(\alpha)}{(\alpha - \theta)(1 - \delta\eta^{\alpha-\theta-1}) \Gamma(\alpha) \Gamma(\alpha - q + 1)},$$

$$B = \frac{(1 - \delta\eta^{\beta-\theta}) \Gamma(\beta - p + 1) + [(\beta - \theta)(1 - \delta\eta^{\beta-\theta-1}) + (\beta - p)(1 + \delta\eta^{\beta-\theta})] \Gamma(\beta)}{(\beta - \theta)(1 - \delta\eta^{\beta-\theta-1}) \Gamma(\beta) \Gamma(\beta - p + 1)},$$

$$\begin{aligned} m &= \max_{t \in J} \int_0^1 |a(s) k_1(t, s)| ds \\ &+ \frac{1}{\Gamma(\alpha - q)(1 - \delta\eta^{\alpha-\theta-1})} \times [(1 - \delta\eta^{\alpha-\theta-1}) \int_0^1 (1 - s)^{\alpha-q-1} a(s) ds \\ &+ \int_0^1 (1 - s)^{\alpha-\theta-1} a(s) ds + \delta \int_0^\eta (\eta - s)^{\alpha-\theta-1} a(s) ds], \end{aligned}$$

$$\begin{aligned} n &= \max_{t \in J} \int_0^1 |b(s) k_1(t, s)| ds \\ &+ \frac{1}{\Gamma(\beta - p)(1 - \delta\eta^{\beta-\theta-1})} \times [(1 - \delta\eta^{\beta-\theta-1}) \int_0^1 (1 - s)^{\beta-p-1} a(s) ds \\ &+ \int_0^1 (1 - s)^{\beta-\theta-1} a(s) ds + \delta \int_0^\eta (\eta - s)^{\beta-\theta-1} a(s) ds], \end{aligned}$$

Define a ball W in the Banach space $X \times Y$ as

$$W = \{ (u(t), v(t)) \mid (u(t), v(t)) \in X \times Y, \|(u(t), v(t))\|_{X \times Y} \leq R, t \in J \},$$

where $R \geq \max \left\{ (3A\epsilon_1)^{\frac{1}{1-\rho_1}}, (3A\epsilon_2)^{\frac{1}{1-\rho_2}}, (3B\delta_1)^{\frac{1}{1-\sigma_1}}, (3B\delta_2)^{\frac{1}{1-\sigma_2}}, 3m, 3n \right\}$

Assume that the assumptions (A₁) and (A₂) hold. Then there exists a solution for the three-point boundary value problem (1)

Proof. As a first step, we prove that $F: W \rightarrow W$.

$$|F_1 v(t)| = \left| \int_0^1 k_1(t, s) f(s, v(s), D^p v(s)) ds \right|$$

$$\begin{aligned}
&\leq \int_0^1 |k_1(t, s) f(s, v(s), D^p v(s)) ds| \\
&\leq \int_0^1 |k_1(t, s) [a(s) + \epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}]| ds \\
&= \int_0^1 |a(s) k_1(t, s) + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) k_1(t, s)| ds \\
&= \int_0^1 |a(s) k_1(t, s)| ds + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \int_0^1 |k_1(t, s)| ds \\
&= \int_0^1 |a(s) k_1(t, s)| ds + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \left[- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds \right. \\
&\quad \left. + \frac{t^{\alpha-1}}{(1-\delta\eta^{\alpha-\theta-1})} \int_0^1 \frac{(1-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} ds - \frac{\delta t^{\alpha-1}}{(1-\delta\eta^{\alpha-\theta-1})} \int_0^\eta \frac{(\eta-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} ds \right] \\
&= \int_0^1 |a(s) k_1(t, s)| ds + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \left[- \frac{t^\alpha}{\Gamma(\alpha+1)} \right. \\
&\quad \left. + \frac{t^{\alpha-1}}{(1-\delta\eta^{\alpha-\theta-1})(\alpha-\theta)\Gamma(\alpha)} - \frac{\delta t^{\alpha-1} \eta^{\alpha-\theta}}{(1-\delta\eta^{\alpha-\theta-1})(\alpha-\theta)\Gamma(\alpha)} \right] \\
&= \int_0^1 |a(s) k_1(t, s)| ds + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \left[- \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha-1} (1-\delta\eta^{\alpha-\theta})}{(1-\delta\eta^{\alpha-\theta-1})(\alpha-\theta)\Gamma(\alpha)} \right] \\
&\leq \int_0^1 |a(s) k_1(t, s)| ds + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \left[\frac{(1-\delta\eta^{\alpha-\theta})}{(1-\delta\eta^{\alpha-\theta-1})(\alpha-\theta)\Gamma(\alpha)} \right] \\
&\text{and} \\
|D^q F_1 v(t)| &= \left| D^q I^\alpha f(t, v(t), D^p v(t)) - \frac{1}{(1-\delta\eta^{\alpha-\theta-1})} \left[\int_0^1 \frac{(1-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} f(s, v(s), D^p v(s)) ds \right. \right. \\
&\quad \left. \left. - \delta \int_0^\eta \frac{(\eta-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} f(s, v(s), D^p v(s)) ds \right] D^q t^{\alpha-1} \right| \\
&= \left| I^{\alpha-q} f(t, v(t), D^p v(t)) - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)(1-\delta\eta^{\alpha-\theta-1})} \left[\int_0^1 \frac{(1-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} f(s, v(s), D^p v(s)) ds \right. \right. \\
&\quad \left. \left. - \delta \int_0^\eta \frac{(\eta-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} f(s, v(s), D^p v(s)) ds \right] t^{\alpha-q-1} \right| \\
&\leq \frac{1}{\Gamma(\alpha-q)} \left[\int_0^t (t-s)^{\alpha-q-1} (a(s) + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2})) ds \right. \\
&\quad \left. + \frac{1}{(1-\delta\eta^{\alpha-\theta-1})} \left\{ \int_0^1 (1-s)^{\alpha-\theta-1} (a(s) + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2})) ds \right. \right. \\
&\quad \left. \left. + \delta \int_0^\eta (\eta-s)^{\alpha-\theta-1} (a(s) + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2})) ds \right\} \right] \\
&= \frac{1}{\Gamma(\alpha-q)} \left[\int_0^t (t-s)^{\alpha-q-1} a(s) ds + \int_0^t (t-s)^{\alpha-q-1} (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) ds \right. \\
&\quad \left. + \frac{1}{(1-\delta\eta^{\alpha-\theta-1})} \left\{ \int_0^1 (1-s)^{\alpha-\theta-1} a(s) ds + \int_0^1 (1-s)^{\alpha-\theta-1} (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) ds \right. \right. \\
&\quad \left. \left. + \delta \int_0^\eta (\eta-s)^{\alpha-\theta-1} a(s) ds + \delta \int_0^\eta (\eta-s)^{\alpha-\theta-1} (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) ds \right\} \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha-q)} \left[\int_0^1 (1-s)^{\alpha-q-1} a(s) ds + \int_0^1 (1-s)^{\alpha-q-1} (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) ds \right. \\
&\quad + \frac{1}{(1-\delta\eta^{\alpha-\theta-1})} \left\{ \int_0^1 (1-s)^{\alpha-\theta-1} a(s) ds + \int_0^1 (1-s)^{\alpha-\theta-1} (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) ds \right. \\
&\quad \left. \left. + \delta \int_0^\eta (\eta-s)^{\alpha-\theta-1} a(s) ds + \delta \int_0^\eta (\eta-s)^{\alpha-\theta-1} (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) ds \right\} \right] \\
&= \frac{1}{\Gamma(\alpha-q)(1-\delta\eta^{\alpha-\theta-1})} \left[(1-\delta\eta^{\alpha-\theta-1}) \int_0^1 (1-s)^{\alpha-q-1} a(s) ds \right. \\
&\quad + \int_0^1 (1-s)^{\alpha-\theta-1} a(s) ds + \delta \int_0^\eta (\eta-s)^{\alpha-\theta-1} a(s) ds \\
&\quad + \frac{(\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2})}{\Gamma(\alpha-q)(1-\delta\eta^{\alpha-\theta-1})} \left[(1-\delta\eta^{\alpha-\theta-1}) \int_0^1 (1-s)^{\alpha-q-1} ds \right. \\
&\quad \left. + \int_0^1 (1-s)^{\alpha-\theta-1} ds + \delta \int_0^\eta (\eta-s)^{\alpha-\theta-1} ds \right] \\
&= \frac{1}{\Gamma(\alpha-q)(1-\delta\eta^{\alpha-\theta-1})} \left[(1-\delta\eta^{\alpha-\theta-1}) \int_0^1 (1-s)^{\alpha-q-1} a(s) ds \right. \\
&\quad + \int_0^1 (1-s)^{\alpha-\theta-1} a(s) ds + \delta \int_0^\eta (\eta-s)^{\alpha-\theta-1} a(s) ds \\
&\quad \left. + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \left[\frac{(\alpha-\theta)(1-\delta\eta^{\alpha-\theta-1}) + (\alpha-q)(1+\delta\eta^{\alpha-\theta})}{(\alpha-\theta)\Gamma(\alpha-q+1)(1-\delta\eta^{\alpha-\theta-1})} \right] \right]
\end{aligned}$$

Thus,

$$\|F_1 v(t)\|_X = \max_{t \in J} |F_1 v(t)| + \max_{t \in J} |D^q F_1 v(t)| \leq \mu + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \varpi_1 \leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R.$$

Similarly, it can be shown that $\|F_2 u(t)\|_Y \leq \nu + (\epsilon_1 |R|^{\rho_1} + \epsilon_2 |R|^{\rho_2}) \varpi_2 \leq R$.

Hence, we conclude that $\|F(u, v)\|_{X \times Y} \leq R$.

Since $F_1 v(t)$, $F_2 u(t)$, $D^q F_1 v(t)$, $D^q F_2 u(t)$ are continuous on J , therefore,

$$F: W \rightarrow W.$$

Now we show that F is a completely continuous operator. For that we fix

$$M = \max_{t \in J} |f(t, v(t), D^p v(t))|, \quad N = \max_{t \in J} |g(t, u(t), D^p u(t))|.$$

For $(u, v) \in W$, $t, \tau \in J$ ($t < \tau$), we have

$$\begin{aligned}
|F_1 v(t) - F_1 v(\tau)| &= \left| \int_0^1 (k_1(t, s) - k_1(\tau, s)) f(s, v(s), D^p v(s)) ds \right| \\
&\leq \int_0^1 |(k_1(t, s) - k_1(\tau, s)) f(s, v(s), D^p v(s))| ds \\
&\leq M \int_0^1 |(k_1(t, s) - k_1(\tau, s))| ds \\
&= M \left[\int_0^t |(k_1(t, s) - k_1(\tau, s))| ds + \int_t^\tau |(k_1(t, s) - k_1(\tau, s))| ds + \right. \\
&\quad \left. + \int_\tau^\eta |(k_1(t, s) - k_1(\tau, s))| ds + \int_\eta^1 |(k_1(t, s) - k_1(\tau, s))| ds \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{M}{\Gamma(\alpha)(1-\delta\eta^{\alpha-\theta-1})} \left[\int_0^t \left\{ \left((\tau-s)^{\alpha-1} - (t-s)^{\alpha-1} \right) (1-\delta\eta^{\alpha-\theta-1}) \right. \right. \\
&\quad \left. \left. + (\tau^{\alpha-1} - t^{\alpha-1}) \left((1-s)^{\alpha-\theta-1} - \delta(\eta-s)^{\alpha-\theta-1} \right) \right\} ds \right. \\
&\quad \left. + \int_t^\tau \left\{ (\tau^{\alpha-1} - t^{\alpha-1}) \left((1-s)^{\alpha-\theta-1} - \delta(\eta-s)^{\alpha-\theta-1} \right) + (\tau-s)^{\alpha-1} (1-\delta\eta^{\alpha-\theta-1}) \right\} ds \right. \\
&\quad \left. + \int_\tau^\eta \left\{ (\tau^{\alpha-1} - t^{\alpha-1}) \left((1-s)^{\alpha-\theta-1} - \delta(\eta-s)^{\alpha-\theta-1} \right) \right\} ds + \int_\eta^1 (\tau^{\alpha-1} - t^{\alpha-1}) (1-s)^{\alpha-\theta-1} ds \right] \\
&= \frac{M}{\Gamma(\alpha)(1-\delta\eta^{\alpha-\theta-1})} \left[\int_0^\tau (\tau-s)^{\alpha-1} (1-\delta\eta^{\alpha-\theta-1}) ds - \int_0^t (t-s)^{\alpha-1} (1-\delta\eta^{\alpha-\theta-1}) ds \right. \\
&\quad \left. + \int_0^\eta (\tau^{\alpha-1} - t^{\alpha-1}) \left((1-s)^{\alpha-\theta-1} - \delta(\eta-s)^{\alpha-\theta-1} \right) ds + \int_\eta^1 (\tau^{\alpha-1} - t^{\alpha-1}) (1-s)^{\alpha-\theta-1} ds \right] \\
&= \frac{M}{\Gamma(\alpha+1)(1-\delta\eta^{\alpha-\theta-1})} \left[(1-\delta\eta^{\alpha-\theta}) (\tau^{\alpha-1} - t^{\alpha-1}) + (1-\delta\eta^{\alpha-\theta-1}) (\tau^\alpha - t^\alpha) \right]
\end{aligned}$$

and

$$\begin{aligned}
|D^q F_1 v(t) - D^q F_1 v(\tau)| &= \left| \begin{aligned} &D^q I^\alpha f(t, v(t), D^p v(t)) \\ &- \frac{1}{(1-\delta\eta^{\alpha-\theta-1})} \left[\int_0^1 \frac{(1-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} f(s, v(s), D^p v(s)) ds \right. \\ &\quad \left. - \delta \int_0^\eta \frac{(\eta-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} f(s, v(s), D^p v(s)) ds \right] D^q t^{\alpha-1} \\ &\quad - D^q I^\alpha f(\tau, v(\tau), D^p v(\tau)) \\ &+ \frac{1}{(1-\delta\eta^{\alpha-\theta-1})} \left[\int_0^1 \frac{(1-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} f(s, v(s), D^p v(s)) ds \right. \\ &\quad \left. - \delta \int_0^\eta \frac{(\eta-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} f(s, v(s), D^p v(s)) ds \right] D^q \tau^{\alpha-1} \end{aligned} \right| \\
&= \left| \begin{aligned} &I^{\alpha-q} f(t, v(t), D^p v(t)) - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)(1-\delta\eta^{\alpha-\theta-1})} \left[\int_0^1 \frac{(1-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} f(s, v(s), D^p v(s)) ds \right. \\ &\quad \left. - \delta \int_0^\eta \frac{(\eta-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} f(s, v(s), D^p v(s)) ds \right] t^{\alpha-q-1} - I^{\alpha-q} f(\tau, v(\tau), D^p v(\tau)) \\ &\quad + \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)(1-\delta\eta^{\alpha-\theta-1})} \left[\int_0^1 \frac{(1-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} f(s, v(s), D^p v(s)) ds \right. \\ &\quad \left. - \delta \int_0^\eta \frac{(\eta-s)^{\alpha-\theta-1}}{\Gamma(\alpha)} f(s, v(s), D^p v(s)) ds \right] \tau^{\alpha-q-1} \end{aligned} \right| \\
&\leq \left| \int_0^t \frac{(t-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} f(s, v(s), D^p v(s)) ds - \int_0^\tau \frac{(\tau-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} f(s, v(s), D^p v(s)) ds \right| \\
&\quad + \frac{M(\tau^{\alpha-q-1} - t^{\alpha-q-1})}{(1-\delta\eta^{\alpha-\theta-1})} \left| \left(\int_0^1 \frac{(1-s)^{\alpha-\theta-1}}{\Gamma(\alpha-q)} ds - \delta \int_0^\eta \frac{(\eta-s)^{\alpha-\theta-1}}{\Gamma(\alpha-q)} ds \right) \right| \\
&= \left| \int_0^t \frac{(t-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} f(s, v(s), D^p v(s)) ds - \int_0^\tau \frac{(\tau-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} f(s, v(s), D^p v(s)) ds \right. \\
&\quad \left. - \int_t^\tau \frac{(\tau-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} f(s, v(s), D^p v(s)) ds \right| \\
&\quad + \frac{M(\tau^{\alpha-q-1} - t^{\alpha-q-1})}{(1-\delta\eta^{\alpha-\theta-1})} \left| \int_0^1 \frac{(1-s)^{\alpha-\theta-1}}{\Gamma(\alpha-q)} ds - \delta \int_0^\eta \frac{(\eta-s)^{\alpha-\theta-1}}{\Gamma(\alpha-q)} ds \right|
\end{aligned}$$

$$\begin{aligned}
&= \left| \int_0^t \frac{(t-s)^{\alpha-q-1} - (\tau-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} f(s, v(s), D^p v(s)) ds \right. \\
&\quad \left. - \int_t^\tau \frac{(\tau-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} f(s, v(s), D^p v(s)) ds \right| \\
&\quad + \frac{M(\tau^{\alpha-q-1} - t^{\alpha-q-1})}{(1 - \delta\eta^{\alpha-\theta-1})} \left| \int_0^1 \frac{(1-s)^{\alpha-\theta-1}}{\Gamma(\alpha-q)} ds - \delta \int_0^\eta \frac{(\eta-s)^{\alpha-\theta-1}}{\Gamma(\alpha-q)} ds \right| \\
&\leq \frac{M}{\Gamma(\alpha-q)} \left| \int_0^t \left((t-s)^{\alpha-q-1} - (\tau-s)^{\alpha-q-1} \right) ds - \int_t^\tau (\tau-s)^{\alpha-q-1} ds \right| \\
&\quad + \frac{M(\tau^{\alpha-q-1} - t^{\alpha-q-1})}{(1 - \delta\eta^{\alpha-\theta-1}) \Gamma(\alpha-q)} \left| \int_0^1 (1-s)^{\alpha-\theta-1} ds - \delta \int_0^\eta (\eta-s)^{\alpha-\theta-1} ds \right| \\
&= \frac{M}{\Gamma(\alpha-q)} \left| \int_0^t \left((t-s)^{\alpha-q-1} - (\tau-s)^{\alpha-q-1} \right) ds - \int_t^\tau (\tau-s)^{\alpha-q-1} ds \right| \\
&\quad + \frac{M(\tau^{\alpha-q-1} - t^{\alpha-q-1})(1 - \delta\eta^{\alpha-\theta})}{(1 - \delta\eta^{\alpha-\theta-1})(\alpha - \theta) \Gamma(\alpha-q)} \\
&= \frac{M(\tau^{\alpha-q} - t^{\alpha-q})}{\Gamma(\alpha-q+1)} + \frac{M(\tau^{\alpha-q-1} - t^{\alpha-q-1})(1 - \delta\eta^{\alpha-\theta})}{(1 - \delta\eta^{\alpha-\theta-1})(\alpha - \theta) \Gamma(\alpha-q)}
\end{aligned}$$

Analogously, it can be proved that

$$\begin{aligned}
|F_2 u(t) - F_2 u(\tau)| &\leq \frac{N}{\Gamma(\beta+1)(1 - \delta\eta^{\beta-\theta-1})} [(1 - \delta\eta^{\beta-\theta})(\tau^{\beta-1} - t^{\beta-1}) + (1 - \delta\eta^{\beta-\theta-1})(\tau^\beta - t^\beta)] \\
|D^p F_2 u(t) - D^p F_2 u(\tau)| &\leq \frac{N(\tau^{\beta-p} - t^{\beta-p})}{\Gamma(\beta-p+1)} + \frac{N(\tau^{\beta-p-1} - t^{\beta-p-1})(1 - \delta\eta^{\beta-\theta})}{(1 - \delta\eta^{\beta-\theta-1})(\beta - \theta) \Gamma(\beta-p)}
\end{aligned}$$

Since the function t^α , $t^{\alpha-1}$, t^β , $t^{\beta-1}$, $t^{\alpha-q}$, $t^{\alpha-q-1}$, $t^{\beta-p}$, $t^{\beta-p-1}$ are uniformly continuous on J , therefore it follows from the above estimates that FW is an equicontinuous set. Also, it is uniformly bounded as $FW \subset W$. Thus, we conclude that F is a completely continuous operator. Hence, by Schauder's fixed point theorem, there exists a solution for the three-point boundary value problem (1). \square

4. EXAMPLE

In this section, we consider an example to illustrate our results.

Consider the three-point boundary value problem:

$$\begin{aligned}
D^{\frac{3}{2}} u(t) &= a_1 + \left(t - \frac{1}{2}\right)^4 \left[(v(t))^{\rho_1} + \left(D^{\frac{1}{4}} v(t)\right)^{\rho_2} \right], \quad t \in (0, 1), \\
D^{\frac{7}{4}} v(t) &= b_1 + \left(t - \frac{1}{2}\right)^4 \left[(u(t))^{\sigma_1} + \left(D^{\frac{1}{2}} u(t)\right)^{\sigma_2} \right], \quad t \in (0, 1), \\
u(0) &= 0, \quad D^{\frac{4}{10}} u(1) = \frac{3}{4} D^{\frac{4}{10}} u\left(\frac{2}{3}\right), \quad v(0) = 0, \quad D^{\frac{4}{10}} v(1) = \frac{3}{4} D^{\frac{4}{10}} v\left(\frac{2}{3}\right),
\end{aligned} \tag{7}$$

Where $0 < \rho_i, \sigma_i < 1$ ($i = 1, 2$) and a_1, b_1 are constants different from 0. Obviously, it follows by theorem 3 that there exists a solution of the coupled system (7)

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