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SPLINE SOLUTION FOR FOURTH ORDER FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. Recently, a large number of applied problems have been formulated on fractional differential equations. Analytical solution of many applications, where the fractional differential equations appear, cannot be established. Therefore, quintic polynomial spline function is considered to find approximate solution for a class of two point fourth order integro-differential equation of fractional order. Convergence analysis of the method is considered. Some illustrative examples are included to demonstrate the practical usefulness of the proposed method.

1. Introduction

In the last few decades, it has been shown that many phenomena cannot be described within the framework of the classical theory using integer order derivatives and there has been a significant interest in fractional differential equations. It is caused both by the intensive development of the theory of fractional calculus and by the application of such constructions in various sciences[1,3,5,6,12] such as electrical circuits , biology, control theory , viscoelasticity, fitting of experimental data, electromagnetic acoustic and material science. For details refer to [1,2-3,13-14,17-18,21]. Boundary value problems of fractional order occur in the description of many physical processes of stochastic transport and in the investigation of liquid filtration in a strongly porous medium [20]. Also, boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They occur also in the mathematical model which is developed for a micro-electro-mechanical system (MEMS) instrument that has been designed primarily to measure the viscosity of fluids that are encountered during oil well exploration [6].

Analysis and design of many systems require solution of fractional differential equations (FDEs). Several methods have recently been proposed to obtain the analytical solution of these equations. These methods include Laplace and Fourier transforms, eigenvector expansion, direct solution based on Grunewald Letnikov

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approximation, truncated Taylor series expansion and power series method [5,9-10,12,14-16]. Also, several algorithms have been developed to solve FDEs numerically such as fractional Adams–Moulton methods, explicit Adams multistep methods, fractional difference method, decomposition method, variational iteration method , least squares finite element solution and extrapolation method [4,7-8,15,20]. In [25], the authors considered the numerical solution of the fractional boundary value problem (FBVP) $D^{-\alpha}y''(x)+p(x)\,y=g(x)$, $0\le\alpha<1$, $x\in[a,b]$,with Dirichlet boundary conditions using quadratic polynomial spline.

In this paper, we consider the numerical solution of the following fractional integro-differential boundary value problem (FIDBVP):

$$D^{-\alpha}y^{(4)}(x)+\eta y(x)+\mu\int_0^x k(t)y(t)dt=g(x)\,,\ 0\le\alpha<1\,,\ \forall x\in[a,b], \tag{1}$$

with the boundary conditions:

$$y(a)=A_1\ ,\ y(b)=A_2\ ,\ \ y''(a)=B_1\ \text{ and } y''(b)=B_2. \tag{2}$$

where the functions $k(x)$ and $g(x)$ are continuous on the interval $[a,b]$, η and μ are constants. The operator D^{α} represents the Caputo fractional derivative. The analytical solution of (1.1-1.2) cannot be obtained for arbitrary choices of $k(x)$ and $g(x)$. When $\alpha=0$, Eq. (1) is reduced to the classical fourth order integro-differential equation.

The main objective of this work is to use polynomial spline function for solving the FBVP (1.1-1.2). This approach has its own advantages. For example, once the solution has been computed, the information needed for spline interpolation between mesh points is available. This is important when the solution of the boundary value problem is required at different locations in the interval $[a,b]$. This approach has added advantage that it not only provides continuous approximations to $y(x)$, but also for $y^{(j)}(x)$, $j=1,2,3,4$ at every point of the range of integration [22-25].

This paper is organized as follows: In section 2, we introduce some definitions and theorems necessary to our work. Derivation of our method is established in section 3. Convergence analysis of the new method is presented in section 4. In section 5, numerical results are included to show the applications and advantages of our method.

2. Preliminaries

In this section, definitions of fractional derivative and integral, used in our work, will be presented. There are different definitions for fractional derivatives, the most commonly used ones are the Riemann-Liouville and the Caputo derivatives.
Let $f(x)$ be a function defined on(a,b), then
Definition 1 [12] The Riemann-Liouville fractional derivative:

$$^{R}D^{\alpha}f(x)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_0^x(x-t)^{m-\alpha-1}f(t)dt\,,\ \alpha>0,\ \ m-1<\alpha<m,$$

where Γ is the gamma function.

Definition 2 [12] The Riemann-Liouville fractional integral:

$$D_a^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt\,, \;\; \alpha > 0.$$

Definition 3 [2] The Caputo fractional derivative:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^{(m)}(s) ds\,, \;\; \alpha > 0, \;\; m-1 < \alpha < m.$$

The relation between the Riemann–Liouville operator and Caputo operator is given by:

$$D^{\alpha} f(x) = {}^{R}D^{\alpha}[f(x) - \sum_{k=0}^{m-1} \frac{1}{k!}\,(x-a)^k f^{(k)}(a)]\,, \quad \alpha > 0, \;\; m-1 < \alpha < m.$$

Lemma 1 [12] If $f(x)$ is continuous and $\alpha, \beta > 0$, then the following relationships hold:

(1) ${}^{R}D^{\alpha}(D^{-\beta} f(x)) = {}^{R}D^{\alpha-\beta} f(x)$
(2) $D^{-\alpha} D^{-\beta} f(x) = D^{-\beta} D^{-\alpha} f(x) = D^{-\alpha-\beta} f(x)$
(3) $D^{-\alpha} x^m = \frac{\Gamma(m+1)}{\Gamma(m+\alpha+1)} x^{m+\alpha}$
(4) $D^{-\alpha} \exp(ax) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \exp(at)\, dt \;\; = x^{\alpha} \exp(ax) \gamma(\alpha, ax)$,

where, $\gamma(\alpha, ax) = \frac{1}{x^{\alpha}\Gamma(\alpha)} \int_0^x t^{\alpha-1} \exp(-a\,t)\, dt$, is called incomplete gamma function.

Theorem 1[17] Let $f \in C^m[0,1]$ and $\alpha \in (m-1, m)$,$m \in N$ and $g \in C[0,1]$. Then for$x \in [0,1]$:

(1) $D^{\alpha} D^{-\alpha} g(x) = g(x)$
(2) $D^{-\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} \frac{x^k}{k!} f^{(k)}(0)$
(3) $\lim\limits_{x \to 0} D^{\alpha} f(x) = \lim\limits_{x \to 0} D^{-\alpha} f(x) = 0$
(4) If $\alpha_i \in (0,1], i = 1, 2, ..., n$ with $\alpha = \sum_{i=1}^{n} \alpha_i$are such that, for each $k = 1, 2, ..., m-1$, there exist $i_k < n$ with $\sum_{j=1}^{i_k} \alpha_j = k$, then the following composition rule holds: $D^{\alpha} f(x) = D^{\alpha_n} ... D^{\alpha_2} D^{\alpha_1} f(x)$.

3. Spline solution for fourth order fractional integro-differential equations

In order to develop a spline approximation for the fourth order fractional integro-differential equation (1) along with the boundary condition (2), we, firstly, use theorem 1 [17] to convert the FIBVPs given by Eq.(1) into the following form:

$$y^{(4)}(x) + \eta D^{\alpha} y(x) + \mu D^{\alpha} \int_0^x k(t) y(t) dt = \tilde{g}(x)\,, \forall x \in [a,b], \tag{3}$$

where $\tilde{g}(x) = D^{\alpha} g(x)$.

Now we introduce a finite set of grid points x_i by dividing the interval $[a,b]$ into n-equal parts.

$$x_i = a + ih\,, \;\; x_0 = a\,, \;\; x_n = b, \;\;\; h = \tfrac{b-a}{n}\;, \;\;\; i = 0, 1, 2, ..., n. \tag{4}$$

Let $y(x)$ be the exact solution of (1) and S_i be an approximation to $y_i = y(x_i)$ obtained by the spline function $P_i(x)$ passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}).
Consider that each quintic polynomial spline segment $P_i(x)$ has the form, see [22]:

$$P_i(x) = a_i(x - x_i)^5 + b_i(x - x_i)^4 + c_i(x - x_i)^3 + d_i(x - x_i)^2 + e_i(x - x_i) + f_i \qquad i = 0, 1, 2, ..., n-1\,, \tag{5}$$

where a_i, b_i, c_i, d_i, e_i and f_i are constants to be determined. The quintic spline $P_i(x)$ satisfies the conditions:

$$\begin{aligned} &(i) P_i(x) \in C^4[a, b], \\ &(ii)\, S(x) = P_i(x),\ \ x \in [x_i, x_{i+1}]\,,\ i = 0, 1, 2, ..., n-1. \end{aligned} \tag{6}$$

We express the six coefficients in Eq.(5) in terms of $S_i,\ S_{i+1}, M_i, M_{i+1}, F_i$ and F_{i+1} where:

$$\begin{gathered} P_i(x_i) = S_i,\ P_i(x_{i+1}) = S_{i+1},\ \ P_i^{(2)}(x_i) = M_i,\ P_i^{(2)}(x_{i+1}) = M_{i+1}, \\ P_i^{(4)}(x_i) = F_i,\ P_i^{(4)}(x_{i+1}) = F_{i+1}. \end{gathered} \tag{7}$$

Thus we obtain:

$$a_i = \frac{1}{120h}(F_{i+1} - F_i), b_i = \frac{1}{24}F_i, c_i = \frac{1}{6h}(M_{i+1} - M_i) - \frac{h}{36}(F_{i+1} + 2F_i), d_i = \frac{1}{2}M_i,$$

$$e_i = \frac{1}{h}(S_{i+1} - S_i) + \frac{h^3}{360}(7F_{i+1} + 8F_i) - \frac{h}{6}(M_{i+1} + 2M_i), f_i = S_i. \tag{8}$$

Now apply the continuity conditions and using Eq.(8), we get the following two relation respectively, see [25]:

$$M_{i+1} + 4M_i + M_{i-1} = \frac{6}{h^2}(S_{i+1} - 2S_i + S_{i-1}) + \frac{h^2}{60}(7F_{i+1} + 16F_i + 7F_{i-1}), \tag{9}$$

and

$$M_{i+1} - 2M_i + M_{i-1} = \frac{h^2}{6}(F_{i+1} + 4F_i + F_{i-1}), \tag{10}$$

From Eqs.(9) and (10) we can deduce that:

$$M_i = \frac{1}{h^2}(S_{i+1} - 2S_i + S_{i-1}) - \frac{h^2}{360}(3F_{i+1} + 24F_i + 3F_{i-1}),\ \ i = 1, 2, 3, ..., n-1. \tag{11}$$

Then substituting from Eq.(11) into Eq.(10) we get:

$$S_{i+2} - 4S_{i+1} + 6S_i - 4S_{i-1} + S_{i-2} = \frac{h^4}{360}[3F_{i+2} + 78F_{i+1} + 198F_i + 78F_{i-1} + 3F_{i-2}, \qquad i = 2, 3, ..., n-2\,. \tag{12}$$

Where $F_i$is determined as:

$$F_i = \tilde{g}_i - \eta D^\alpha\, S(x)|_{\,x = x_i} - \mu D^\alpha \int_0^{x_i} k(t) S(t) dt\ \ ,\ i = 0, 1, 2, ..., n, \tag{13}$$

where $\tilde{g}_i = \tilde{g}(x_i)$, Eq.(12) gives $n-3$ linear algebraic equations in $n-1$ unknown. We need two more equations one at each end. Following [36], the two end conditions are:

$$S_3-4S_2+5S_1=2S_0+h^2M_0+\frac{h^4}{360}[3F_3+78F_2+195F_1+54F_0], \tag{14}$$

and

$$S_{n-3}-4S_{n-2}+5S_{n-1}=2S_n+h^2M_n+\frac{h^4}{360}[3F_{n-3}+78F_{n-2}+195F_{n-1}+54F_n]. \tag{15}$$

Lemma 3 Let $y\in\ C^6[a,b]$ then the local truncation errors $t_i,\ i=1,2,...,n-1$ associated with the scheme (12), (14) and (15) are:

$$t_i=\begin{cases}\frac{-28}{360}h^6y_0^{(6)}+O(h^8), & i=1\\ \frac{-24}{360}h^6y_i^{(6)}+O(h^8), & i=2,3,...,n-2\\ \frac{-28}{360}h^6y_n^{(6)}+O(h^8), & i=n-1\,.\end{cases} \tag{16}$$

Proof To obtain the local truncation errors $t_i,\ i=1,2,...,n-1$ of Eqs.(12) , (14) and (15), we may refer to[22].

Returning to Eq.(13), we use the Grunewald definition of the fractional derivative for discretizing the fractional terms $D^\alpha\ S(x)|_{\ x=x_i}$ and $D^\alpha\int_0^{x_i}k(t)S(t)dt\ \ ,i=0,1,2,...,n$, in order to obtain a numerical solution for Eq.(1). The Grunewald definition for fractional derivative is:

$${}^GD^\alpha y(x)=\lim_{N\to\infty}\frac{1}{h^\alpha}\sum_{k=0}^{N}g_{\alpha,k}\,y(x-kh), \tag{17}$$

Where the Grunewald weights are:

$$g_{\alpha,k}=\frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\,\Gamma(k+1)}. \tag{18}$$

These normalized weights depend only on the fractional order α and the index k. We have that: $g_{\alpha,0}=1,\ g_{\alpha,1}=-\alpha$ and

$$g_{\alpha,k}=\frac{(-\alpha)(-\alpha+1)...(-\alpha+k-1)}{k!},\ \ \forall k\geq 2. \tag{19}$$

It is well known that:

$$(1+z)^p=\sum_{k=0}^{\infty}\left(\begin{array}{c}p\\k\end{array}\right)\ z^k\,,\qquad \forall\,|z|\leq 1\ ,p>0\,, \tag{20}$$

where

$$\left(\begin{array}{c}p\\k\end{array}\right)=\frac{(-1)^k\Gamma(k-p)}{\Gamma(-p)\,\Gamma(k+1)}. \tag{21}$$

Then for $z=-1$, we have

$$\sum_{k=0}^{\infty}\frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\,\Gamma(k+1)}=0. \tag{22}$$

Then from the above we can approximate the fractional term $D^\alpha y(x_i)\ ,i=0,1,2,...,n$ by:

$$D^\alpha\ S(x)|_{\ x=x_i}=\frac{1}{h^\alpha}\sum_{k=0}^{i}g_{\alpha,k}\,S(x-kh)\ \ ,i=0,1,2,...,n. \tag{23}$$

Also, we use the Grunewald definition of the fractional derivative for discretizing the fractional term $D^\alpha \int_0^{x_i} k(t)S(t)dt$, $i = 0, 1, 2, ..., n$ as follows:
Let

$$D^\alpha I(x) = D^\alpha \int_0^x k(t)S(t)dt, \tag{24}$$

then:

$$D^\alpha I(x_i) = \frac{1}{h^\alpha} \sum_{k=0}^{i} g_{\alpha,k}\, I(x - kh) \ \ , i = 0, 1, 2, ..., n. \tag{25}$$

For $x \in [x_{i-1}, x_i]$, we can use the trapezoidal rule to approximate the integration $I(x_i)$. Then we have:

$$I(x_i) = \frac{h}{2}(k(x_i)S(x_i) + k(x_{i-1})S(x_{i-1})),\ i = 1, 2, ..., n. \tag{26}$$

Note that $I(x_0) = 0$.
Then from Eqs.(13) , (23) and (26) we can get the value of F_i , $i = 0, 1, 2, ..., n$.
Remark 1:

The above technique can be used to develop a spline approximation for the following fourth order fractional differential equation:

$$y^{(4)}(x) + (\eta D^\alpha + \mu)y(x) = g(x)\, , m - 1 < \alpha < m\ , \forall x \in [a, b]. \tag{27}$$

with the boundary conditions given by Eq.(2), $m = 1, 2$.
In this case, the value of F_i is determined from Eq.(27) and have :

$$F_i = g_i - \mu S_i - \eta D^\alpha\ S(x)|_{\ x=x_i}\ , i = 0, 1, 2, ..., n. \tag{28}$$

4. Convergence analysis of the method

In the following let $Y = (y_i)$, $S = (S_i)$, $C = (C_i)$, $T = (t_i)$ and $E = (e_i) = Y - S$ be (n-1) dimensional column vectors, where Y, S, T and E are the exact, approximate, truncation error and error column vectors respectively.

We can write the system given by (12) and the end formulas determined by (14) and (15) as follows:

$$NS = h^4 BF + C, \tag{29}$$

where the matrices N, B and the vector C are given below:

$$N = \begin{pmatrix} 5 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ & 1 & -4 & 6 & -4 & 1 & \\ & & & \ddots & & & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 5 \end{pmatrix}, \tag{30}$$

$$B=\frac{1}{360}\begin{pmatrix} 195 & 78 & 3 & & & & \\ 78 & 198 & 78 & 3 & & & \\ 3 & 78 & 198 & 78 & 3 & & \\ & & & \ddots & & & \\ & & 3 & 78 & 198 & 78 & 3 \\ & & & 3 & 78 & 198 & 78 \\ & & & & 3 & 78 & 195 \end{pmatrix}, \tag{31}$$

and

$$C=\begin{pmatrix} 2A_1+\frac{54}{360}h^4F_0-h^2B_1 \\ -A_1+\frac{3}{360}h^4F_0 \\ \vdots \\ -A_2+\frac{3}{360}h^4F_n \\ 2A_2+\frac{54}{360}h^4F_n-h^2B_2 \end{pmatrix}. \tag{32}$$

From Eq.(13) the vector F can be written as

$$F=G_1-\eta h^{-\alpha}(GS+G_0)-\tfrac{1}{2}\mu h^{1-\alpha}(GLS-GL_0), \tag{33}$$

where the vectors G_1, G_0, L_0 and the matrices G and L are given below respectively:

$$G_1=\begin{bmatrix} \tilde{g}_1 & \tilde{g}_2 & \cdots & \tilde{g}_{n-2} & \tilde{g}_{n-1} \end{bmatrix}^t, \tag{34}$$

$$G_0=A_1\begin{bmatrix} g_{\alpha,1} & g_{\alpha,2} & \cdots & g_{\alpha,n-2} & g_{\alpha,n-1} \end{bmatrix}^t, \tag{35}$$

$$L_0=\begin{bmatrix} A_1k(a) & 0 & \cdots & 0 \end{bmatrix}^t, \tag{36}$$

$$G=\begin{pmatrix} g_{\alpha,0} & & & & & \\ g_{\alpha,1} & g_{\alpha,0} & & & & \\ g_{\alpha,2} & g_{\alpha,1} & g_{\alpha,0} & & & \\ \vdots & \vdots & & \ddots & & \\ g_{\alpha,n-3} & g_{\alpha,n-4} & \cdots & g_{\alpha,1} & g_{\alpha,0} & \\ g_{\alpha,n-2} & g_{\alpha,n-3} & \cdots & g_{\alpha,2} & g_{\alpha,1} & g_{\alpha,0} \end{pmatrix}, \tag{37}$$

and

$$L=\begin{pmatrix} k_1 & & & & & \\ k_1 & k_2 & & & & \\ & k_2 & k_3 & & & \\ & & & \ddots & & \\ & & & k_{n-3} & k_{n-2} & \\ & & & & k_{n-2} & k_{n-1} \end{pmatrix}. \tag{38}$$

Substituting from Eq.(32) into Eq.(29) we get:

$$(N+\tfrac{1}{2}\mu h^{5-\alpha}BGL+\eta h^{4-\alpha}BG)S=h^4B(G_1-h^{-\alpha}G_0-\tfrac{1}{2}\mu h^{1-\alpha}GL_0)+C. \tag{39}$$

and

$$(N+\tfrac{1}{2}\mu h^{5-\alpha}BGL+\eta h^{4-\alpha}BG)Y=h^4B(G_1-h^{-\alpha}G_0-\tfrac{1}{2}\mu h^{1-\alpha}GL_0)+C+T. \tag{40}$$

Then the error equation can be written as:

$$(N + \tfrac{1}{2}\mu h^{5-\alpha} BGL + \eta h^{4-\alpha} BG)E = T. \tag{41}$$

Our aim is to drive a bound on $\|E\|$ (the infinite norm). In order to achieve this, we need the following lemma.

Lemma 4 [11,19] If M is square matrix of order n and $\|M\| < 1$, then $(I + M)^{-1}$ exists and $\left\|(I + M)^{-1}\right\| < 1/(1 - \|M\|)$.

Rewrite the error equation Eq.(40), we get

$$E = (I + \tfrac{1}{2}\mu h^{5-\alpha} N^{-1} BGL + \eta h^{4-\alpha} N^{-1} BG)^{-1} N^{-1} T, \tag{42}$$

Using Lemma 4, we get

$$\|E\| \leq \frac{\left\|N^{-1}\right\| \|T\|}{1 - \|N^{-1}\| \left[\bar{\mu} h^{5-\alpha} \|B\| \|G\| \|U\| + \eta h^{4-\alpha} \|B\| \|G\|\right]}. \tag{43}$$

Provided that $\left\|N^{-1}\right\| \left[\bar{\mu} h^{5-\alpha} \|G\| \|L\| + \eta h^{4-\alpha} \|G\|\right]$ <1, where $\bar{\mu} = \frac{\mu}{2}$and $\|B\| = 1$ and

$$\|G\| = \sum_{i=0}^{n-2} |g_{\alpha,i}| \,. \tag{44}$$

We have that:

(1) When $0 < \alpha < 1$, we have $g_{\alpha,0} = 1$ and $g_{\alpha,i} < 0$ $\forall i$ and $i \neq 0$. Then from Eq.(18), we get that $\sum_{k=1}^{\infty} \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\,\Gamma(k+1)} = -1$ which leads to $\|G\| \leq 2$.

(2) When $1 < \alpha < 2$, we have $g_{\alpha,1} = -\alpha$ and $g_{\alpha,i} > 0$ $\forall i$ and $i \neq 1$. Then from Eq.(18), we get that $\sum_{\substack{k = 0 \\ k \neq 1}}^{\infty} \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\,\Gamma(k+1)} = \alpha$ which leads to $\|G\| \leq 2\alpha$.

Then from the above we can conclude that:

$$\|G\| \leq 2m, \qquad \forall (m-1) < \alpha < m. \tag{45}$$

According to [23], the matrix N is nonsingular and its inverse satisfies the inequality

$$\left\|N^{-1}\right\| = \frac{5(b-a)^4 + 4(b-a)^2 h^2}{384 h^4} = \lambda\, h^{-4} = O(h^{-4}), \tag{46}$$

$$\lambda = \frac{5(b-a)^4 + 4(b-a)^2 h^2}{384}.$$

Also, from Eq.(16) we have: $\|T\| = T_0 h^6 M_6$ where

$$M_6 = \max_{a \leq x \leq b} \left| y^{(6)}(x) \right|. \tag{47}$$

Then from Eqns.(44),(45),(46) and (47) , into Eq. (43), we obtain that:

$$\|E\| \leq O(h^2). \tag{48}$$

We summarize the above results in the next theorem.

Theorem 5

Let $y(x)$ be the exact solution of the continuous boundary value problems (1) and (2) and let $y(x_i), i = 1, 2, ..., n-1$, satisfy the discrete boundary value problem (29). Further, if $e_i = y(x_i) - S_i$,then $\|E\| \cong O\left(h^2\right)$ second order convergent method, which is given by Eq. (48).

5. Numerical examples

We will consider some numerical examples illustrating the solution using quintic spline methods. All calculations are implemented with MATLAB 7.0.1.

Example 5.1

Consider the fractional boundary value problem:

$$y^{(4)}(x) + D^{\alpha}y(x) + y(x) = g(x)\,, \quad \forall x \in [0,1],$$

$$y(0) = y(1) = y''(0) = y''(1) = 0\,. \tag{49}$$

where, $g(x) = x(840x^2 - 120) + x^7(1 + \frac{7!x^{-\alpha}}{\Gamma(8-\alpha)}) - x^5(\frac{5!x^{-\alpha}}{\Gamma(6-\alpha)} + 1)$.
The exact solution of Eq.(49) is:

$$y(x) = x^5(x^2 - 1). \tag{50}$$

The numerical solution is represented in Table 5.1 for $\alpha = 0, 0.4$ and 0.8 respectively.

Table 5.1, Observed maximum absolute errors for example 5.1 and order of convergence

h	α=0		α=0.4		α=0.8	
	Error	O. C.	Error	O. C.	Error	O. C.
1/8	3.65E-2		3.65E-2		3.66E-2	
1/16	1.02E-2	1.84	1.02E-2	1.84	1.02E-2	1.84
1/32	2.62E-3	1.96	2.61E-3	1.97	2.61E-3	1.97
1/64	6.61E-4	1.97	6.56E-4	1.99	6.57E-4	1.99
1/128	1.65E-4	2.002	1.63E-4	2.01	1.65E-4	1.99

Example 5.2

Consider the fractional boundary value problem:

$$y^{(4)}(x) + D^{\alpha}y(x) + y(x) = g(x)\,, \quad \forall x \in [0,1],$$

$$y(0) = y(1) = y''(0) = 0 \text{ and } y''(1) = -4e. \tag{51}$$

where, $g(x) = (-8-6x-2x^2)\,e^x + \sum_{k=0}^{\infty} \frac{k+1}{\Gamma(k+2-\alpha)} x^{k+1-\alpha} - \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{\Gamma(k+3-\alpha)} x^{k+2-\alpha}$.
The exact solution of Eq.(51) is:

$$y(x) = x(1-x)\,e^x. \tag{52}$$

Note that in this example in order to find the fractional derivative of the exponential term we approximated this term by Taylor series.

The numerical solution is represented in Table 5.2 represents the numerical approximation of example 5.2 for $\alpha = 0, 0.2$ and 0.5 respectively.

Table 5.2, Numerical solutions of example 5.2 for $\alpha = 0\,, 0.2$ and $\alpha = 0.5$

x	Exact solution	Approximate solution		
		α=0	α=0.2	α=0.5
0	0	0	0	0
0.125	0.123938	0.124053	0.123551	0.1234731
0.250	0.240755	0.240876	0.239818	0.239823
0.375	0.341014	0.341077	0.339701	0.339734
0.500	0.41218	0.412170	0.410686	0.410756
0.625	0.437870	0.437806	0.436438	0.436537
0.750	0.396938	0.396860	0.395812	0.395915
0.875	0.262377	0.262328	0.261758	0.261827
1	0	0	0	0

Example 5.3

Consider the fractional integro-differential BVP:

$$D^{-\alpha}y^{(4)}(x) + y(x) + \int_0^x t\,y(t)\,dt = g(x) \quad , \forall x \in [0,1],$$

$$y(0) = y''(0) = 0\ , y(1) = 1 \text{ and } y''(1) = 20. \tag{53}$$

where, $g(x) = \frac{5!}{\Gamma(2+\alpha)}x^{1+\alpha} + x^5(1 + \frac{1}{7}x^2)$.

The exact solution of Eq.(53) is:

$$y(x) = x^5. \tag{54}$$

The numerical solution is represented in Table 5.3 for $\alpha = 0, 0.2 \,\text{and}\, 0.6$.

Table 5.3, Numerical solutions of example 5.3 for $\alpha = 0.3\,.0.5$ and 0.7

x	Exact solution	The corresponding error		
		α=0.3	α=0.5	α=0.7
0	0	0	0	0
0.125	3.05E-5	1.55E-3	2.19E-3	3.09E-3
0.250	9.77E-4	2.93E-3	4.15E-3	5.84E-3
0.375	7.42E-3	3.99E-3	5.63E-3	7.92E-3
0.500	0.03125	4.55E-3	6.42E-3	9.02E-3
0.625	0.09537	4.48E-3	6.31E-3	8.84E-3
0.750	0.23731	3.65E-3	5.14E-3	7.19E-3
0.875	0.51291	2.08E-3	2.93E-3	4.09E-3
1	1	0	0	0

Example 5.4

Consider the fractional integro-differential BVP:

$$D^{-\alpha}y^{(4)}(x) + y(x) + \int_0^x y(t)\,dt = g(x) \quad , \forall x \in [0,1],$$

$$y(0) = 1\ \ , y''(0) = 2\ \ , y(1) = 1 + e \text{ and } y''(1) = 3e. \tag{55}$$

where, $g(x) = x(1 + e^x) + 3\, e^x$.
The exact solution of Eq.(54) when $\alpha = 0$ is:

$$y(x) = 1 + x\, e^x. \tag{56}$$

This example was solved by Momani and Noor [14] using Adomian decomposition method. Tables 5.4 represents a comparison between the solution of this example using our method and Momani and Noor method for $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.75$ while Table 5.5 represents the numerical results for$\alpha = 0$.

Table 5.4, Approximate solutions of example 5.4 for
$\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.75$

x	α=0.25		α=0.5		α=0.75	
	Our method	Momani and Noor [14]	Our method	Momani and Noor [14]	Our method	Momani and Noor [14]
0	1	1	1	1	1	1
0.1	1.11456	1.110409	1.121805	1.119372	1.128759	1.120249
0.2	1.243597	1.244072	1.256704	1.260797	1.269653	1.262401
0.3	1.396721	1.404663	1.418272	1.427218	1.435697	1.429356
0.4	1.583766	1.596372	1.610648	1.622404	1.630746	1.624858
0.5	1.809867	1.823967	1.838554	1.850938	1.859381	1.853478
0.6	2.080538	2.092878	2.107362	2.118260	2.126972	2.120655
0.7	2.401769	2.409274	2.423204	2.430730	2.439782	2.432762
0.8	2.780138	2.780162	2.793091	2.795712	2.805094	2.797190
0.9	3.22293	3.213494	3.225055	3.221670	3.231355	3.222450
1	3.718282	3.718282	3.718282	3.718282	3.718282	3.718282

Table 5.5, Numerical solutions of example 5.4 for $\alpha = 0$

x	Exact solution	Approximate solution	Error
0	1	1	0
0.1	1.11000	1.11000	0
0.2	1.24000	1.25032	0.00605
0.3	1.40495	1.41342	0.00846
0.4	1.59673	1.60687	0.01014
0.5	1.82436	1.83525	0.01089
0.6	2.09327	2.10386	0.01059
0.7	2.40962	2.41883	0.00921
0.8	2.78043	2.78724	0.00681
0.9	3.21364	3.21726	0.00362
1	3.71828	3.71828	0

6. Conclusion

In this paper, we used quintic polynomial spline based method to present an approximate solution for two point fourth order integro-differential equations of fractional order. Our approach depends on approximating the fractional term using the Grunewald definition of the fractional derivative. Convergence analysis of the method is presented. Some numerical examples were included to illustrate the practical usefulness of the proposed methods.

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