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THE AMPLITUDE EQUATION FOR THE STOCHASTIC KURAMOTO-SHIVASHINSKY EQUATION

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ABSTRACT. In this paper we derive rigorously amplitude equation of the stochastic Kuramoto-Shivashinsky equation near a change of stability. We discuss the impact of degenerate noise on the dominant behaviour, and see that additive noise has the potential to stabilize the dynamics of the dominant modes.

1. INTRODUCTION

In this article we consider the Kuramoto-Sivashinsky (KS) equation with additive noise forcing near change of stability, where the order of the noise strength is less than the order of the distance from the change of stability. The KS equation dates to the mid-1970s. The first derivation was by Kuramoto in the study of reaction-diffusion equations modelling the Belousov-Zabotinskii reaction. The equation was also developed by Sivashinsky in higher space dimensions in modelling small thermal diffusive instabilities in laminar flamence Poiseuille flow of a film layer on an inclined plane. In one space dimension it is also used as a model for the problem of Benard convection in an elongated box, and it may be used to describe long waves on the interface between two viscous fluids and unstable drift waves in plasmas. The KS equation has some application for instance in the control of surface roughness in the growth of thin solid films by sputtering, step dynamics in epitaxy, the growth of amorphous films, and models in population dynamics [5, 6, 11, 12, 13, 16, 17]. The stochastic Kuramoto-Sivashinsky equation takes the form

$$\partial_t u = -(\partial_x^4 + \partial_x^2)u + \nu\varepsilon^2\partial_x^4 u - u\partial_x u + \varepsilon\partial_t W(t), \tag{1}$$

where $\varepsilon^2\partial_x^4 u$ represents a linear instability term that can destabilize the dominant modes of the equation, and W is finite dimensional noise. The initial condition for (1) is usually taken to satisfy

$$\int_0^L u(x,0)dx = 0 \quad \text{for some } L > 0. \tag{2}$$

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We will work in some Hilbert space $\mathcal{H}$ equipped with scalar product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$. For short, let $\mathcal{A} = -(\partial_x^4 + \partial_x^2)$, $\mathcal{L} = \partial_x^4$ and $B(u,u) = -u\partial_x u$. So, we can rewrite the Equation (1) as follows

$$\partial_t u = \mathcal{A}u + \nu\varepsilon^2\mathcal{L}u + B(u,u) + \varepsilon\partial_t W(t). \tag{3}$$

First aim of this paper is to derive rigorously an amplitude equation of (3) in this form

$$\begin{aligned}\partial_T b(T) &= \mathcal{L}_c b(T) + \mathcal{F}(b) + \frac{2\alpha_k^2}{\lambda_k^2}B_c(B_c(b,e_k),e_k) - \frac{\alpha_k^2}{\lambda_k}B_c(b,\mathcal{A}_s^{-1}B_s(e_k,e_k)) \\ &\quad + \sum_{\ell\neq k}\frac{\alpha_k^2}{\lambda_k(\lambda_k+\lambda_\ell)}B_c(B_\ell(b,e_k)e_\ell,e_k) + \frac{2\alpha_k}{\lambda_k}B_c(b,e_k)\partial_T\tilde{\beta}_k, \end{aligned}\tag{4}$$

where

$$\mathcal{F}(b) = -2B_c(b,\mathcal{A}_s^{-1}B_s(b,b)), \tag{5}$$

and show that near a change of stability on a time-scale of order ε^{-2} the solution of (3) is of the type

$$u(t) \simeq \varepsilon b(\varepsilon^2 t) + \varepsilon\mathcal{Z}_k(\varepsilon^2 t)e_k + \mathcal{O}(\varepsilon^{2-})\ , \tag{6}$$

where b is the solution of the amplitude equation (4) and $\mathcal{Z}_k(T)$ is a fast real-valued Ornstein-Uhlenbeck process (OU, for short) defined by

$$\mathcal{Z}_k(T) := \alpha_k\varepsilon^{-1}\int_0^T e^{-\varepsilon^{-2}\lambda_k(T-s)}d\tilde{\beta}_k(s), \tag{7}$$

Second aim of this paper is to investigate whether additive degenerate noise (i.e. noise that does not act directly to the dominant mode) can lead to stabilization of the solution of (3).

Near a change of stability, we can depend on the natural separation of time-scales, in order to derive simpler equations for the evolution of the dominant mode. As these equations describe the amplitudes of dominant pattern, they are referred to as amplitude equations. When the order of the noise strength is comparable to the order of the distance from the change of stability, the degenerate additive noise is transported via nonlinear interaction to the dominant pattern. Examples are [2, 3, 7, 8, 9, 14, 15].

The rest of this paper is organized as follows. In Section 2 we state the assumptions that we make, while in section 3 give the formal derivation of the amplitude equation and we state the main theorem. In Section 4 we give bounds for high modes. In Section 5 we give averaging over the fast OU-process. In Section 6 we give the proof of the main results for the first order estimate. Finally, we study the Kuramoto-Sivashinsky equation in one dimension with either Dirichlet or periodic boundary conditions.

2. Assumptions

This section summarizes all assumptions necessary for our results. For the linear operator $\mathcal{A}$ in (3) we assume the following:

Assumption 1. *(Linear operator $\mathcal{A}$) Suppose $\mathcal{A}$ is a non-positive operator on $\mathcal{H}$ with eigenvalues $0\leq\lambda_1\leq....\leq\lambda_k\leq....$and $\lambda_k \geq Ck^m$ for all sufficiently large k, and a complete orthonormal system of eigenvectors $\{e_k\}_{k=1}^{\infty}$ such that $\mathcal{A}e_k = -\lambda_k e_k$. Suppose that $\mathcal{N} := \ker\mathcal{A}$ has finite dimension n with basis $(e_1,....,e_n)$. Define $S =$*

$\mathcal{N}^{\perp}$ the orthogonal complement of $\mathcal{N}$ in $\mathcal{H}$, and P_c for the projection $P_c : \mathcal{H} \to \mathcal{N}$ and define $P_s := \mathcal{I} - P_c$ where $\mathcal{I}$ is the identity operator on $\mathcal{H}$

As the dimension of $\mathcal{N}$ is finite, it is well known that both P_c and P_s are bounded linear operators on $\mathcal{H}$ (cf. Weidmann[18]).

Definition 2. *For $\alpha \in \mathbb{R}$, we define the space $\mathcal{H}^\alpha$ as*

$$\mathcal{H}^\alpha = \left\{ \sum_{k=0}^{\infty} \gamma_k e_k : \sum_{k=1}^{\infty} \gamma_k^2 k^{2\alpha} < \infty \right\} \quad \text{with norm} \ \Big\| \sum_{k=1}^{\infty} \gamma_k e_k \Big\|_\alpha^2 = \sum_{k=0}^{\infty} \gamma_k^2 k^{2\alpha}.$$

The operator $\mathcal{A}$ given by Assumption 1 generates an analytic semigroup $\{e^{t\mathcal{A}}\}_{t\geq 0}$ defined by

$$e^{\mathcal{A}t}\Big(\sum_{k=1}^{\infty} \gamma_k e_k \Big) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \gamma_k e_k \quad \forall\, t \geq 0,$$

and has the following property for all $t > 0$, $\beta \geq \alpha$, $\lambda_n < \omega \leq \lambda_{n+1}$and all $u \in \mathcal{H}^\beta$

$$\left\| e^{t\mathcal{A}} P_s u \right\|_\alpha \leq M t^{-\frac{\alpha-\beta}{m}} e^{-\omega t} \left\| P_s u \right\|_\beta \ , \tag{8}$$

where M depends only on α, β and ω.

For the operator $\tilde{\mathcal{L}}$, will be defined later in Lemma 9, we assume that:

Assumption 3. *(Operator $\tilde{\mathcal{L}}$) Let $\tilde{\mathcal{L}} : \mathcal{H}^\alpha \to \mathcal{H}^{\alpha-\beta}$ for some $\beta \in [0, m)$ be a linear continuous mapping that commutes with P_c and P_s.*

Assumption 4. ***(Bilinear Operator** B) With α, β from Assumption 3 let B be a bounded bilinear mapping from $\mathcal{H}^\alpha \times \mathcal{H}^\alpha$ to $\mathcal{H}^{\alpha-\beta}$. Suppose without loss of generality that B is symmetric, i.e. $B(u, v) = B(v, u)$, and satisfies $P_c B(u, u) = 0$ for $u \in \mathcal{H}$.*

For the noise we suppose:

Assumption 5. *Let W be a cylindrical Wiener process on $\mathcal{H}$. Suppose for $t \geq 0$,*

$$W(t) = \alpha_k \beta_k(t) e_k \ \ \text{for one } k \text{ in } \{n+1, n+2.....\},$$

where $(\beta_k)_k$ are independent, standard Brownian motions in $\mathbb{R}$ and $(\alpha_k)_k$ are real numbers.

For our result we rely on a cut off argument. We consider only solutions (a, ψ) that are not too large, as given by the next definition.

Definition 6. *For the $\mathcal{N} \times S$-valued stochastic process (a, ψ) defined later in (11) we define, for some $T_0 > 0$ and $\kappa \in (0, \frac{1}{7})$, the stopping time τ^* as*

$$\tau^* := T_0 \wedge \inf \left\{ T > 0 : \| a(T) \|_\alpha > \varepsilon^{-\kappa} \ \text{or} \ \| \psi(T) \|_\alpha > \varepsilon^{-\kappa} \right\}. \tag{9}$$

Definition 7. *For a real-valued family of processes $\{X_\varepsilon(t)\}_{t\geq 0}$ we say $X_\varepsilon = \mathcal{O}(f_\varepsilon)$, if for every $p \geq 1$ there exists a constant C_p such that*

$$\mathbb{E} \sup_{t\in[0,\tau^*]} |X_\varepsilon(t)|^p \leq C_p f_\varepsilon^p. \tag{10}$$

We use also the analogous notation for time-independent random variables.

3. Formal Derivation and Main Result

In this section we present a short formal derivation of the main result We interest here the studying behavior of solution to (3) on time-scales of order ε^{-2}. So, we split the solution u into

$$u(t) = \varepsilon a(\varepsilon^2 t) + \varepsilon\psi(\varepsilon^2 t) \tag{11}$$

where $a \in \mathcal{N}$ and $\psi \in S$. After rescaling to the slow time-scale $T = \varepsilon^2 t$, we obtain the following system of equations:

$$da = \left[\mathcal{L}_c a + 2\varepsilon^{-1} B_c(a,\psi) + \varepsilon^{-1} B_c(\psi,\psi)\right] dT, \tag{12}$$

and

$$d\psi = \left[\varepsilon^{-2}\mathcal{A}_s\psi + \mathcal{L}_s\psi + \varepsilon^{-1} B_s(a+\psi, a+\psi)\right] dT + \varepsilon^{-1}\alpha_k d\beta_k(\tilde{T})e_k \ , \tag{13}$$

where $\tilde{\beta_k}(T) := \varepsilon\beta_k(\varepsilon^{-2}T)$ is a rescaled version of the Brownian motion. Integrating (12) from 0 to T, yields

$$a(T) = a(0) + \int_0^T \mathcal{L}_c a(\tau)d\tau + 2\varepsilon^{-1}\int_0^T B_c(a,\psi)d\tau + \varepsilon^{-1}\int_0^T B_c(\psi,\psi)d\tau. \tag{14}$$

In order to remove ε^{-1} from the front of last two terms in the above equation, we Apply Itô's formula to $B_c(a, \mathcal{A}_s^{-1}\psi)$ and $B_c(\psi_k e_k, \psi_\ell e_\ell)$ to obtain, respectively,

$$\begin{aligned}
\varepsilon^{-1}\int_0^T B_c(a,\psi)d\tau &= \varepsilon B_c(a(T), \mathcal{A}_s^{-1}\psi(T)) - \varepsilon\int_0^T B_c(\mathcal{L}_c a, \mathcal{A}_s^{-1}\psi)d\tau \\
&\quad -\varepsilon\int_0^T B_c(a, \mathcal{A}_s^{-1}\mathcal{L}_c\psi)d\tau - \int_0^T B_c(a, \mathcal{A}_s^{-1}B_s(a, a+2\psi))d\tau \\
&\quad -\int_0^T B_c(a, \mathcal{A}_s^{-1}d\tilde{W}_s) - \int_0^T B_c(a, \mathcal{A}_s^{-1}B_s(\psi,\psi))d\tau \\
&\quad -\int_0^T B_c(B_c(2a+\psi,\psi), \mathcal{A}_s^{-1}\psi)d\tau
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
\varepsilon^{-1}\int_0^T B_c(\psi,\psi)d\tau &= \varepsilon\sum_{\ell,k}\frac{1}{(\lambda_k+\lambda_\ell)}\int_0^T B_c(\psi_k e_k, \psi_\ell e_\ell)d\tau \\
&\quad +\varepsilon\sum_{\ell,k}\frac{k^4+\ell^4}{(\lambda_k+\lambda_\ell)}\int_0^T B_c(\psi_k e_k, \psi_\ell e_\ell)d\tau \\
&\quad +\sum_{\ell,k}\frac{2}{(\lambda_k+\lambda_\ell)}\int_0^T B_c(B_\ell(a+\psi, a+\psi)e_\ell, \psi_k e_k)d\tau \\
&\quad +\sum_{\ell,k}\frac{2\alpha_k}{(\lambda_k+\lambda_\ell)}\int_0^T B_c(\psi_\ell e_\ell, e_k)d\tilde{\beta}_k.
\end{aligned} \tag{16}$$

Substituting from (15) and (16) into (14) we obtain

$$\begin{aligned} a(T) &= a(0) + \int_0^T \mathcal{L}_c a d\tau + \int_0^T \mathcal{F}(a) d\tau - 2\int_0^T B_c(B_c(\psi,\psi), \mathcal{A}_s^{-1}\psi) d\tau \\ &\quad -4\int_0^T B_c(B_c(a,\psi), \mathcal{A}_s^{-1}\psi) d\tau - 4\int_0^T B_c(a, \mathcal{A}_s^{-1} B_s(a,\psi)) d\tau \\ &\quad -2\int_0^T B_c(a, \mathcal{A}_s^{-1} B_s(\psi,\psi)) d\tau - 2\int_0^T B_c(a, \mathcal{A}_s^{-1} d\tilde{W}_s) \\ &\quad + \sum_{\ell,k} \frac{2}{(\lambda_k + \lambda_\ell)} \int_0^T B_c(B_\ell(a+\psi, a+\psi) e_\ell, \psi_k e_k) d\tau \\ &\quad + \sum_{\ell,k} \frac{2\alpha_k}{(\lambda_k + \lambda_\ell)} \int_0^T B_c(\psi_\ell e_\ell, e_k) d\tilde{\beta}_k + \mathcal{O}(\varepsilon^{1-2\kappa}). \end{aligned} \tag{17}$$

where the cubic term $\mathcal{F}(a)$ is defined in (5).

To illustrate our approximation result of Theorem 8 here, we consider the stochastic Kuramoto-Shivashinsky equation (1) with Dirichlet boundary condition on $[0, \pi]$ and forced by noise acting on second mode. In this case our main theorem states that the solution (1) of is given by

$$u(t, x) = \varepsilon b(\varepsilon^2 t) \sin(x) + \varepsilon \mathcal{Z}_2(\varepsilon^2 t) \sin(2x) + \mathcal{O}(\varepsilon^{2-}) \ ,$$

where b is the solution of the amplitude equation of Stratonovic type

$$\partial_T b = (\nu - \frac{\sigma^2}{2688}) b - \tfrac{1}{48} b^3 + \frac{\sigma}{24} b \circ \partial_T \tilde{\beta}_2 \ ,$$

with a rescaled standard Brownian motion $\tilde{\beta}_2$.

The main result is:

Theorem 8. *(Approximation) Under Assumptions 1, 3, 4 and 5 let u be a solution of (3) defined in (11) with the initial condition $u(0) = \varepsilon a(0) + \varepsilon\psi(0)$ with $a(0) \in \mathcal{N}$ and $\psi(0) \in S$ where $a(0)$ and $\psi(0)$ are of order one, and b is a solution of (4) with $b(0) = a(0)$. Then for all $p > 1$ and $T_0 > 0$ and all $\kappa \in (0, \frac{1}{7})$, there exists $C > 0$ such that*

$$\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]} \Big\| u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon \mathcal{Q}(\varepsilon^2 t) \Big\|_\alpha > \varepsilon^{2-8\kappa} \Big) \le C\varepsilon^p, \tag{18}$$

where

$$\mathcal{Q}(T) = e^{\varepsilon^{-2} T \mathcal{A}_s} \psi(0) + \mathcal{Z}_k(T) e_k, \tag{19}$$

with $\mathcal{Z}_k$ defined in (7). We see that the first part of (19) decays exponentially fast on time-scale $\mathcal{O}(\varepsilon^2)$ and the second part of (19) is small noise (see above).

4. Bounds for the high modes

In next lemma , we will approximate ψ by the fast Ornstein-Uhlenbeck process $\mathcal{Z}$ as follows

Lemma 9. *Under Assumption 1, 3 and 4, there is a constant $C > 0$ such that, for $\kappa > 0$ from the definition of τ^* and $p \ge 1$,*

$$\mathbb{E} \sup_{T\in[0,\tau^*]} \Big\| \psi(T) - \mathcal{Q}(T) \Big\|_\alpha^p \le C\varepsilon^{p-2p\kappa}, \tag{20}$$

where $\mathcal{Q}(T)$ is defined in (19).

Proof. Define

$$\mathcal{Z}(T) := \mathcal{Z}_k(T)e_k, \tag{21}$$

where $\mathcal{Z}_k(T)$ is defined in (7). The mild solution of (13) is

$$\psi(T) = e^{\varepsilon^{-2}T\mathcal{A}_s}\psi(0) + \int_0^T e^{\varepsilon^{-2}(T-\tau)\mathcal{A}_s}[\mathcal{L}_s\psi + \varepsilon^{-1}B_s(a+\psi, a+\psi)]\,(\tau)\,d\tau + \mathcal{Z}(T).$$

Let $\tilde{\mathcal{A}} = \varepsilon^{-2}\mathcal{A}_s$ and $\tilde{\mathcal{L}}_s = P_s(\partial_x^2)$. So, we can rewrite the above equation as follows

$$\begin{aligned}\psi(T) - \mathcal{Q}(T) &= -\varepsilon^2 \int_0^T e^{(T-\tau)\tilde{\mathcal{A}}}\tilde{\mathcal{A}}\psi d\tau - \int_0^T e^{(T-\tau)\tilde{\mathcal{A}}}\tilde{\mathcal{L}}_s\psi d\tau \\ &\quad + \int_0^T e^{(T-\tau)\tilde{\mathcal{A}}}B_s(a+\psi, a+\psi)d\tau \\ &= \varepsilon^2(e^{T\tilde{\mathcal{A}}}\psi - \psi) - \int_0^T e^{(T-\tau)\tilde{\mathcal{A}}}\tilde{\mathcal{L}}_s\psi d\tau \\ &\quad + \varepsilon^{-1}\int_0^T e^{(T-\tau)\tilde{\mathcal{A}}}B_s(a+\psi, a+\psi)d\tau.\end{aligned}$$

Taking the α-norm of both sides and using triangle inequality to obtain

$$\begin{aligned}\Big\|\psi(T) - \mathcal{Q}(T)\Big\|_\alpha &\leq \varepsilon^2\Big\|e^{T\tilde{\mathcal{A}}}\psi\Big\|_\alpha + \varepsilon^2\Big\|\psi\Big\|_\alpha \\ &\quad \Big\|\int_0^T e^{(T-\tau)\tilde{\mathcal{A}}}\tilde{\mathcal{L}}_s\psi d\tau\Big\|_\alpha \\ &\quad + \varepsilon^{-1}\Big\|\int_0^T e^{(T-\tau)\tilde{\mathcal{A}}}B_s(a+\psi, a+\psi)d\tau\Big\|_\alpha \\ &\leq \varepsilon^{2-\kappa} + I_1 + I_2,\end{aligned}$$

where we used $\Big\|e^{T\tilde{\mathcal{A}}}\psi\Big\|_\alpha \leq \Big\|\psi\Big\|_\alpha$ and the definition of τ^*. To bound the second term, we obtain by using (8) and Assumption 3

$$\begin{aligned}I_1 &\leq C\varepsilon^{\frac{2\beta}{m}}\int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\Big\|\tilde{\mathcal{L}}_s\psi(\tau)\Big\|_{\alpha-\beta}d\tau \\ &\leq C\varepsilon^{\frac{2\beta}{m}}\int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\|\psi(\tau)\|_\alpha d\tau \\ &\leq C\varepsilon^2\sup_{\tau\in[0,\tau^*]}\|\psi(\tau)\|_\alpha\int_0^{\varepsilon^{-2}\omega T} e^{-\eta}\eta^{-\frac{\beta}{m}}d\eta \leq C\varepsilon^{2-\kappa},\end{aligned}$$

where we used the definition of τ^*. For the third term, we obtain

$$\begin{aligned} I_2 &\leq C\varepsilon^{\frac{2\beta}{m}-1}\int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\left\|B_s(a(\tau)+\psi(\tau))\right\|_{\alpha-\beta}d\tau \\ &\leq C\varepsilon^{\frac{2\beta}{m}-1}\int_0^T e^{-\varepsilon^{-2}\omega(T-\tau)}(T-\tau)^{-\frac{\beta}{m}}\left\|a(\tau)+\psi(\tau)\right\|_{\alpha}^2 d\tau \\ &\leq C\varepsilon\sup_{\tau\in[0,\tau^*]}\left\|a(\tau)+\psi(\tau)\right\|_{\alpha}^2\int_0^{\varepsilon^{-2}\omega T}e^{-\eta}\eta^{-\frac{\beta}{m}}d\eta \\ &\leq C\varepsilon\Big(\sup_{[0,\tau^*]}\|a\|_\alpha^2+\sup_{[0,\tau^*]}\|\psi\|_\alpha^2\Big) \\ &\leq C\varepsilon^{1-2\kappa}, \end{aligned}$$

where we used again the definition of τ^*. Combining all results, yields (20). □

Lemma 10. *Under Assumption 1 and 5, for every $\kappa_0 > 0$ and $p \geq 1$, there is a constant C, depending on p, α_k, λ_k, κ_0 and T_0, such that*

$$\mathbb{E}\sup_{T\in[0,T_0]}|\mathcal{Z}_k(T)|^p \leq C\varepsilon^{-\kappa_0},$$

where $\mathcal{Z}_k(T)$ is defined in (7).

Proof. See the first part of the proof of Lemma 14 in [3] with $\lambda_k = 1$. □

The following Corollary states that $\psi(T)$ is with high probability much smaller than $\varepsilon^{-\kappa}$ as asserted by the Definition 6 for $T \leq \tau^*$. We will show later $\tau^* \geq T_0$ with high probability (cf. Proof of Theorem 8).

Corollary 11. *Under the assumptions of Lemmas 9 and 10, if $\psi(0) = \mathcal{O}(1)$, then for $p > 0$ and for all $\kappa_0 > 0$ there exist a constant $C > 0$ such that*

$$\mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\|\psi(T)\|_\alpha^p\Big) \leq C\varepsilon^{-\kappa_0}\ . \tag{22}$$

Proof. From (20), by triangle inequality and Lemma 10, we obtain

$$\mathbb{E}\Big(\sup_{T\in[0,\tau^*]}\|\psi(T)\|_\alpha^p\Big) \leq C + C\varepsilon^{-\kappa_0} + C\varepsilon^{p-2p\kappa},$$

for $\kappa < \frac{2}{3}$ we obtain (22). □

Lemma 12. *If Assumption 1 hold and $\psi(0) = \mathcal{O}(1)$, then for $q \geq 1$ there exist a constant $C > 0$ such that*

$$\int_0^T \left\|e^{\tau\varepsilon^{-2}\mathcal{A}_s}\psi(0)\right\|_\alpha^q d\tau \leq C\varepsilon^2.$$

Proof. Using (8) we obtain

$$\int_0^T \left\|e^{\varepsilon^{-2}\mathcal{A}_s\tau}\psi(0)\right\|_\alpha^q d\tau \leq c\int_0^T e^{-q\varepsilon^{-2}\omega\tau}\|\psi(0)\|_\alpha^q d\tau \leq \frac{\varepsilon^2}{q\omega}\|\psi(0)\|_\alpha^q\,.$$

Hence,

$$\mathbb{E}\sup_{T\in[0,T_0]}\int_0^T \left\|e^{\varepsilon^{-2}\mathcal{A}_s\tau}\psi(0)\right\|_\alpha^q d\tau \leq C\varepsilon^2.$$

□

5. Averaging over the fast OU-process

Lemma 13. *Let X be a real valued stochastic process and $\|X(0)\|_\infty = \mathcal{O}(\varepsilon^{-r})$ for $r \geq 0$. If $dX = GdT$, with $\|G\|_\infty = \mathcal{O}(\varepsilon^{-r})$, then, for $\kappa_0 \in (0,1)$,*

$$\int_0^T X\mathcal{Z}_k^n d\tau = \frac{(n-1)\sigma^2}{2}\int_0^T X\mathcal{Z}_k^{n-2}d\tau + \mathcal{O}(\varepsilon^{1-r-n\kappa_0}), \tag{23}$$

and we can write (23) with higher order correction as

$$\int_0^T X\mathcal{Z}_k^n d\tau = \frac{(n-1)\sigma^2}{2}\int_0^T X\mathcal{Z}_k^{n-2}d\tau + \varepsilon\sigma\int_0^T X\mathcal{Z}_k^{n-1}d\tilde{\beta} + \mathcal{O}(\varepsilon^{2-r-n\kappa_0}). \tag{24}$$

Proof. We note first that

$$\begin{aligned}\mathbb{E}\sup_{[0,T_0]}|X|^p &\leq C\mathbb{E}\,|X(0)|^p + C\mathbb{E}\sup_{[0,T_0]}\left|\int_0^T Gd\tau\right|^p \\ &\leq C\varepsilon^{-pr},\end{aligned}$$

Applying Itô formula to $X\mathcal{Z}_k^n$ and integrating from 0 to T in order to obtain

$$\begin{aligned}\int_0^T X\mathcal{Z}_k^n d\tau &= \frac{(n-1)\sigma^2}{2}\int_0^T X\mathcal{Z}_k^{n-2}d\tau + \varepsilon\sigma\int_0^T X\mathcal{Z}_k^{n-1}d\tilde{\beta} \\ &\quad -\frac{\varepsilon^2}{n}X(T)\mathcal{Z}_k^n(T) + \frac{\varepsilon^2}{n}\int_0^T G\mathcal{Z}_k^n d\tau.\end{aligned}$$

To prove the first part, taking the absolute value and using Burkholder-Davis-Gundy theorem yields (23). For the second part, we leave the first and the second terms on the right hand side and bound the other terms to obtain (24). □

Remark 14. *The above Lemma is true, even if X is a stochastic process in $\mathcal{N}$ or $\mathbb{C}$.*

6. Proof of the main result

This section is devoted to the proof of Theorem 8 for the approximation of the solution (6) of the SPDE (3). Let us first derive the amplitude equation of the Equation (3) with error.

Lemma 15. *If Assumptions 1, 4 and 5 hold and $\psi(0) = \mathcal{O}(1)$, then*

$$\begin{aligned}a(T) &= a(0) + \int_0^T \mathcal{L}_c a(\tau)d\tau + \int_0^T \mathcal{F}(a(\tau))d\tau + \frac{2\alpha_k^2}{\lambda_k^2}\int_0^T B_c(B_c(a,e_k),e_k)d\tau \\ &\quad -\frac{\alpha_k^2}{\lambda_k}\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(e_k,e_k))d\tau + \sum_{\ell\neq k}\frac{\alpha_k^2}{\lambda_k(\lambda_k+\lambda_\ell)}B_c(B_\ell(a,e_k)e_\ell,e_k) \\ &\quad +\frac{2\alpha_k}{\lambda_k}\int_0^T B_c(a,e_k)d\tilde{\beta}_k + R(T),\end{aligned} \tag{25}$$

where

$$R = \mathcal{O}(\varepsilon^{1-6\kappa}), \tag{26}$$

for $\kappa > 0$ from the definition of τ^*.

Proof. From the mild solution of equation (13) and Lemma 9 we obtain

$$\psi(T) = y_\varepsilon(T) + \mathcal{Z}(T) + \mathcal{O}(\varepsilon^{1-3\kappa}), \tag{27}$$

where

$$y_\varepsilon(T) = e^{\varepsilon^{-2}T\mathcal{A}_s^{-1}}\psi(0).$$

Substituting from (27) into (17) and using Assumption 4 to obtain

$$\begin{aligned} a(T) &= a(0) + \int_0^T \mathcal{L}_c a d\tau + \int_0^T \mathcal{F}(a)d\tau - 4\int_0^T \mathcal{Z}_k^2 B_c(B_c(a,e_k),\mathcal{A}_s^{-1}e_k)d\tau \\ &\quad -4\int_0^T \mathcal{Z}_k B_c(a,\mathcal{A}_s^{-1}B_s(a,e_k))d\tau - 2\int_0^T \mathcal{Z}_k^2 B_c(a,\mathcal{A}_s^{-1}B_s(e_k,e_k))d\tau \\ &\quad -2\int_0^T B_c(a,\mathcal{A}_s^{-1}d\tilde{W}_s) + \sum_\ell \frac{2}{(\lambda_k+\lambda_\ell)}\int_0^T \mathcal{Z}_k B_c(B_\ell(a,a)e_\ell,e_k)d\tau \\ &\quad +\sum_\ell \frac{4}{(\lambda_k+\lambda_\ell)}\int_0^T \mathcal{Z}_k^2 B_c(B_\ell(a,e_k)e_\ell,e_k)d\tau \\ &\quad +\sum_\ell \frac{2}{(\lambda_k+\lambda_\ell)}\int_0^T \mathcal{Z}_k^3 B_c(B_\ell(e_k,e_k)e_\ell,e_k)d\tau \\ &\quad +\sum_\ell \frac{2\alpha_k}{(\lambda_k+\lambda_\ell)}\int_0^T \mathcal{Z}_\ell B_c(e_\ell,e_k)d\tilde{\beta}_k + R_1, \end{aligned} \tag{28}$$

where

$$\begin{aligned} R_1 &= 2\varepsilon B_c(a(T),\mathcal{A}_s^{-1}\psi(T)) - 2\varepsilon\int_0^T B_c(\mathcal{L}_c a,\mathcal{A}_s^{-1}\psi)d\tau \\ &\quad -2\varepsilon\int_0^T B_c(a,\mathcal{A}_s^{-1}L_s\psi)d\tau - 4\int_0^T B_c(B_c(\mathcal{Z},y_\varepsilon),\mathcal{A}_s^{-1}\mathcal{Z})d\tau \\ &\quad -4\int_0^T B_c(B_c(a,y_\varepsilon),\mathcal{A}_s^{-1}\mathcal{Z})d\tau - 4\int_0^T B_c(B_c(a,\mathcal{Z}),\mathcal{A}_s^{-1}y_\varepsilon)d\tau \\ &\quad -4\int_0^T B_c(B_c(a,y_\varepsilon),\mathcal{A}_s^{-1}y_\varepsilon)d\tau - 4\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(\mathcal{Z},y_\varepsilon))d\tau \\ &\quad -2\int_0^T B_c(a,\mathcal{A}_s^{-1}B_s(y_\varepsilon,y_\varepsilon))d\tau + \varepsilon^{-1}\int_0^T B_c(2\mathcal{Z}+y_\varepsilon,y_\varepsilon)d\tau \\ &\quad +\sum_{\ell,k}\frac{2\alpha_k}{(\lambda_k+\lambda_\ell)}\int_0^T B_c(B_\ell(\langle y_\varepsilon,e_\ell\rangle e_\ell,e_k)d\tilde{\beta}_k + \mathcal{O}(\varepsilon^{1-6\kappa}). \end{aligned} \tag{29}$$

It is easy to prove that

$$R_1 = \mathcal{O}(\varepsilon^{1-6\kappa}). \tag{30}$$

We note that

$$\sum_\ell \frac{2\alpha_k}{(\lambda_k+\lambda_\ell)}\int_0^T \mathcal{Z}_\ell B_c(e_\ell,e_k)d\tilde{\beta}_k = 0,$$

because $\mathcal{Z}_\ell = \alpha_\ell\varepsilon^{-1}\int_0^T e^{-\varepsilon^{-2}\lambda_\ell(T-s)}d\tilde{\beta}_\ell(s)$ contains on α_ℓ. So, the above term contains α_ℓ and α_k in the same time and one of them equal zero. Using Assumption 3, the definition of τ^*, Lemma 12 and the equivalence of $\mathcal{H}^\alpha$-norms on $\mathcal{N}$. Applying finally Lemma 13 to (28), we obtain (25). □

Lemma 16. *Let Assumptions 1 and 4, hold. Define $b(t)$ in $\mathcal{N}$ as the solution of (4). If the initial condition satisfies $\mathbb{E}\,|b(0)|^p \leq C$ for some $p > 1$, then for all $T_0 > 0$ there exists another constant C such that*

$$\mathbb{E} \sup_{T\in[0,T_0]} |b(T)|^{2p} \leq C. \tag{31}$$

Proof. Applying Ito's formula to $|b(T)|^{2p}$ to get

$$\begin{aligned} |b(T)|^{2p} &= |b(0)|^{2p} + 2p\int_0^T |b(T)|^{2(p-1)} \langle b(s), db(s)\rangle \\ &\quad + p\int_0^T |b(T)|^{2(p-1)} \langle db(s), db(s)\rangle \\ &\quad + 2p(p-1)\int_0^T |b(T)|^{2(p-2)} \langle b(s), db(s)\rangle^2 . \end{aligned}$$

From (4) we have

$$\begin{aligned} |b(T)|^{2p} &= |b(0)|^{2p} + 2p\int_0^T |b(s)|^{2p-2} \langle b(s), \mathcal{L}_c b(T) + \mathcal{F}(b(s))\rangle\, ds \\ &\quad + C_1\int_0^T |b(s)|^{2p-2} \langle b, B_c(B_c(b,e_k),e_k)\rangle\, ds \\ &\quad - C_2\int_0^T |b(s)|^{2p-2} \left\langle b, B_c(b, \mathcal{A}_s^{-1}B_s(e_k,e_k))\right\rangle ds \\ &\quad + \sum_{\ell\neq k}\frac{2p\alpha_k^2}{\lambda_k(\lambda_k+\lambda_\ell)}\int_0^T |b(s)|^{2p-2} \langle b, B_c(B_\ell(b,e_k)e_\ell,e_k)\rangle\, ds \\ &\quad + C_3\int_0^T |b(s)|^{2p-2} \langle b, B_c(b,e_k)\rangle\, d\tilde{\beta}_k \\ &\quad + C_4\int_0^T |b(s)|^{2p-2} \langle B_c(b,e_k), B_c(b,e_k)\rangle\, ds \\ &\quad + C_5\int_0^T |b(s)|^{2p-4} \langle b, B_c(b,e_k)\rangle^2\, ds. \end{aligned}$$

Using Cauchy-Schwarz inequality and Assumption 4, we obtain

$$\begin{aligned} |b(T)|^{2p} &= |b(0)|^{2p} + C\int_0^T |b(s)|^{2p}\, ds - 2p\int_0^T |b(s)|^{2p+2}\, ds \\ &\quad + C\int_0^T |b(s)|^{2p+1}\, ds + C\int_0^T |b(s)|^{2p-1}\, ds \\ &\quad + C_5\int_0^T |b(s)|^{2p-2} \langle b, B_c(b,e_k)\rangle\, d\tilde{\beta}_k. \end{aligned}$$

If we use the inequality $|b(T)|^q \leq \delta |b(T)|^{2p+2} + C_{\delta,q,p}$ for $q \in (0, 2p+2)$, then

$$\begin{aligned} |b(T)|^{2p} &= |b(0)|^{2p} + C_{\delta,q,p}T + C\int_0^T |b(s)|^{2p}\, ds - \tilde{C}\int_0^T |b(s)|^{2p+2}\, ds \\ &\quad + C_6\int_0^T |b(s)|^{2p-2} \langle b, B_c(b, e_k)\rangle\, d\tilde{\beta}_k \\ &\leq |b(0)|^{2p} + C_{\delta,q,p}T + C\int_0^T |b(s)|^{2p}\, ds \\ &\quad + C_5\int_0^T |b(s)|^{2p-2} \langle b, B_c(b, e_k)\rangle\, d\tilde{\beta}_k. \end{aligned} \tag{32}$$

Taking the expectations on both sides, yields

$$\mathbb{E}\,|b(T)|^{2p} \leq C + C\int_0^T \mathbb{E}\,|b(s)|^{2p}\, ds,$$

where we used $\mathbb{E}\int_0^T |b(s)|^{2p-2} \langle b, B_c(b, e_k)\rangle\, d\tilde{\beta}_k = 0$. Applying now Gronwall's lemma to obtain

$$\mathbb{E}\,|b(T)|^{2p} \leq C. \tag{33}$$

Taking expectation after supremum on both sides of (32)

$$\begin{aligned} \mathbb{E}\sup_{T\in[0,T_0]} |b(T)|^{2p} &\leq \mathbb{E}\,|b(0)|^{2p} + C_{\delta,q,p}T_0 + C\mathbb{E}\sup_{T\in[0,T_0]}\int_0^T |b(s)|^{2p}\, ds \\ &\quad + C_5\mathbb{E}\sup_{T\in[0,T_0]}\int_0^T |b(s)|^{2p-2}\langle b, B_c(b, e_k)\rangle\, d\tilde{\beta}_k. \end{aligned}$$

Using Burkholder-Davis-Gundy inequality (cf. Theorem A.7 in [1])

$$\mathbb{E}\sup_{T\in[0,T_0]} |b(T)|^{2p} \leq C + C\int_0^{T_0} \mathbb{E}\,|b(s)|^{2p}\, ds + C_5\mathbb{E}\left(\int_0^{T_0} |b(s)|^{4p}\, ds\right)^{1/2}.$$

Using our first bound (33) on b, yields (31). □

Theorem 17. *Assume that Assumptions 1, 4 and 5 hold and suppose $a(0) = \mathcal{O}(1)$ and $\psi(0) = \mathcal{O}(1)$. Let b be a solution of (4) and a as defined in (25). If the initial condition satisfies $a(0) = b(0)$, then for $\kappa < \frac{1}{7}$*

$$\mathbb{E}\sup_{T\in[0,\tau^*]} |a(T) - b(T)| \leq C\varepsilon^{1-7\kappa}. \tag{34}$$

Proof. Subtracting (4) from (25) and defining $h(T) := a(T) - b(T)$, we obtain

$$\begin{aligned} h(T) &= \int_0^T \mathcal{L}_c h(\tau)d\tau + \int_0^T \mathcal{F}(h)d\tau + 3\int_0^T \mathcal{F}(h,h,b)d\tau + 3\int_0^T \mathcal{F}(h,b,b)d\tau \\ &\quad + \frac{2\alpha_k^2}{\lambda_k^2}\int_0^T B_c(B_c(h,e_k),e_k)d\tau + \sum_{\ell\neq k}\frac{\alpha_k^2}{\lambda_k(\lambda_k+\lambda_\ell)}\int_0^T B_c(B_\ell(h,e_k)e_\ell,e_k)d\tau \\ &\quad - \frac{\alpha_k^2}{\lambda_k}\int_0^T B_c(h,\mathcal{A}_s^{-1}B_s(e_k,e_k))d\tau + \frac{2\alpha_k}{\lambda_k}\int_0^T B_c(h,e_k)d\tilde{\beta}_k + R(T), \end{aligned}$$

Taking $|\cdot|^2$ on both sides

$$\begin{aligned}|h(T)|^2 \leq & \int_0^T |\mathcal{L}_c h(\tau)|^2 d\tau + \int_0^T |\mathcal{F}(h)|^2 d\tau + 3\int_0^T |\mathcal{F}(h,h,b)|^2 d\tau + 3\int_0^T |\mathcal{F}(h,b,b)|^2 d\tau \\ & + C\int_0^T |B_c(B_c(h,e_k),e_k)|^2 d\tau + \sum_{\ell\neq k} \frac{\alpha_k^2}{\lambda_k(\lambda_k+\lambda_\ell)} \int_0^T |B_c(B_\ell(h,e_k)e_\ell,e_k)|^2 d\tau \\ & + C\int_0^T \left|B_c(h,\mathcal{A}_s^{-1}B_s(e_k,e_k))\right|^2 d\tau + C\left|\int_0^T B_c(h,e_k)d\tilde{\beta}_k\right|^2 + |R(T)|^2,\end{aligned}$$

Using Ito isometry and Assumptions 4 to obtain

$$\begin{aligned}|h(T)|^2 \leq & \int_0^T |h|^6 d\tau + 3\int_0^T |b|^2 |h|^4 d\tau \\ & + 3\int_0^T |b|^4 |h|^2 d\tau + C\int_0^T |h|^2 d\tau + |R(T)|^2 \\ \leq & C\int_0^T |h|^2 d\tau + \int_0^T |h|^6 d\tau + C\int_0^T |h|^4 d\tau + |R(T)|^2,\end{aligned}$$

where we used (**??**). As long as $|h| \leq 1$, we obtain for $T \leq \tau^*$

$$|h(T)|^2 \leq C\int_0^T |h|^2 d\tau + \int_0^T |h|^6 d\tau + C\int_0^T |h|^4 d\tau.$$

Using Gronwall's Lemma, we obtain for $T \leq \tau^* \leq T_0$

$$|h(T)|^2 \leq \varepsilon^{2-14\kappa} e^{CT_0} \leq 1,$$

for ε sufficiently small as $\kappa < \frac{1}{7}$. Thus

$$\mathbb{E} \sup_{[0,\tau^*]} |a-b| = \mathbb{E} \sup_{[0,\tau^*]} |h| \leq C\varepsilon^{1-7\kappa} .$$

□

Now, we can use the results previously obtained to prove the main result of Theorem 8 for the approximation of the solution (6) of the SPDE (3).

Proof of Theorem 8. For the stopping time, we note that

$$\Omega \supset \{\tau^* = T_0\} \supseteq \{ \sup_{T\in[0,T_0]} \|a(T)\|_\alpha < \varepsilon^{-\kappa}, \sup_{T\in[0,T_0]} \|\psi(T)\|_\alpha < \varepsilon^{-\kappa}\}.$$

Hence

$$\mathbb{P}\{\tau^* < T_0\} \leq \{ \sup_{[0,\tau^*]} \|a\|_\alpha > \varepsilon^{-\kappa}, \sup_{[0,\tau^*]} \|\psi\|_\alpha > \varepsilon^{-\kappa}\} \leq C\varepsilon^{q\kappa-\kappa_0}, \tag{35}$$

where we used Chebychev's inequality and (22). Now let us turn to the approximation result. Using (11) and triangle inequality, yields

$$\mathbb{E} \sup_{T\in[0,\tau^*]} \left\|u(\varepsilon^{-2}T) - \varepsilon b(T) - \varepsilon\mathcal{Q}(T)\right\|_\alpha \leq \varepsilon\mathbb{E} \sup_{[0,\tau^*]} \|a-b\|_\alpha + \varepsilon\mathbb{E} \sup_{[0,\tau^*]} \left\|\psi - \mathcal{Q}\right\|_\alpha.$$

From (20) and (34), we obtain

$$\begin{aligned}&\mathbb{E} \sup_{t\in[0,\varepsilon^{-2}T_0]} \left\|u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon\mathcal{Q}(\varepsilon^2 t)\right\|_\alpha \\ &\quad = \mathbb{E} \sup_{t\in[0,\varepsilon^{-2}\tau^*]} \left\|u(t) - \varepsilon b(\varepsilon^2 t) - \varepsilon\mathcal{Q}(\varepsilon^2 t)\right\|_\alpha \leq C\varepsilon^{2-7\kappa} .\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}T_0]}\Big\|u(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha>\varepsilon^{2-8\kappa}\Big)\\
&\quad\le\ \mathbb{P}\Big(\sup_{t\in[0,\varepsilon^{-2}\tau^*]}\Big\|u(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha>\varepsilon^{2-8\kappa}\Big)+\mathbb{P}\{\tau^*<T_0\}\\
&\quad\le\ \frac{1}{\varepsilon^{2q-8q\kappa}}\mathbb{E}\Big(\sup_{t\in[0,\varepsilon^{-2}\tau^*]}\Big\|u(t)-\varepsilon b(\varepsilon^2 t)-\varepsilon\mathcal{Q}(\varepsilon^2 t)\Big\|_\alpha^q+\mathbb{P}\{\tau^*<T_0\}\\
&\quad\le\ C\varepsilon^{q\kappa}+C\varepsilon^{q\kappa-\kappa_0}\\
&\quad\le\ C\varepsilon^{q\kappa-\kappa_0},
\end{aligned}$$

where we used again Chebychev's inequality and (35). Let $p=q\kappa-\kappa_0$, yields (18). □

7. Amplitude Equation for SK Equation

We consider the Kuramoto-Sivashinsky equation in one dimension with either Dirichlet or periodic boundary conditions.

First case: Consider (1) with Dirichlet boundary condition on $[0,\pi]$. In this case, we take

$$\mathcal{H}=L^2([0,\pi]),\quad e_k(x)=\sqrt{\tfrac{2}{\pi}}\sin(kx)\quad\text{and}\quad\mathcal{N}=\text{span}\{\sin(x)\}.$$

If we fix P_c to be the $\mathcal{H}$-orthogonal projection onto $\mathcal{N}$, then both P_c and P_s commute with $\mathcal{A}$.

Moreover, all conditions of Assumption 4 are satisfied with

$$B(u,v)=-\tfrac{1}{2}\partial_x(uv),$$

as follows:

$$P_cB(u,u)=P_c\left[\gamma^2\sin(x)\cos(x)\right]=0\quad\text{for } u=\gamma\sin\in\mathcal{N},$$

and for $\alpha=\frac{1}{4}$ and $\beta=\frac{5}{4}<m$, we obtain

$$\begin{aligned}
2\|B(u,v)\|_{\mathcal{H}^{-1}}&=\|\partial_x(uv)\|_{\mathcal{H}^{-1}}\le\|uv\|_{L^2}\\
&\le C\|u\|_{L^4}\|v\|_{L^4}\le C\|u\|_{\mathcal{H}^{\frac14}}\|v\|_{\mathcal{H}^{\frac14}},
\end{aligned}$$

where we used Sobolev embedding from $\mathcal{H}^{1/4}$ into L^4.

In this case we study two cases depend on the type of the noise:

First case: If the noise acts on the second mode, then our main theorem states that

$$u(t,x)=\varepsilon b(\varepsilon^2 t)\sin(x)+\varepsilon\mathcal{Z}_2(\varepsilon^2 t)\sin(2x)+\mathcal{O}(\varepsilon^{2-})\ ,$$

where b is the solution of the amplitude equation of Stratonovic type

$$\partial_T b=(\nu-\frac{\sigma^2}{2688})b-\tfrac{1}{48}b^3+\frac{\sigma}{24}b\circ\partial_T\tilde{\beta}_2\ ,$$

with a rescaled standard Brownian motion $\tilde{\beta}_2$. If $\sigma^2>2688\nu$, then $(\nu-\frac{\sigma^2}{2688})<0$, in this case we can say the degenerate additive noise has the potential to stabilize the domiant mode.

second case: If the noise takes this form $W(t)=\sigma\beta_3(t)\sin(3x)$, then the amplitude equation b solves this deterministic equation

$$\partial_T b=(\nu+\frac{\sigma^2}{104832})b-\tfrac{1}{48}b^3.$$

Second case: Consider (1), which statisfies the initial condition (2), with periodic boundary condition on $[0,2\pi]$. In this case, we take

$$\begin{aligned}
\mathcal{H} &= \{u\in L^2([0,2\pi]):\int_0^{2\pi} udx=0\},\\
\mathcal{N} &= \text{span}\{\sin(x),\cos(x)\},\\
e_k(x) &= \begin{cases}\sqrt{\frac{1}{\pi}}\sin(kx) & \text{if } k\geq 0,\\ \sqrt{\frac{1}{\pi}}\cos(kx) & \text{if } k<0,\end{cases} \qquad \text{and}\\
W(t) &= \sigma\beta_2(t)\sin(2x).
\end{aligned}$$

In this case our main theorem states that

$$u(t,x)=\varepsilon b_1(\varepsilon^2 t)\sin(x)+\varepsilon b_2(\varepsilon^2 t)\cos(x)+\varepsilon\mathcal{Z}_2(\varepsilon^2 t)\sin(2x)+\mathcal{O}(\varepsilon^{2-}),$$

where b_1 and b_2 are solutions of the amplitude equation

$$\begin{aligned}
\partial_T b_1 &= (\nu-\frac{\sigma^2}{2688})b_1-\tfrac{1}{48}b_1(b_1^2+b_2^2)+\frac{\sigma}{24}b_1\circ\partial_T\tilde{\beta}_2,\\
\partial_T b_2 &= (\nu-\frac{\sigma^2}{2688})b_2-\tfrac{1}{48}b_2(b_1^2+b_2^2)-\frac{\sigma}{24}b_2\circ\partial_T\tilde{\beta}_2.
\end{aligned}$$

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