

MEAN SQUARE CONVERGENT FINITE DIFFERENCE SCHEME FOR RANDOM FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, the random finite difference method is used in solving random partial differential equations problems of first order. The conditions of the mean square convergence of the numerical solutions are studied. The numerical solutions are computed through numerical case studies.

1. INTRODUCTION

Random partial differential equations (RPDE) are defined as partial differential equations involving random inputs. Various numerical methods and approximation schemes for RPDEs have also been developed, analyzed, and tested (see [[1],[2],[3],[4],[5],[6],[7],[8],[9],[10],[11],[12],[13],[14],[15],[16]]). This paper is interested in studying the following random partial differential problem of the form:

$$\begin{aligned}u_t(x, t) &= \beta u_x(x, t) \quad , t \in [0, T], x \in [0, X] \\u(x, 0) &= u_0(x) \quad , x \in [0, X]\end{aligned}\tag{1.1}$$

Randomness may exist in the initial condition, in the differential equation it self or both. The random finite difference method is used to obtain an approximate solution for problem 1.1.

In this paper the random finite difference method is used to obtain an approximation solution for random partial differential equations problems of the first order. This paper is organized as follows. In Section 2, some important preliminaries are discussed. In Section 3, the Consistent of (RFDS), Stability of (RFDS) and the Convergence of (RFDS) are discussed. Section 4, presents some results. Section 5, presents the solution of some numerical examples of first order random partial differential equations using random finite difference method. The general conclusions are presented in the end section.

2000 *Mathematics Subject Classification.* 34A12, 34A30, 34D20.

Key words and phrases. Random Partial Differential Equations (RPDEs), Mean Square Sense (m.s), Second Order Random Variable, Random Finite Difference Method (RFDM)..

Proc. of the 4th. Symb. of Frac. Calcu. Appl. Faculty of Science Alexandria University, Alexandria, Egypt July, 11, 2012.

2. PRELIMINARIES

2.1. Mean Square Calculus.

Definition 2.1. [15]. *Let us consider the properties of a class of real r.v.'s*

$X_{11}, X_{12}, \dots, X_{21}, X_{22}, \dots, X_{nk}, \dots$ Whose second moments, $E(X_{11}^2), E(X_{12}^2), \dots, E(X_{21}^2), E(X_{22}^2), \dots, E(X_{nk}^2), \dots$ are finite. In this case, they are called "second order random variables", (2.r.v's).

Definition 2.2. [15]. *The linear vector space of second order random variables with inner product, norm and distance, is called an L_2 -space. A s.p. $\{X(t), t \in T\}$ is called a "second order stochastic process" (2.s.p) if for $t_1, t_2, t_3, \dots, t_n$ the r.v's $\{X(t_1), X(t_2), \dots, X(t_n)\}$ are elements of L_2 -space. A second order s.p. $\{X(t), t \in T\}$ is characterized by :*

$$\|X(t)\|^2 = E(X^2(t)) < \infty, t \in T$$

2.1.1. *The convergence in mean square*[15]. A sequence of r.v's $\{X_{nk}, n, k > 0\}$ converges in mean square (m.s) to a random variable X if

$$\lim_{n,k \rightarrow \infty} \|X_{nk} - X\| = 0$$

i.e.

$$X_{nk} \xrightarrow{m.s} X \text{ or } L.i.m_{n,k \rightarrow \infty} X_{nk} = X$$

Where L.i.m is the limit in mean square sense.

3. RANDOM FINITE DIFFERENCE SCHEME (RFDS)

In this current work, we extend one kind of the finite difference methods to random case in order to approximate of random first order partial differential equations of the form:

$$u_t(x, t) = \beta u_x(x, t) \quad , t \in [0, T], x \in [0, X] \quad (3.1)$$

$$u(x, 0) = u_0(x) \quad , x \in [0, X] \quad (3.2)$$

For difference method, consider a uniform mesh with step size Δx and Δt on x-axis and t-axis. Notational, u_k^n will be approximate of $u(x, t)$ at point $(k\Delta x, n\Delta t)$ (Hence $u_k^0 = u_0(k\Delta x)$). On this mesh, we have

$$u_t(x, t) \simeq \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t}$$

then

$$u_t(k\Delta x, n\Delta t) \simeq \frac{u_k^{n+1} - u_k^n}{\Delta t}$$

Similarly

$$u_x(x, t) \simeq \frac{u(x + \Delta x, t) - u(x, t)}{\Delta x},$$

then

$$u_x(k\Delta x, n\Delta t) \simeq \frac{u_{k+1}^n - u_k^n}{\Delta x}$$

Hence for 3.1

$$u_k^{n+1} = u_k^n + r\beta(u_{k+1}^n - u_k^n) \quad (3.3)$$

$$u_k^0 = u_0(x_k) \quad (3.4)$$

where $r = \frac{\Delta t}{\Delta x}$

Above scheme is a random version of 3.1. For a RPDE, say:

$$Lv = G$$

where L be a differential operator and $G \in L^2(R)$ in the other hand, we represent finite difference scheme at the point $(k\Delta x, n\Delta t)$ by $L_k^n u_k^n = G_k^n$.

3.1. Consistent Of (RFDS).

Definition 3.1. [7],[10],[12]. A random difference scheme $L_k^n u_k^n = G_k^n$ approximating RPDE $Lv = G$ is consistent in mean square at time $t = (n+1)\Delta t$, if for any continuously differentiable function $\Phi = \Phi(x, t)$, we have in mean square:

$$E |(L\Phi - G)_k^n - (L_k^n \Phi(k\Delta x, n\Delta t) - G_k^n)|^2 \rightarrow 0$$

as $\Delta t \rightarrow 0$, $\Delta x \rightarrow 0$, and $(k\Delta x, n\Delta t) \rightarrow (x, t)$

Theorem 3.2. The random difference scheme (3.3-3.4) is consistent in mean square sense

Proof. Assume that be a smooth function then

$$L(\Phi)_k^n = \Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) - \beta_{n\Delta t}^{(n+1)\Delta t} \Phi_x(k\Delta x, s) ds$$

And

$$L_k^n \Phi = \Phi(k\Delta x, (n+1)\Delta t) - \Phi(k\Delta x, n\Delta t) - r\beta(\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t))$$

Then we have

$$\begin{aligned} E |(L\Phi)_k^n - L_k^n \Phi|^2 &= E \left[-\beta_{n\Delta t}^{(n+1)\Delta t} \Phi_x(k\Delta x, s) ds + \frac{\Delta t}{\Delta x} \beta(\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t)) \right]^2 \\ &= E \left[-\beta_{n\Delta t}^{(n+1)\Delta t} \Phi_x(k\Delta x, s) ds - \frac{\Delta t}{\Delta x} (\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t)) \right]^2 \\ &= E \left[\beta_{n\Delta t}^{(n+1)\Delta t} \Phi_x(k\Delta x, s) ds - \frac{\Delta t}{\Delta x} (\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t)) \right]^2 \\ &= E \left[\beta_{n\Delta t}^{2(n+1)\Delta t} \Phi_x(k\Delta x, s) ds]^2 - 2\beta_{n\Delta t}^{(n+1)\Delta t} [\Phi_x(k\Delta x, s) ds] \left[\frac{k\Delta t}{k\Delta x} (\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t)) \right]^2 \right. \\ &\quad \left. + \beta_{n\Delta t}^{2(n+1)\Delta t} \left[\frac{k\Delta t}{k\Delta x} (\Phi((k+1)\Delta x, n\Delta t) - \Phi(k\Delta x, n\Delta t)) \right]^2 \right] \end{aligned}$$

if $\Delta t \rightarrow 0$, $\Delta x \rightarrow 0$, and $(k\Delta x, n\Delta t) \rightarrow (x, t)$, then

$$E |(L\Phi)_k^n - (L_k^n \Phi(k\Delta x, n\Delta t))|^2 \rightarrow 0$$

Hence the random difference scheme (3.3-3.4) is consistent in mean square sense. \square

3.2. Stability Of (RFDS).

Definition 3.3. [[7],[10],[12]]. A random difference scheme is stable in mean square if there exist some positive constants ε, δ and constants k, b such that

$$E |u_k^{n+1}|^2 \leq ke^{bt} \sup_k E |u^0|^2$$

For all $0 \leq t = (n+1)\Delta t$, $0 \leq \Delta x \leq \varepsilon$ and $0 \leq \Delta t \leq \delta$.

Theorem 3.4. The random difference scheme (3.3-3.4) is stable in mean square sense.

Proof. Since $u_k^{n+1} = u_k^n + r\beta(u_{k+1}^n - u_k^n)$ then:

$$\begin{aligned} E |u_k^{n+1}|^2 &= E |u_k^n + r\beta(u_{k+1}^n - u_k^n)|^2 \\ &= E [(u_k^n)^2 + 2r\beta(u_{k+1}^n - u_k^n)(u_k^n) + r^2\beta^2(u_{k+1}^n - u_k^n)^2] \\ &= E(u_k^n)^2 + 2r\beta(E(u_{k+1}^n u_k^n) - E(u_k^n)^2) + r^2\beta^2[E(u_{k+1}^n)^2 - 2E(u_{k+1}^n u_k^n) + E(u_k^n)^2] \\ &\leq \sup_k [E(u_k^n)^2 + 2r\beta(E(u_{k+1}^n u_k^n) - E(u_k^n)^2) + r^2\beta^2[E(u_{k+1}^n)^2 - 2E(u_{k+1}^n u_k^n) + E(u_k^n)^2]] \\ &= \sup_k E(u_k^n)^2 + 2r\beta \sup_k E(u_k^n)^2 - 2r\beta \sup_k E(u_k^n)^2 \\ &\quad + r^2\beta^2 \sup_k E(u_{k+1}^n)^2 - 2r^2\beta^2 \sup_k E(u_k^n)^2 + r^2\beta^2 E(u_k^n)^2 \\ &= \sup_k E(u_k^n)^2. \end{aligned}$$

Hence

$$\sup_k E |u_k^{n+1}|^2 \leq \sup_k E |u_k^n|^2 \leq \sup_k E |u_k^{n-1}|^2 \leq \dots \leq \sup_k E |u_k^0|^2$$

Then

$$E |u_k^{n+1}|^2 \leq \sup_k E |u^0|^2$$

where $k = 1$ and $b = 0$ then the random difference scheme (3.3-3.4) is stable in mean square sense. \square

3.3. Convergence of (RFDS).

Definition 3.5. [[7],[10],[12]]. A random difference scheme $L_k^n u_k^n = G_k^n$ approximating RPDE $Lv = G$ is convergent in mean square at time $t = (n+1)\Delta t$, if

$$E |u_k^n - u|^2 \rightarrow 0$$

as $\Delta t \rightarrow 0$, $\Delta x \rightarrow 0$ and $(k\Delta x, n\Delta t) \rightarrow (x, t)$ or as $n \rightarrow 0$, $k \rightarrow 0$ and $(k\Delta x, n\Delta t) \rightarrow (x, t)$

3.3.1. A Stochastic Version of Lax-Richtmyer Theorem [12]. A random difference scheme $L_k^n u_k^n = G_k^n$ approximating SPDE $Lv = G$ is convergent in mean square at time $t = (n+1)\Delta t$, if it is consistent and stable.

Theorem 3.6. The random difference scheme (3.3-3.4) is convergent in mean square sense

Proof.

$$E |u_k^n - u|^2 = E |(L_k^n)^{-1}(L_k^n u_k^n - L_k^n u)|^2$$

□

Proof. Since the scheme is consistent then we have:

$$L_k^n u_k^n \xrightarrow{m.s} L_k^n u.$$

Then we obtain

$$E |L_k^n u_k^n - L_k^n u|^2 \rightarrow 0$$

as $\Delta t \rightarrow 0$, $\Delta x \rightarrow 0$, and $(k\Delta x, n\Delta t) \rightarrow (x, t)$. And since the scheme is stable. Then $(L_k^n)^{-1}$ is bounded, Hence $E |u_k^n - u|^2 \rightarrow 0$ as $\Delta t \rightarrow 0$, $\Delta x \rightarrow 0$. Then the random difference scheme (3.3-3.4) is convergent in mean square sense. □

4. SOME RESULTS

Theorem 4.1. *Let be $\{X_{nk}, n, k > 0\}$, $\{Y_{nk}, n, k > 0\}$ sequences of 2-r.v's over the same probability space and suppose that:*

$$\begin{aligned} \lim_{n,k \rightarrow \infty} \|X_{nk} - X\| &= 0, \\ \lim_{n,k \rightarrow \infty} \|Y_{nk} - Y\| &= 0, \end{aligned}$$

then

- (1) $\lim_{n,k \rightarrow \infty} E \{X_{nk}\} = E \{X\}$
- (2) $\lim_{n,k \rightarrow \infty} E \{X_{nk}^2\} = E \{X^2\}$
- (3) $\lim_{n,k \rightarrow \infty} Var \{X_{nk}\} = Var \{X\}$
- (4) $\lim_{n,k \rightarrow \infty} PDF \{X_{nk}\} = PDF \{X\}$

Proof. (1) From Schwarz inequality:

$$E |XY| \leq (E \{X^2\})^{\frac{1}{2}} (E \{Y^2\})^{\frac{1}{2}}$$

we have:

$$|E \{X_{nk}\} \cdot 1| \leq 1 \cdot (E \{X_{nk}^2\}) = \|X_{nk}\|$$

then:

$$|E \{X_{nk}\}| \leq E \{|X_{nk}\}| \leq \|X_{nk}\| < \infty \quad (4.1.1)$$

In (4.1.1) put $X_{nk} - X$ instead of X_{nk} , then we have:

$$|E \{X_{nk} - X\}| = |E(X_{nk}) - E(X)| \leq E \{|X_{nk} - X\}| \leq \|X_{nk} - X\|$$

As $n, k \rightarrow \infty$ then

$$|E \{X_{nk}\} - E \{X\}| = 0$$

i.e

$$\lim_{n,k \rightarrow \infty} E \{X_{nk}\} = E \{X\}$$

(2) From successive applications of the triangle and Schwarz inequalities:

$$\begin{aligned} |E(XY) - E(X_{nk}Y_{nk})| &= |E(XY) - E(X_{nk}Y_{nk}) \pm E(XY_{nk}) \pm E(XY) \pm E(X_{nk}Y)| \\ &= |E(Y - Y_{nk})X + E(X - X_{nk})Y + E((Y - Y_{nk})(X - X_{nk}))| \\ &\leq |E(Y - Y_{nk})X| + |E(X - X_{nk})Y| + |E((Y - Y_{nk})(X - X_{nk}))| \\ &\leq |X| \|Y_{nk} - Y\| + |Y| \|X_{nk} - X\| + \|Y_{nk} - Y\| \|X_{nk} - X\| \end{aligned}$$

But each of the terms on the right-hand side tends to zero by hypothesis as $n, k \rightarrow \infty$ then:

$$\lim_{n,k \rightarrow \infty} E \{Y_{nk}X_{nk}\} = E \{XY\}$$

As $X_{nk} = X$ then we obtain

$$\lim_{n,k \rightarrow \infty} E \{X_{nk}^2\} = E \{X^2\}$$

(3) Since

$$\begin{aligned} \lim_{n,k \rightarrow \infty} E \{X_{nk}\} &= E \{X\}, \\ \lim_{n,k \rightarrow \infty} E \{X_{nk}^2\} &= E \{X^2\}, \end{aligned}$$

and

$$Var \{X_{nk}\} = E \{X_{nk}^2\} - (E(X_{nk}))^2$$

$$\begin{aligned} \lim_{n,k \rightarrow \infty} Var \{X_{nk}\} &= \lim_{n,k \rightarrow \infty} (E \{X_{nk}^2\} - (E(X_{nk}))^2) \\ &= \lim_{n,k \rightarrow \infty} E \{X_{nk}^2\} - \lim_{n,k \rightarrow \infty} (E(X_{nk}))^2 \\ &= E \{X^2\} - (E \{X\})^2 = Var(X) \end{aligned}$$

Then we obtain

$$\lim_{n,k \rightarrow \infty} Var \{X_{nk}\} = Var \{X\}$$

□

Definition 4.2. [15]. "The convergence in probability"

A sequence of r.v's $\{X_{nk}\}$ converges in probability to a random variable X as $n, k \rightarrow \infty$ if:

$$\lim_{n,k \rightarrow \infty} p \{|X_{nk} - X| > \varepsilon\} = 0 \quad \forall \varepsilon > 0$$

Definition 4.3. [15]. "The convergence in distribution"

A sequence of r.v's $\{X_{nk}\}$ converges in distribution to a random variable X as $n, k \rightarrow \infty$ if:

$$\lim_{n,k \rightarrow \infty} F_{x_{nk}}(x) = F_x(x)$$

Lemma 4.4. [15]

The convergence in m.s implies convergence in probability

Lemma 4.5. [15]

The convergence in probability implies convergence in distribution

Theorem 4.6. if $X_{nk} \xrightarrow{m,s} X$ then $PDF\{X_{nk}\} \xrightarrow{m,s} PDF\{X\}$

$$\lim_{n,k \rightarrow \infty} f_{x_{nk}}(x) = f_x(x)$$

Proof. Since we have shown that If $X_{nk} \xrightarrow{m,s} X$ then $X_{nk} \xrightarrow{d} X$ i.e. $X_{nk} \xrightarrow{m,s} X$ then $\lim_{n,k \rightarrow \infty} F_{x_{nk}}(x) = F_x(x)$, Then

$$\lim_{n,k \rightarrow \infty} \frac{d}{dx} F_{x_{nk}}(x) = \frac{d}{dx} F_x(x),$$

hence

$$\lim_{n,k \rightarrow \infty} f_{x_{nk}}(x) = f_x(x)$$

□

5. NUMERICAL EXAMPLES

Example 5.1. Solve the random first order partial differential equation:

$$u_t = \beta u_x, \beta \sim N(0,1), \quad , t \in [0, T], x \in [0, X] \quad (5.1)$$

$$u(x, 0) = \frac{1}{2} e^{-2x}, x \in [0, X] \quad (5.2)$$

The exact solution

$$u(x, t) = \frac{1}{2} e^{-2\beta t - 2x}$$

The numerical solution

For the difference method, consider a uniform mesh with step size Δx and Δt on x-axis and t-axis. Where $\Delta x = \frac{X}{M}$, $\Delta t = \frac{T}{N}$. and $M, N > 0$. Notational u_k^n , will be approximate of $u(x, t)$ at point $(k\Delta x, n\Delta t)$, $u_k^0 = u_0(k\Delta x)$. On this mesh, the difference scheme for this problem is given by

$$u_k^{n+1} = u_k^n + r\beta(u_{k+1}^n - u_k^n) \quad (5.3)$$

$$u_k^0 = \frac{1}{2} e^{-2x_k} \quad (5.4)$$

where $r = \frac{\Delta t}{\Delta x}$, $x_k = k\Delta x$. First from the initial condition we have:

$$\begin{aligned}
 u(0, 0) &= u_{00} = \frac{1}{2} \\
 u(1, 0) &= u_{10} = \frac{1}{2}e^{-2\Delta x} \\
 u(2, 0) &= u_{20} = \frac{1}{2}e^{-4\Delta x} \\
 u(k, 0) &= u_{k0} = \frac{1}{2}e^{-2k\Delta x}
 \end{aligned} \tag{5.5}$$

From 5.3

$$\begin{aligned}
 u(1, 0) &= u_{01} = u_{00} + r\beta(u_{10} - u_{00}) \\
 &= (1 - r\beta)u_{00} + r\beta u_{10} \\
 &= (1 - r\beta)\frac{1}{2} + r\beta \left[\frac{1}{2}e^{-2\Delta x} \right] \\
 u(1, 1) &= u_{11} = u_{10} + r\beta(u_{20} - u_{10}) \\
 &= (1 - r\beta)u_{10} + r\beta u_{20} \\
 &= (1 - r\beta)\left[\frac{1}{2}e^{-2\Delta x}\right] + r\beta \left[\frac{1}{2}e^{-4\Delta x}\right] \\
 u(2, 1) &= u_{21} = u_{20} + r\beta(u_{30} - u_{20}) \\
 &= (1 - r\beta)u_{20} + r\beta u_{30} \\
 &= (1 - r\beta)\left[\frac{1}{2}e^{-4\Delta x}\right] + r\beta \left[\frac{1}{2}e^{-6\Delta x}\right]
 \end{aligned}$$

Finally

$$\begin{aligned}
 u(k, 1) &= u_{k1} = (1 - r\beta)\left[\frac{1}{2}e^{-2k\Delta x}\right] + r\beta \left[\frac{1}{2}e^{-(2k+2)\Delta x}\right] \\
 &= \left[\frac{1}{2}e^{-2k\Delta x}\right][(1 - r\beta) + r\beta e^{-2\Delta x}]
 \end{aligned} \tag{5.6}$$

$$\begin{aligned}
u(0, 2) &= u_{02} = u_{01} + r\beta(u_{11} - u_{01}) \\
&= (1 - r\beta) \left[(1 - r\beta) \frac{1}{2} + r\beta \left[\frac{1}{2} e^{-2\Delta x} \right] \right] + r\beta \left[(1 - r\beta) \left[\frac{1}{2} e^{-2\Delta x} \right] + r\beta \left[\frac{1}{2} e^{-4\Delta x} \right] \right] \\
&= (1 - r\beta)^2 \left[\frac{1}{2} \right] + 2(1 - r\beta)r\beta \left[\frac{1}{2} e^{-2\Delta x} \right] + (r\beta)^2 \left[\frac{1}{2} e^{-4\Delta x} \right] \\
&= \left[\frac{1}{2} \right] [(1 - r\beta) + (r\beta e^{-2\Delta x})]^2
\end{aligned}$$

$$\begin{aligned}
u(1, 2) &= u_{12} = u_{11} + r\beta(u_{21} - u_{11}) \\
&= (1 - r\beta) \left[(1 - r\beta) \left[\frac{1}{2} e^{-2\Delta x} \right] + r\beta \left[\frac{1}{2} e^{-4\Delta x} \right] \right] + r\beta \left[(1 - r\beta) \left[\frac{1}{2} e^{-4\Delta x} \right] + r\beta \left[\frac{1}{2} e^{-6\Delta x} \right] \right] \\
&= (1 - r\beta)^2 \left[\frac{1}{2} e^{-2\Delta x} \right] + 2(1 - r\beta)r\beta \left[\frac{1}{2} e^{-4\Delta x} \right] + (r\beta)^2 \left[\frac{1}{2} e^{-6\Delta x} \right] \\
&= \left[\frac{1}{2} e^{-2\Delta x} \right] [(1 - r\beta) + (r\beta e^{-2\Delta x})]^2
\end{aligned}$$

$$\begin{aligned}
u(2, 2) &= u_{22} = u_{21} + r\beta(u_{31} - u_{21}) \\
&= (1 - r\beta) \left[(1 - r\beta) \left[\frac{1}{2} e^{-4\Delta x} \right] + r\beta \left[\frac{1}{2} e^{-6\Delta x} \right] \right] + r\beta \left[(1 - r\beta) \left[\frac{1}{2} e^{-6\Delta x} \right] + r\beta \left[\frac{1}{2} e^{-8\Delta x} \right] \right] \\
&= (1 - r\beta)^2 \left[\frac{1}{2} e^{-4\Delta x} \right] + 2(1 - r\beta)r\beta \left[\frac{1}{2} e^{-6\Delta x} \right] + (r\beta)^2 \left[\frac{1}{2} e^{-8\Delta x} \right] \\
&= \left[\frac{1}{2} e^{-4\Delta x} \right] [(1 - r\beta) + (r\beta e^{-2\Delta x})]^2
\end{aligned}$$

finally

$$u_{k2} = \left[\frac{1}{2} e^{-2k\Delta x} \right] [(1 - r\beta) + (r\beta e^{-2\Delta x})]^2 \quad (5.7)$$

$$\begin{aligned}
u(0, 3) &= u_{03} = u_{02} + r\beta(u_{12} - u_{02}) \\
&= (1 - r\beta) \left[(1 - r\beta)^2 \left[\frac{1}{2} \right] + 2(1 - r\beta)r\beta \left[\frac{1}{2} e^{-2\Delta x} \right] + (r\beta)^2 \left[\frac{1}{2} e^{-4\Delta x} \right] \right] + \\
&\quad r\beta \left[(1 - r\beta)^2 \left[\frac{1}{2} e^{-2\Delta x} \right] + 2(1 - r\beta)r\beta \left[\frac{1}{2} e^{-4\Delta x} \right] + (r\beta)^2 \left[\frac{1}{2} e^{-6\Delta x} \right] \right] \\
&= (1 - r\beta)^3 \left[\frac{1}{2} \right] + 3(1 - r\beta)r\beta \left[\frac{1}{2} e^{-2\Delta x} \right] + 3(r\beta)^2(1 - r\beta) \left[\frac{1}{2} e^{-4\Delta x} \right] + (r\beta)^3 \left[\frac{1}{2} e^{-6\Delta x} \right] \\
&= \left[\frac{1}{2} \right] [(1 - r\beta) + (r\beta e^{-2\Delta x})]^3
\end{aligned}$$

$$\begin{aligned}
u(1, 3) &= u_{13} = u_{12} + r\beta(u_{22} - u_{12}) \\
&= (1 - r\beta) \left[(1 - r\beta)^2 \left[\frac{1}{2} e^{-2\Delta x} \right] + 2(1 - r\beta)r\beta \left[\frac{1}{2} e^{-4\Delta x} \right] + (r\beta)^2 \left[\frac{1}{2} e^{-6\Delta x} \right] \right] + \\
&\quad r\beta \left[(1 - r\beta)^2 \left[\frac{1}{2} e^{-4\Delta x} \right] + 2(1 - r\beta)r\beta \left[\frac{1}{2} e^{-6\Delta x} \right] + (r\beta)^2 \left[\frac{1}{2} e^{-8\Delta x} \right] \right] \\
&= (1 - r\beta)^3 \left[\frac{1}{2} e^{-2\Delta x} \right] + 3(1 - r\beta)r\beta \left[\frac{1}{2} e^{-4\Delta x} \right] + 3(r\beta)^2(1 - r\beta) \left[\frac{1}{2} e^{-6\Delta x} \right] + (r\beta)^3 \left[\frac{1}{2} e^{-8\Delta x} \right] \\
&= \left[\frac{1}{2} e^{-2\Delta x} \right] [(1 - r\beta) + (r\beta e^{-2\Delta x})]^3
\end{aligned}$$

finally

$$u_{k3} = \left[\frac{1}{2} e^{-2k\Delta x} \right] [(1 - r\beta) + (r\beta e^{-2\Delta x})]^3 \quad (5.8)$$

Finally the general numerical form is

$$u_{kn} = \left[\frac{1}{2} e^{-2k\Delta x} \right] [(1 - r\beta) + (r\beta e^{-2\Delta x})]^n \quad (5.9)$$

we can prove that

$$u_k^n \xrightarrow{m,s} u$$

Proof. Since

$$\lim_{n,k \rightarrow \infty} u_k^n = \lim_{\substack{\Delta x, \Delta t \rightarrow 0 \\ k\Delta x, n\Delta t \rightarrow x, t}} u_k^n = u$$

if and only if

$$\lim_{n,k \rightarrow \infty} E |u_k^n - u|^2 = 0 \text{ or,}$$

$$\lim_{\substack{\Delta x, \Delta t \rightarrow 0 \\ k\Delta x, n\Delta t \rightarrow x, t}} E |u_k^n - u|^2 = 0$$

$$u_k^n - u = \left[\frac{1}{2} e^{-2k\Delta x} \right] [(1 - r\beta) + (r\beta e^{-2\Delta x})]^n - \frac{1}{2} e^{-2\beta t - 2x}$$

$$\begin{aligned} |u_k^n - u|^2 &= \left| \left[\frac{1}{2} e^{-2k\Delta x} \right] [(1 - r\beta) + (r\beta e^{-2\Delta x})]^n - \frac{1}{2} e^{-2\beta t - 2x} \right|^2 \\ &= \left[\left[\frac{1}{2} e^{-2k\Delta x} \right] [(1 - r\beta) + (r\beta e^{-2\Delta x})]^n \right]^2 - \left[\frac{1}{2} e^{-2k\Delta x} \right] \\ &\quad [(1 - r\beta) + (r\beta e^{-2\Delta x})]^n e^{-2\beta t - 2x} + \left[\frac{1}{2} e^{-2\beta t - 2x} \right]^2 \end{aligned}$$

$$\begin{aligned} E |u_k^n - u|^2 &= E \left\{ \frac{\left[\frac{1}{4} e^{-4k\Delta x} \right] [(1 - r\beta) + (r\beta e^{-2\Delta x})]^{2n} - \left[\frac{1}{2} e^{-2k\Delta x - 2\beta t - 2x} \right]}{\left[(1 - r\beta) + (r\beta e^{-2\Delta x}) \right]^n + \left[\frac{1}{4} e^{-4\beta t - 4x} \right]} \right\} \\ &= E \left\{ \frac{\left[\frac{1}{4} e^{-4k\Delta x} \right] \left[\left(1 - \frac{k\Delta t}{k\Delta x} \beta \right) + \left(\frac{k\Delta t}{k\Delta x} \beta e^{-2\Delta x} \right) \right]^{2n} - \left[\frac{1}{2} e^{-2k\Delta x - 2\beta t - 2x} \right]}{\left[\left(1 - \frac{k\Delta t}{k\Delta x} \beta \right) + \left(\frac{k\Delta t}{k\Delta x} \beta e^{-2\Delta x} \right) \right]^n + \left[\frac{1}{4} e^{-4\beta t - 4x} \right]} \right\} \end{aligned}$$

At time $t = (n + 1)\Delta t$, then

$$E |u_k^n - u|^2 = E \left\{ \frac{\left[\frac{1}{4} e^{-4k\Delta x} \right] \left[\left(1 - \frac{k\Delta t}{k\Delta x} \beta \right) + \left(\frac{k\Delta t}{k\Delta x} \beta e^{-2\Delta x} \right) \right]^{2n}}{- \left[\frac{1}{2} e^{-2k\Delta x - 2\beta(n+1)\Delta t - 2x} \right] \left[\left(1 - \frac{k\Delta t}{k\Delta x} \beta \right) + \left(\frac{k\Delta t}{k\Delta x} \beta e^{-2\Delta x} \right) \right]^n + \left[\frac{1}{4} e^{-4\beta(n+1)\Delta t - 4x} \right]} \right\}$$

as $\Delta t \rightarrow 0$, $\Delta x \rightarrow 0$, and $(k\Delta x, n\Delta t) \rightarrow (x, t)$ and $E(\beta) = 0$ then we obtain

$$E |u_k^n - u|^2 = 0,$$

then

$$u_k^n \xrightarrow{m,s} u$$

□

Example 5.2. Solve the random first order partial differential equation:

$$u_t = u_x, \quad \beta \sim \exp(1), \quad t \in [0, T], x \in [0, X] \quad (5.10)$$

$$u(x, 0) = \frac{1}{2}e^{-\beta x}, \quad x \in [0, X] \quad (5.11)$$

The exact solution

$$u(x, t) = \frac{1}{2}e^{-\beta(x+t)}$$

The numerical solution

For the difference method, consider a uniform mesh with step size Δx and Δt on x-axis and t-axis. Where $\Delta x = \frac{X}{M}$, $\Delta t = \frac{T}{N}$ and $M, N > 0$. Notational u_k^n , will be approximate of $u(x, t)$ at point $(k\Delta x, n\Delta t)$, $u_k^0 = u_0(k\Delta x)$. On this mesh, the difference scheme for this problem is

$$u_k^{n+1} = u_k^n + r(u_{k+1}^n - u_k^n) \quad (5.12)$$

$$u_k^0 = \frac{1}{2}e^{-\beta x_k} \quad (5.13)$$

where $r = \frac{\Delta t}{\Delta x}$, $x_k = k\Delta x$ and $t_n = n\Delta t$.

First from the initial condition we have

$$\begin{aligned} u(0, 0) &= u_{00} = \frac{1}{2} \\ u(1, 0) &= u_{10} = \frac{1}{2}e^{-\beta\Delta x} \\ u(2, 0) &= u_{20} = \frac{1}{2}e^{-2\beta\Delta x} \end{aligned}$$

From 5.3

$$u(k, 0) = u_{k0} = \frac{1}{2}e^{-\beta k\Delta x} \quad (5.14)$$

$$\begin{aligned} u(1, 0) &= u_{01} = u_{00} + r(u_{10} - u_{00}) \\ &= (1-r)u_{00} + ru_{10} \\ &= (1-r)\frac{1}{2} + r\left[\frac{1}{2}e^{-\beta\Delta x}\right] \\ u(1, 1) &= u_{11} = u_{10} + r(u_{20} - u_{10}) \\ &= (1-r)u_{10} + ru_{20} \\ &= (1-r)\left[\frac{1}{2}e^{-\beta\Delta x}\right] + r\left[\frac{1}{2}e^{-2\beta\Delta x}\right] \\ u(2, 1) &= u_{21} = u_{20} + r(u_{30} - u_{20}) \\ &= (1-r)u_{20} + ru_{30} \\ &= (1-r)\left[\frac{1}{2}e^{-2\beta\Delta x}\right] + r\left[\frac{1}{2}e^{-3\beta\Delta x}\right] \end{aligned}$$

Finally

$$\begin{aligned} u(k, 1) &= u_{k1} = (1-r)\left[\frac{1}{2}e^{-\beta k\Delta x}\right] + r\beta\left[\frac{1}{2}e^{-\beta(k+1)\Delta x}\right] \\ &= \left[\frac{1}{2}e^{-\beta k\Delta x}\right][(1-r) + re^{-\beta\Delta x}] \end{aligned} \quad (5.15)$$

$$\begin{aligned}
u(0, 2) &= u_{02} = u_{01} + r(u_{11} - u_{01}) \\
&= (1-r) \left[(1-r) \frac{1}{2} + r \left[\frac{1}{2} e^{-\beta \Delta x} \right] \right] + r \left[(1-r) \left[\frac{1}{2} e^{-\beta \Delta x} \right] + r \left[\frac{1}{2} e^{-2\beta \Delta x} \right] \right] \\
&= (1-r)^2 \left[\frac{1}{2} \right] + 2(1-r)r \left[\frac{1}{2} e^{-\beta \Delta x} \right] + (r)^2 \left[\frac{1}{2} e^{-2\beta \Delta x} \right] \\
&= \left[\frac{1}{2} \right] \left[(1-r) + (r e^{-\beta \Delta x}) \right]^2 \\
u(1, 2) &= u_{12} = u_{11} + r(u_{21} - u_{11}) \\
&= (1-r) \left[(1-r) \left[\frac{1}{2} e^{-\beta \Delta x} \right] + r \left[\frac{1}{2} e^{-2\beta \Delta x} \right] \right] + r \left[(1-r) \left[\frac{1}{2} e^{-2\beta \Delta x} \right] + r \left[\frac{1}{2} e^{-3\beta \Delta x} \right] \right] \\
&= (1-r)^2 \left[\frac{1}{2} e^{-\beta \Delta x} \right] + 2(1-r)r \left[\frac{1}{2} e^{-2\beta \Delta x} \right] + (r)^2 \left[\frac{1}{2} e^{-3\beta \Delta x} \right] \\
&= \left[\frac{1}{2} e^{-\beta \Delta x} \right] \left[(1-r) + (r e^{-\beta \Delta x}) \right]^2 \\
u(2, 2) &= u_{22} = u_{21} + r(u_{31} - u_{21}) \\
&= (1-r) \left[(1-r) \left[\frac{1}{2} e^{-2\beta \Delta x} \right] + r \left[\frac{1}{2} e^{-3\beta \Delta x} \right] \right] + r \left[(1-r) \left[\frac{1}{2} e^{-3\beta \Delta x} \right] + r \left[\frac{1}{2} e^{-4\beta \Delta x} \right] \right] \\
&= (1-r)^2 \left[\frac{1}{2} e^{-2\beta \Delta x} \right] + 2(1-r)r \left[\frac{1}{2} e^{-3\beta \Delta x} \right] + (r)^2 \left[\frac{1}{2} e^{-4\beta \Delta x} \right] \\
&= \left[\frac{1}{2} e^{-2\beta \Delta x} \right] \left[(1-r) + (r e^{-\beta \Delta x}) \right]^2.
\end{aligned}$$

Finally

$$u_{k2} = \left[\frac{1}{2} e^{-\beta k \Delta x} \right] \left[(1-r) + (r e^{-\beta \Delta x}) \right]^2 \quad (5.16)$$

$$\begin{aligned}
u(0, 3) &= u_{03} = u_{02} + r(u_{12} - u_{02}) \\
&= (1-r) \left[(1-r)^2 \left[\frac{1}{2} \right] + 2(1-r)r \left[\frac{1}{2} e^{-\beta \Delta x} \right] + (r)^2 \left[\frac{1}{2} e^{-2\beta \Delta x} \right] \right] \\
&\quad + r \left[(1-r)^2 \left[\frac{1}{2} e^{-\beta \Delta x} \right] + 2(1-r)r \left[\frac{1}{2} e^{-2\beta \Delta x} \right] + (r)^2 \left[\frac{1}{2} e^{-3\beta \Delta x} \right] \right] \\
&= (1-r)^3 \left[\frac{1}{2} \right] + 3(1-r)r \left[\frac{1}{2} e^{-\beta \Delta x} \right] + 3(r)^2(1-r) \left[\frac{1}{2} e^{-2\beta \Delta x} \right] + (r)^3 \left[\frac{1}{2} e^{-3\beta \Delta x} \right] \\
&= \left[\frac{1}{2} \right] \left[(1-r) + (r e^{-\beta \Delta x}) \right]^3 \\
u(1, 3) &= u_{13} = u_{12} + r(u_{22} - u_{12}) \\
&= (1-r) \left[(1-r)^2 \left[\frac{1}{2} e^{-\beta \Delta x} \right] + 2(1-r)r \left[\frac{1}{2} e^{-2\beta \Delta x} \right] + (r)^2 \left[\frac{1}{2} e^{-3\beta \Delta x} \right] \right] \\
&\quad + r \left[(1-r)^2 \left[\frac{1}{2} e^{-2\beta \Delta x} \right] + 2(1-r)r \left[\frac{1}{2} e^{-3\beta \Delta x} \right] + (r)^2 \left[\frac{1}{2} e^{-4\beta \Delta x} \right] \right] \\
&= (1-r)^3 \left[\frac{1}{2} e^{-\beta \Delta x} \right] + 3(1-r)r \left[\frac{1}{2} e^{-2\beta \Delta x} \right] + 3(r)^2(1-r) \left[\frac{1}{2} e^{-3\beta \Delta x} \right] + (r)^3 \left[\frac{1}{2} e^{-4\beta \Delta x} \right] \\
&= \left[\frac{1}{2} e^{-\beta \Delta x} \right] \left[(1-r) + (r e^{-\beta \Delta x}) \right]^3.
\end{aligned}$$

Hence

$$u_{k3} = \left[\frac{1}{2} e^{-\beta k \Delta x} \right] [(1-r) + (r e^{-\beta \Delta x})]^3 \quad (5.17)$$

Finally the general numerical form is

$$u_{kn} = \left[\frac{1}{2} e^{-\beta k \Delta x} \right] [(1-r) + (r e^{-\beta \Delta x})]^n \quad (5.18)$$

We can prove that

$$u_k^n \xrightarrow{m,s} u$$

Proof. Since $\lim_{n,k \rightarrow \infty} u_k^n = \lim_{\substack{\Delta x, \Delta t \rightarrow 0 \\ k \Delta x, n \Delta t \rightarrow x, t}} u_k^n = u$ (if and only if) $\lim_{n,k \rightarrow \infty} E |u_k^n - u|^2 = 0$

or $\lim_{\substack{\Delta x, \Delta t \rightarrow 0 \\ k \Delta x, n \Delta t \rightarrow x, t}} E |u_k^n - u|^2 = 0$

$$\begin{aligned} u_k^n - u &= \left[\frac{1}{2} e^{-\beta k \Delta x} \right] [(1-r) + (r e^{-\beta \Delta x})]^n - \frac{1}{2} e^{-\beta(x+t)} \\ |u_k^n - u|^2 &= \left| \left[\frac{1}{2} e^{-\beta k \Delta x} \right] [(1-r) + (r e^{-\beta \Delta x})]^n - \frac{1}{2} e^{-\beta t - \beta x} \right|^2 \\ &= \left[\left[\frac{1}{2} e^{-\beta k \Delta x} \right] [(1-r) + (r e^{-\beta \Delta x})]^n \right]^2 - \left[\frac{1}{2} e^{-\beta k \Delta x} \right] \\ &\quad [(1-r) + (r e^{-\beta \Delta x})]^n e^{-\beta t - \beta x} + \left[\frac{1}{2} e^{-\beta t - \beta x} \right]^2 \end{aligned}$$

$$E |u_k^n - u|^2 = E \left\{ \begin{array}{c} \left[\frac{1}{4} e^{-2\beta k \Delta x} \right] [(1-r) + (r e^{-\beta \Delta x})]^{2n} - \left[\frac{1}{2} e^{-\beta k \Delta x - \beta t - \beta x} \right] \\ \left[(1-r) + (r e^{-\beta \Delta x}) \right]^n + \left[\frac{1}{4} e^{-2\beta t - 2\beta x} \right] \end{array} \right\}$$

$$E |u_k^n - u|^2 = E \left\{ \begin{array}{c} \left[\frac{1}{4} e^{-2\beta k \Delta x} \right] \left[\left(1 - \frac{k \Delta t}{k \Delta x}\right) + \left(\frac{k \Delta t}{k \Delta x} e^{-\beta \Delta x}\right) \right]^{2n} - \left[\frac{1}{2} e^{-\beta k \Delta x - \beta t - \beta x} \right] \\ \left[\left(1 - \frac{k \Delta t}{k \Delta x}\right) + \left(\frac{k \Delta t}{k \Delta x} e^{-\beta \Delta x}\right) \right]^n + \left[\frac{1}{4} e^{-2\beta t - 2\beta x} \right] \end{array} \right\}$$

At time $t = (n+1)\Delta t$, then

$$E |u_k^n - u|^2 = E \left\{ \begin{array}{c} \left[\frac{1}{4} e^{-2\beta k \Delta x} \right] \left[\left(1 - \frac{k \Delta t}{k \Delta x}\right) + \left(\frac{k \Delta t}{k \Delta x} e^{-\beta \Delta x}\right) \right]^{2n} - \left[\frac{1}{2} e^{-\beta k \Delta x - \beta(n+1)\Delta t - \beta x} \right] \\ \left[\left(1 - \frac{k \Delta t}{k \Delta x}\right) + \left(\frac{k \Delta t}{k \Delta x} e^{-\beta \Delta x}\right) \right]^n + \left[\frac{1}{4} e^{-2\beta(n+1)\Delta t - 2\beta x} \right] \end{array} \right\}$$

as $\Delta t \rightarrow 0$, $\Delta x \rightarrow 0$, and $(k \Delta x, n \Delta t) \rightarrow (x, t)$ and $E(\beta) = 1$ then we obtain

$$E |u_k^n - u|^2 = 0,$$

then

$$u_k^n \xrightarrow{m,s} u$$

□

6. CONCLUSIONS AND FUTURE WORKS

The first order random partial differential equations can be solved numerically using the random difference method in mean square sense. Through some cases, the convergence of the solution scheme to the exact one is proved. The general theory is still absent and the applications to more complex differential equations are still waiting for development.

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