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GENERALIZED COMPOSITION OPERATORS ON $Q_K(p,q)$ SPACES

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Abstract. In this paper, we study generalized composition operators on α-Bloch and $Q_K(p,q)$ spaces. Moreover, we study boundedness and compactness of the generalized composition operator C_ϕ^g acting between two different Möbius invariant spaces $Q_{K_1}(p,q)$ and $Q_{K_2}(p,q)$.

1. Introduction

Let ϕ be an analytic self-map of the unit disk $\Delta = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$ and let $d\sigma(z)$ be the Euclidean area element on Δ. Associated with ϕ, the composition operator C_ϕ is defined by

$$C_\phi = f \circ \phi,$$

for f analytic on Δ. It maps analytic functions f to analytic functions. The problem of boundedness and compactness of C_ϕ has been studied in many function spaces. The first setting was in the Hardy space H^2, the space of functions analytic on Δ (see [10]). Madigan and Matheson (see [8]) gave a characterization of the compact composition operators on the Bloch space $\mathcal{B}$. Tjani (see [14]) gave a Carleson measure characterization of compact operators C_ϕ on Besov spaces B_p $(1 < p < \infty)$. Bourdon, Cima and Matheson in [4] and Smith in [11] investigated the same problem on $BMOA$. Li and Wulan in [6] gave a characterization of compact operators C_ϕ on Q_K and $F(p,q,s)$ spaces. Also, very recently in [1, 2], there are some characterizations for the composition operators C_ϕ in holomorphic $F(p,q,s)$ spaces. For $a \in \Delta$ the Möbius transformations $\varphi_a(z)$ is defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad \text{for } z \in \Delta.$$

The following identity is easily verified:

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2} = (1-|z|^2)|\varphi_a'(z)|. \tag{1}$$

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Note that $\varphi_a(\varphi_a(z)) = z$ and thus $\varphi_a^{-1}(z) = \varphi_a(z)$. For $a, z \in \Delta$ and $0 < r < 1$, the pseudo-hyperbolic disc $\Delta(a, r)$ is defined by $\Delta(a, r) = \{z \in \Delta : |\varphi_a(z)| < r\}$. Denote by

$$g(z, a) = \log \left| \frac{1 - \bar{a}z}{z - a} \right| = \log \frac{1}{|\varphi_a(z)|}$$

the Green's function of Δ with logarithmic singularity at $a \in \Delta$.

Definition 1.1. [17] *Let f be an analytic function in Δ and let $0 < \alpha < \infty$. If*

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z \in \Delta} (1 - |z|^2)^\alpha |f'(z)| < \infty,$$

then f belongs to the $\alpha-$Bloch space $\mathcal{B}^\alpha$. The space $\mathcal{B}^1$ is called the Bloch space $\mathcal{B}$.

Definition 1.2. [12, 13] *Let f be an analytic function in Δ and let $1 < p < \infty$. If*

$$\|f\|_{B_p}^p = \sup_{z \in \Delta} \int_\Delta |f'(z)|^p (1 - |z|^2)^{p-2} dA(z) < \infty,$$

then f belongs to the Besov space B_p.

In [16] Zhao gave the following definition:

Definition 1.3. *Let f be an analytic function in Δ and let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If*

$$\|f\|_{F(p,q,s)}^p = \sup_{a \in \Delta} \int_\Delta |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \infty,$$

then $f \in F(p, q, s)$. Moreover, if

$$\lim_{|a| \to 1} \int_\Delta |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) = 0,$$

then $f \in F_0(p, q, s)$.

The spaces $F(p, q, s)$ were intensively studied by Zhao in [16] and Rättyä in [9]. It is known from ([16], Theorem 2.10) that, for $p \geq 1$, the spaces $F(p, q, s)$ are Banach spaces under the norm

$$\|f\| = \|f\|_{F(p,q,s)} + |f(0)|.$$

Li and Stević in [7] defined the generalized composition operator C_ϕ^g as the follows:

$$(C_\phi^g)(z) = \int_0^z f'(\phi(\xi)) g(\xi) d\xi.$$

When $g = \phi'$, we see that this operator is essentially the composition operator C_ϕ. Therefore, C_ϕ^g is a generalization of the composition operator C_ϕ.
In this paper we study generalized compact composition operator on the spaces $Q_K(p, q)$, we will define and discuss properties of these spaces. A particular class of Möbius-invariant function spaces, the so-called Q_K spaces, has attracted a lot of attention in recent years.

Definition 1.4. *Let $K : [0, \infty) \to [0, \infty)$ be a right continuous and nondecreasing function in Δ. A function f in Δ is said to belong to the space Q_K if*

$$\|f\|_{Q_K}^2 = \left\{ f : f \text{ analytic in } \Delta \text{ and } \sup_{a \in \Delta} \int_\Delta \left| f'(z) \right|^2 K(g(z, a)) d\sigma(z) < \infty \right\}.$$

Through this paper, we assume that $K:[0,\infty)\to[0,\infty)$ is a right continuous and nondecreasing function. For $0<p<\infty, -2<q<\infty$, we say that a function f analytic in Δ belongs to the space $Q_K(p,q)$ if

$$\|f\|^p_{K,p,q} \quad = \quad \sup_{a\in\Delta}\int_\Delta \left|f'(z)\right|^p\left(1-|z|^2\right)^q K(g(z,a))d\sigma(z)<\infty \tag{2}$$

where $d\sigma(z)$ is the Euclidean area element on Δ. It is clear that $Q_K(p,q)$ is a Banach space with the norm $\|f\| = |f(0)| + \|f\|_{K,p,q}$ where $p\geq 1$. If $q+2=p$, $Q_K(p,q)$ is Möbius invariant, i.e., $\|f\circ\varphi_a\| = \|f\|_{K,p,q}$ for all $a\in\Delta$. Since every Möbius map φ can be written as $\varphi(z)=e^{i\theta}\varphi_a(z)$, where θ is real.
We assume throughout the paper that

$$\int_0^1(1-r^2)^q K(\log\frac{1}{r})rdr<\infty. \tag{3}$$

The author [15] collected the following immediate relations of $Q_K(p,q)$ and $Q_{K,0}(p,q)$
(i) $\quad Q_K(p,q)\subset\mathcal{B}^{\frac{q+2}{p}}$.
(ii) $\quad Q_K(p,q)=\mathcal{B}^{\frac{q+2}{p}}$ if and only if

$$\int_0^1(1-r^2)^{-2} K(\log\frac{1}{r})rdr<\infty.$$

(iii) $\quad F(p,q,0)=Q_K(p,q)$, if $K(0)>0$.
The following lemma is useful for our study (see [15]).

Lemma 1.1. *let $0<p<\infty, -2<q<\infty$, and $K:[0,\infty)\to[0,\infty)$. Then*
(A) $\quad$ *$f\in\mathcal{B}^{\frac{q+2}{p}}$ if and only if there exist $\rho\in(0,1)$ such that*

$$\sup_{a\in\Delta}\int_\Delta \left|f'(z)\right|^p\left(1-|z|^2\right)^q K(g(z,a))d\sigma(z)<\infty;$$

(B) $\quad$ *$f\in\mathcal{B}_0^{\frac{q+2}{p}}$ if and only if there exist $\rho\in(0,1)$ such that*

$$\lim_{|z|\to 1^-}\int_\Delta \left|f'(z)\right|^p\left(1-|z|^2\right)^q K(g(z,a))d\sigma(z)=0.$$

Recall that a linear operator $T:X\to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X and Y of the space of all analytic functions $H(\Delta)$, we call that T is compact from X to Y if and only if for each bounded sequence $\{x_n\}$ in X, the sequence $(Tx_n)\in Y$ contains a subsequence converging to some limit in Y.

2. Composition operators $C^g_\phi : Q_{K_1}(p,q)\to Q_{K_2}(p,q)$

In this section, we characterize boundedness and compactness of the generalized composition operator C^g_ϕ from $Q_{K_1}(p,q)$ spaces to $Q_{K_2}(p,q)$ spaces.
Now we are ready to state and prove the main results in this section.

Theorem 2.1. *Let $g\in H(\Delta)$ and ϕ be an analytic self-map of Δ. If $C^g_\phi(Q_{K_1}(p,q))\subset Q_{K_2}(p,q)$. Then*
$C^g_\phi : Q_{K_1}(p,q)\to Q_{K_2}(p,q)$ is compact if and only if

$$\lim_{t\to 1^-}\sup_{a\in\Delta}\int_{|\phi(z)|>t}|f'(\phi(z))\phi'(z)g(z)|^p(1-|z|^2)^q K(g(z,a))d\sigma(z)=0, \quad \textit{where } \ f\in\mathcal{B}_{Q_{K_1}(p,q)}. \tag{4}$$

Proof. First assume that (4) holds. To show that C_ϕ^g is compact we consider $\{f_n\} \subset \mathcal{B}_{Q_{K_1}(p,q)}$. It suffices to prove that $\{C_\phi^g f_n\}$ has a subsequence which converges in $Q_{K_2}(p,q)$. Since $f_n \subset Q_{K_1}(p,q) \subset \mathcal{B}^{\frac{q+2}{p}}$ (cf. [15]), for $z \in \Delta$

$$\begin{aligned} \Big|f_n(z) - f_n(0)\Big| &= \Big|\int_0^1 f'(zt)zdt\Big| \le \int_0^1 |f'(zt)||z|dt \\ &\le \|f_n\|_{\mathcal{B}^{\frac{q+2}{p}}} \int_0^1 \frac{|z|dt}{(1-t^2|z|^2)^{\frac{q+2}{p}}} \\ &\le C\|f_n\|_{\mathcal{B}^{\frac{q+2}{p}}} \\ &\le \frac{C}{\pi r^2 K(\log\frac{1}{r})}\|f\|_{Q_{K_1}(p,q)} . \end{aligned}$$

We know that $\{f_n\}$ is a normal family. By passing to a subsequence, we may assume, without loss of generality, that $\{f_n\}$ converges to 0 uniformly on compact subsets of Δ. We must show that $\{C_\phi^g f_n\}$ converges to 0 in the norm $\| \,.\, \|_{Q_{K_2}(p,q)}$. Given $\epsilon \in (0,1)$, by (4), there is a $t \in (0,1)$ such that for all functions f_n and for all $a \in \Delta$,

$$\int_{|\phi(z)|>t} |f_n'(\phi(z))\phi'(z)g(z)|^p(1-|z|^2)^q K_2(g(z,a))d\sigma(z) < \epsilon \tag{5}$$

By (4) and the fact that $\Delta_t = \{z \in \Delta : |z| \le t\}$ is a compact subset of Δ, we see that $\phi \in Q_{K_2}(p,q)$, since $z \in Q_{K_1}(p,q)$, and also that $\{f_n'\}$ converges to 0 uniformly on Δ_t. Therefore, there exists an integer $N > 1$ such that for $n \ge N$,

$$\int_{|\phi(z)|\le t} |f_n'(\phi(z))\phi'(z)g(z)|^p(1-|z|^2)^q K_2(g(z,a))d\sigma(z) < \epsilon\|\phi\|^p_{Q_{K_2}(p,q)} . \tag{6}$$

Thus (5) and (6) give

$$\int_{|\phi(z)|\le t} |f_n'(\phi(z))\phi'(z)|^p|g(z)|^p(1-|z|^2)^q K_2(g(z,a))d\sigma(z) < \epsilon(1+\|\phi\|^p_{Q_{K_2}(p,q)}) ,$$

when $n \ge N$. That is, $\|C_\phi^g f_n\|_{Q_{K_2}(p,q)} \to 0$ as $n \to \infty$.

Now suppose that $C_\phi^g : Q_{K_1}(p,q) \to Q_{K_2}(p,q)$ is compact. To verify (4) consider $f \in \mathcal{B}_{Q_{K_1}(p,q)}$ and let $f_s(z) = f(sz)$ for $s \in (0,1)$ and $z \in \Delta$. Note that $f_s \to f$ uniformly on compact subsets of Δ as $s \to 1$. By [3] we know that $\{f_s, 0 < s < 1\}$ is bounded in $Q_{K_1}(p,q)$. Since C_ϕ is compact, $\|C_\phi^g f_s - C_\phi f\|_{Q_{K_2}(p,q)} \to 0$ as $s \to 1$. That is, for given $\epsilon > 0$ there exists $s_0 \in (0,1)$ such that

$$\sup_{a\in\Delta}\int_\Delta |f_{s_0}'(\phi(z)) - f'(\phi(z))|^p|\phi'(z)|^p|g(z)|^p(1-|z|^2)^q K_2(g(z,a))d\sigma(z) < \epsilon.$$

For $t \in (0,1)$ and the above s_0 the triangle inequality gives

$$\begin{aligned} &\sup_{a\in\Delta}\int_{|\phi(z)|>t} |f'(\phi(z))|^p|\phi'(z)|^p|g(z)|^p(1-|z|^2)^q K_2(g(z,a))d\sigma(z) \\ &\le \epsilon + \|f_{s_0}'\|_\infty^p \sup_{a\in\Delta}\int_{|\phi(z)|>t} \big|\phi'(z)\big|^p|g(z)|^p\big(1-|z|^2\big)^q K_2(g(z,a))d\sigma(z). \end{aligned} \tag{7}$$

We know that

$$\sup_{a\in\Delta}\int_{|\phi(z)|>t}\left|\phi'(z)\right|^p|g(z)|^p\left(1-|z|^2\right)^q K_2(g(z,a))d\sigma(z)\le\|\phi\|^p_{Q_K(p,q)}<\infty$$

since $C^g_\phi(Q_{K_1}(p,q))\subset Q_{K_2}(p,q)$. It will be shown that for given $\epsilon>0$ and $\|f'_{s_0}\|^p_\infty>0$ there exists a $\delta\in(0,1)$ such that for $\delta<t<1$

$$\|f'_{s_0}\|^p_\infty\sup_{a\in\Delta}\int_{|\phi(z)|>t}\left|\phi'(z)\right|^p|g(z)|^p\left(1-|z|^2\right)^q K_2(g(z,a))d\sigma(z)<\epsilon.$$

Let $n=2^j, j=1,2,....$ Choose $h_n(z)=n^{\frac{-1}{2}}z^n$, and we know that $h_n\in\mathcal{B}^{\frac{q+2}{p}}$. It is easy to check that $\{h_n\}$ is a bounded family in $Q_{K_1}(p,q)$ since $\mathcal{B}^{\frac{q+2}{p}}\subseteq Q_{K_1}(p,q)$ (see [15]). Since C^g_ϕ is compact and h_n converges uniformly to 0 on compact subsets of Δ, we have

$$\lim_{n\to\infty}\|h_n\circ\phi\|_{Q_{K_2}(p,q)}=0.$$

Thus, for any given $\epsilon>0$, there exists an integer $N>1$ such that for all $a\in\Delta$

$$n\int_{|\phi(z)|>t}\left|\phi'(z)\right|^p|\phi(z)|^{pn-p}\left(1-|z|^2\right)^q|g(z)|^pK_2(g(z,a))d\sigma(z)<\epsilon \tag{8}$$

whenever $n\ge N$. Given $t\in(0,1)$, (8) yields

$$Nt^{pN-p}\int_{|\phi(z)|>t}\left|\phi'(z)\right|^p|g(z)|^p\left(1-|z|^2\right)^q K_2(g(z,a))d\sigma(z)<\epsilon \tag{9}$$

Taking $t=e^{-\frac{\log N}{P(N-1)}}$, we get

$$\|f'_{s_0}\|^p_\infty\sup_{a\in\Delta}\int_{|\phi(z)|>t}\left|\phi'(z)\right|^p\left(1-|z|^2\right)^q|g(z)|^pK_2(g(z,a))d\sigma(z)<\epsilon.$$

Hence by (7) and (8) we have already proved that for any $\epsilon>0$ and for $f\in\mathcal{B}_{Q_{K_1}}(p,q)$, there exists a $\delta=\delta(\epsilon,f)$ such that

$$\sup_{a\in\Delta}\int_{|\phi(z)|>t}\left|(f\circ\phi)'(z)g(z)\right|^p\left(1-|z|^2\right)^q K_2(g(z,a))d\sigma(z)<\epsilon$$

whenever $\delta<t<1$.
the above $\delta=\delta(\epsilon,f)$, in fact, is independent of $f\in\mathcal{B}_{Q_{K_1}(p,q)}$. Since $C^g_\phi:Q_{K_1}(p,q)\to Q_{K_2}(p,q)$ is compact, $C^g_\phi(\mathcal{B}_{Q_{K_1}(p,q)})$ is a relatively compact subset of $Q_{K_2}(p,q)$. It means that there is a finite collection of functions $f_1,f_2,...,f_n$ in $\mathcal{B}_{Q_{K_1}(p,q)}$ such that for any $\epsilon>0$ and $f\in\mathcal{B}_{Q_{K_1}(p,q)}$ there is a $k, 1\le k\le n$, satisfying

$$\sup_{a\in\Delta}\int_\Delta|f'(\phi(z))-f'_k(\phi(z))|^p|\phi'(z)|^p|g(z)|^p(1-|z|^2)^qK_2(g(z,a))d\sigma(z)<\epsilon\ . \tag{10}$$

On the other hand, if $\rho=\max_{1\le k\le n}\delta(\epsilon,f_k)<t<1$, we have from the previous observation that for all $k=1,2,...,n$,

$$\sup_{a\in\Delta}\int_{|\phi(z)|>t}|f'_k(\phi(z))|^p|\phi'(z)|^p|g(z)|^p(1-|z|^2)^qK_2(g(z,a))d\sigma(z)<\epsilon\ . \tag{11}$$

The triangle inequality, together with (10) and (11), gives

$$\sup_{a\in\Delta}\int_{|\phi(z)|>t}|(f\circ\phi)'(z)g(z)|^p(1-|z|^2)^qK_2(g(z,a))d\sigma(z)<2\epsilon$$

whenever $\rho < t < 1$. The proof is complete.
Although Theorem 2.1 can be viewed as a characterization of compact composition operators $C_\phi^g : Q_{K_1}(p,q) \to Q_{K_2}(p,q)$, by condition (4) it is not easy to check compactness of C_ϕ^g. The following theorem gives a characterization of C_ϕ^g directly in terms of ϕ.

Theorem 2.2. *Let $g \in H(\Delta)$, ϕ be an analytic self-map of Δ and $C_\phi^g : Q_{K_1}(p,q) \subset Q_{K_2}(p,q)$. Let two functions $K_1, K_2 : [0,\infty) \to [0,\infty)$ be right-continuous and nondecreasing, satisfying*

$$\int_0^1 (1-r^2)^{-2} K_1(\log \frac{1}{r}) r dr < \infty. \tag{12}$$

If

$$\lim_{t\to 1^-} \sup_{a\in\Delta} \int_{|\phi(z)|>t} \frac{|\phi'(z)|^p |g(z)|^p}{(1-|\phi(z)|^2)^{2p}} (1-|z|^2)^q K_2(g(z,a)) d\sigma(z) = 0. \tag{13}$$

Then, $C_\phi^g : Q_{K_1}(p,q) \to Q_{K_2}(p,q)$ is compact. Conversely, if $C_\phi^g : Q_{K_1}(p,q) \to Q_{K_2}(p,q)$ is compact, then (13) holds.

Proof. Consider $\{f_n\} \in \mathcal{B}_{Q_{K_1}(p,q)}$ which converges to 0 uniformly on compact subsets of Δ. We must show that $\{C_\phi^g f_n\}$ converges to 0 in the norm $\| \, . \, \|_{Q_K(p,q)}$. Thus

$$\begin{aligned}
\|C_\phi^g f_n\|_{Q_K(p,q)}^p &= \sup_{a\in\Delta} \int_\Delta \big|(f\circ\phi)'(z)g(z)\big|^p \big(1-|z|^2\big)^q K_2(g(z,a)) d\sigma(z) \\
&= \sup_{a\in\Delta} \bigg(\int_{|\phi(z)|\le t} + \int_{|\phi(z)|>t} \bigg) \big|f_n'(\phi)(z)\big|^p |\phi'(z)|^p |g(z)|^p \big(1-|z|^2\big)^q K_2(g(z,a)) d\sigma(z) \\
&\le \sup\{|f_n'(w)g(w)|^p : |w|\le t\} \|\phi\|_{Q_{K_2}(p,q)}^p \\
&+ \ const. \|f_n\|_{\mathcal{B}^{\frac{q+2}{p}}}^p \int_{|\phi(z)|>t} | \frac{|\phi'(z)g(z)|^p}{(1-|\phi(z)|^2)^{2p}} K_2(g(z,a)) d\sigma(z) = I_1 + I_2.
\end{aligned}$$

Since $\{f_n\}$ converges to 0 uniformly on compact sets and $\phi \in Q_{K_2}(p,q)$, we have $I_1 \to 0$ as $n \to \infty$. In the second term I_2 we know that

$$\|f_n\|_{\mathcal{B}^{\frac{q+2}{p}}}^p \le C \|f_n\|_{Q_{K_1}(p,q)}^p$$

since every function in $Q_{K_1}(p,q)$ must be $\frac{q+2}{p}-$Bloch. Thus, I_2 goes to 0 when $t \to 1$ by our assumption. Therefore, C_ϕ^g is compact.
Conversely, let $C_\phi^g : Q_{K_1}(p,q) \to Q_{K_2}(p,q)$ be compact. By [5] we know that (12) ensures

$$f_\theta(z) = \log \frac{1}{1-e^{-i\theta}z} \in Q_{K_1}(p,q) \quad \text{for all } \ \theta \in [0,2\pi).$$

By Theorem 2.1,

$$\lim_{t\to 1^-} \int_{|\phi(z)|>t} \frac{|\phi'(z)|^p |g(z)|^p}{(1-|\phi(z)|^2)^{2p}} K_2(g(z,a)) d\sigma(z) = 0$$

holds for all $a \in \Delta$ and $\theta \in [0,2\pi)$. Thus, we obtain (13) by integrating with respect to θ, the Fubini theorem and the Poisson formula.

3. Composition operators $C_\phi^g : Q_{K_1}(p,q) \to Q_{K_2,0}(p,q)$

In this section, we consider compactness of the generalized composition operators $C_\phi^g : Q_{K_1}(p,q) \to Q_{K_2,0}(p,q)$, where $Q_{K,0}(p,q)$ is a subspace of $Q_K(p,q)$ satisfying

$$\lim_{|a|\to 1^-} \int_\Delta \left|f'(z)\right|^p \left(1-|z|^2\right)^q K(g(z,a))d\sigma(z) = 0.$$

By [15], we know that $Q_{K,0}(p,q) \subset \mathcal{B}_0^{\frac{q+2}{p}}$ and that $Q_{K,0}(p,q) = \mathcal{B}_0^{\frac{q+2}{p}}$ if and only if

$$\int_0^1 (1-r^2)^{-2} K(\log\frac{1}{r}) r dr < \infty.$$

We should mention that the generalized composition operator C_ϕ^g is compact from $Q_{K_1}(p,q)$ to $Q_{K_2,0}(p,q)$ if $\phi \in Q_{K_2,0}(p,q)$ and C_ϕ^g is compact from $Q_{K_1}(p,q)$ to $Q_{K_2}(p,q)$.

Theorem 3.1. *Let $g \in H(\Delta)$ and ϕ be an analytic self-map of Δ such that*

$$C_\phi^g(Q_{K_1}(p,q)) \subset Q_{K_2,0}(p,q).$$

Then $C_\phi^g : Q_{K_1}(p,q) \to Q_{K_2,0}(p,q)$ is compact if and only if

$$\lim_{|a|\to 1^-} \sup_{\|f\|_{Q_{K_1,p,q}}<1} \int_\Delta \left|f'(\phi(z))\right|^p |\phi'(z)|^p |g(z)|^p (1-|z|^2)^q K_2(g(z,a))d\sigma(z) = 0. \quad (14)$$

Proof. First suppose that $C_\phi^g : Q_{K_1}(p,q) \to Q_{K_2,0}(p,q)$ is compact. Then $A = cl(\{(f\circ\phi)g \in Q_{K_2,0}(p,q) : \|f\|_{Q_{K_1,p,q}} < 1\})$, the $Q_{K_2,0}(p,q)$ closure of the image under C_ϕ^g of the unit ball of $Q_{K_1}(p,q)$, is a compact subset of $Q_{K_2,0}(p,q)$. For given $\epsilon > 0$, since a compact set in a metric space is completely bounded, there exist $f_1, f_2, ..., f_N \in Q_{K_1}(p,q)$ such that each function f in A lies at most ϵ distant from

$$B = \{(f_1\circ\phi)g, (f_2\circ\phi)g, (f_3\circ\phi)g, ..., (f_N\circ\phi)g\}.$$

That is, there exists $j \in J = \{1,2,...,N\}$ such that

$$\|(f\circ\phi)g - (f_j\circ\phi)g\|_{Q_{K_2}(p,q)} < \frac{\epsilon}{4}. \quad (15)$$

On the other hand, since $\{(f_j\circ\phi)g : j \in J\} \subset Q_{K,0}(p,q)$, there exists a $\delta > 0$ such that for all $j \in J$ and $|a| > 1-\delta$,

$$\int_\Delta \left|(f_j\circ\phi)'(z)g(z)\right|^p (1-|z|^2)^q K_2(g(z,a))d\sigma(z) < \frac{\epsilon}{4}. \quad (16)$$

Therefore by (15) and (16), we obtain that for each $|a| > 1-\delta$ and $f \in Q_{K_1}(p,q)$ with $\|f\|_{Q_{K_1,p,q}} < 1$ there exists $j \in J$ such that

$$\begin{aligned} &\int_\Delta \left|(f\circ\phi)'(z)g(z)\right|^p (1-|z|^2)^q K_2(g(z,a))d\sigma(z) \\ &\le 2\int_\Delta \left|(f\circ\phi - f_j\circ\phi)'(z)g(z)\right|^p (1-|z|^2)^q K_2(g(z,a))d\sigma(z) \\ &+ \int_\Delta \left|(f_j\circ\phi)'(z)g(z)\right|^p (1-|z|^2)^q K_2(g(z,a))d\sigma(z) < \epsilon. \end{aligned}$$

This proves (14).

Now let (14) hold and let $\{f_n\}$ be a sequence in the unit ball of $Q_{K_1}(p,q)$. By Montel's theorem, there exists a subsequence $\{f_{n_k}\}$ which converges to a function

f analytic in Δ and both $f_{n_k} \to f$ and $f'_{n_k} \to f'$ uniformly on compact subsets of Δ. By hypothesis and Fatou's lemma, we see that $C_\phi^g \in Q_{K_2,0}(p,q)$. Since $z \in Q_{K_1}(p,q)$, $\phi \in Q_{K_2,0}(p,q)$. Thus we remark that C_ϕ^g is a compact composition operator by showing that

$$\|C_\phi^g(f_{n_k} - f)\|_{Q_{K_2}(p,q)} \to 0 \quad \text{as} \quad k \to \infty.$$

In order to simplify the notation we additionally assume, without loss of generality, that $f = 0$. Hence it remains to show that

$$\lim_{|n|\to\infty} \|C_\phi^g f_n\|_{Q_{K_2}(p,q)} = 0.$$

Let $\epsilon > 0$. By (14), we can choose $r \in (0,1)$ for all n,

$$\sup_{r<|a|<1} \int_\Delta \left|(f \circ \phi)'(z) g(z)\right|^p (1 - |z|^2)^q K_2(g(z,a)) d\sigma(z) < \epsilon. \tag{17}$$

For $a \in \Delta$ and $t \in (0,1)$, define $t\Delta = \{z \in \Delta : |z| \le t\}$ and set

$$I_t(a) = \int_{\Delta \setminus t\Delta} \left|(f \circ \phi)'(z) g(z)\right|^p (1 - |z|^2)^q K_2(g(z,a)) d\sigma(z).$$

By using the same way as in [6] we know that for each $t \in (0,1)$, $I_t(a)$ is a continuous function of a. Since

$$\int_\Delta \left|(f_n \circ \phi)'(z) g(z)\right|^p (1 - |z|^2)^q K_2(g(z,a)) d\sigma(z) < \infty$$

for each $a \in \Delta$, we can choose $t(a) \in (r,1)$ such that $I_{t(a)}(a) < \frac{\epsilon}{2}$. Moreover, there is a neighborhood $U(a) \subset \Delta$ of a such that $I_{t(a)}(b) < \epsilon$ for every $b \in U(a)$, by the continuity of $I_t(a)$. Thus, using the compactness of $\{a : |a| \le r\}$, there exists $t_0 \in (0,1)$ such that $I_{t_0}(a) < \epsilon$ if $|a| \le r$, and so

$$\sup_{|a|\le r} \int_{\Delta \setminus t_0\Delta} \left|(f_n \circ \phi)'(z) g(z)\right|^p (1 - |z|^2)^q K_2(g(z,a)) d\sigma(z) < \epsilon. \tag{18}$$

Also, by the uniform convergence of $\{(f'_n \circ \phi) g\}$ to 0 on compact subsets of Δ, there exists N such that,

$$\int_{t_0\Delta} \left|(f_n \circ \phi)'(z) g(z)\right|^p (1 - |z|^2)^q K_2(g(z,a)) d\sigma(z) < \epsilon,$$

if $n \ge N$. Thus, for any such n, we have

$$\sup_{|a|\le r} \int_\Delta \left|(f_n \circ \phi)'(z) g(z)\right|^p (1 - |z|^2)^q K_2(g(z,a)) d\sigma(z) < 2\epsilon. \tag{19}$$

Combining (17) and (19), we obtain that

$$\lim_{|n|\to\infty} \|C_\phi^g f_n\|_{Q_{K_2}(p,q)} = 0.$$

The proof of Theorem 3.1 is complete.

Theorem 3.2. *Let $g \in H(\Delta)$ and ϕ be an analytic self-map of Δ such that*

$$C_\phi^g(Q_{K_1}(p,q)) \subseteq Q_{K_2,0}(p,q).$$

Assume that

$$\int_0^1 (1 - r^2)^{-2} K_1(\log \frac{1}{r}) r dr < \infty. \tag{20}$$

If

$$\lim_{|a|\to 1^-}\int_{\Delta}\frac{|\phi'(z)|^p|g(z)|^p}{(1-|\phi(z)|^2)^{2p}}(1-|z|^2)^q K_2(g(z,a))d\sigma(z)=0, \tag{21}$$

then $C_\phi^g : Q_{K_1}(p,q) \to Q_{K_2,0}(p,q)$ *is compact. Conversely assume that* $C_\phi^g : Q_{K_1}(p,q) \to Q_{K_2,0}(p,q)$ *is compact, (21) holds.*

Proof. The proof is very similar as the proof of Theorem 2.2, so it will be omitted.

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