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AN ACCELERATED HOMOTOPY PERTURBATION METHOD FOR SOLVING NONLINEAR EQUATIONS

I. L. EL-KALLA

Abstract. Based on homotopy perturbation method (HPM), a new approach for solving nonlinear equations is introduced. In this approach, a new formula of the so-called He's polynomials is used and this approach is called an accelerated homotopy perturbation method (AHPM). Using this approach, the rate of convergence is accelerated. Some numerical examples are introduced to verify the efficiency of this approach.

1. Introduction

Our life is nonlinear; so, mathematician always search for a better and easy methods for solving the nonlinear equations illuminating the nonlinear phenomena of our life. Among these methods, the series solution methods such as Taylor method [1], Adomian decomposition method (ADM) [2, 3], homotopy analysis method (HAM) [4, 5] and homotopy perturbation method (HPM) [6, 7]. Using HPM, proposed by Ji-Huan He in [8], the solution is considered as the summation of an infinite series which assumed to be convergent to the exact solution. Application of the HPM to various kinds of nonlinear equations has become a hot topic see for example [9, 10]. In recent years, HPM has been applied with a great success; so, relations and algorithms have been deduced and continuously improved to obtain an accurate solution for a large variety of linear and nonlinear problems for example [11, 12]. In this paper, based on HPM a new approach is introduced for solving functional equations of various kinds in the form

$$y - N(y) = g \tag{1}$$

where N is a nonlinear operator from Hilbert space H to H, y is an unknown function, and g is a known function in H. To explain the HPM, we reconstitute (1) as

$$L(u) = u(x) - g(x) - N(u) = 0, \tag{2}$$

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with solution $u(x) = y(x)$ and we define the homotopy $H(u,p)$ by

$$H(u,0) = F(u), \quad H(u,1) = L(u), \tag{3}$$

where $F(u)$ is a functional operator with solution, say u_0, which can be obtained easily. We may choose a convex homotopy

$$H(u,p) = (1-p)F(u) + pL(u) = 0, \tag{4}$$

which continuously trace an implicitly defined curve from a starting point $H(u_0,0)$ to a solution function $H(y,1)$. The embedding parameter p monotonically increases from zero to one as the problem $F(u) = 0$ is continuously deformed to the original problem $L(u) = 0$. The embedding parameter $p \in [0,1]$ can be considered as an expanding parameter such that

$$u = \sum_{n=0}^{\infty} p^n u_n, \tag{5}$$

when $p \to 1$, equation (4) corresponds to equation (2) and equation (5) becomes the approximate solution of equation (2); i.e.,

$$y(x) = \lim_{p\to 1} u = \sum_{n=0}^{\infty} u_n(x). \tag{6}$$

Taking $F(u) = u(x) - g(x)$ and substituting (2) into (4), we have

$$H(u,p) = u - g - pN(u) = 0. \tag{7}$$

The nonlinear term $N(u)$ can be expressed in the so-called He's polynomials [13]

$$N(u) = \sum_{n=0}^{\infty} p^n H_n(u_0, u_1, ..., u_n), \tag{8}$$

where, the traditional formula of H_n is

$$H_n(u_0, u_1, ..., u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{k=0}^{n} p^k u_k\right)\right]_{p=0}, n \geq 0. \tag{9}$$

Substituting (5) and (8) into (7) and equate the terms with identical powers of p, we obtain the recursive relation

$$\begin{aligned} p^0 \quad &: \quad u_0(x) = g(x), \\ p^n \quad &: \quad u_n(x) = H_{n-1}, n \geq 1. \end{aligned} \tag{10}$$

Clearly, He's polynomials (9) are exactly the same as the well known Adomian polynomials [14]. In the next section, formula (9) will be replaced by another accelerated simple formula to obtain the AHPM. In section three, some numerical examples are introduced to verify the efficiency of the AHPM.

2. The AHPM

By rearranging the terms of the Adomian polynomials, the author in [15] deduced another mathematical formula to H_n called accelerated polynomials ($\tilde{H}_n$) and the author proved that: $N(u) = \sum_{n=0}^{\infty} H_n = \sum_{n=0}^{\infty} \tilde{H}_n$; in which $\tilde{H}_n$ can be written in the new mathematical form

$$\tilde{H}_n(u_0, u_1, ..., u_n) = N(S_n) - \sum_{k=0}^{n-1} \tilde{H}_k, n \geq 1, \tag{11}$$

where the partial sum $S_n = \sum_{i=0}^{n} u_i(x)$ and $\tilde{H}_0 = N(u_0)$. Substituting $u(x) = \sum_{n=0}^{\infty} p^n u_n$ and $N(u) = \sum_{n=0}^{\infty} p^n \tilde{H}_n$ into (7) and equate the terms with identical powers of p, we obtain the following accelerated recursive formula

$$\begin{aligned} p^0 &: \quad u_0(x) = g(x), \\ p^n &: \quad u_n(x) = \tilde{H}_{n-1}, n \geq 1. \end{aligned} \tag{12}$$

The use of the accelerated formula (11) has the following main advantages:

i)- Absence of any derivative terms in the recursion, in contrast of formula (9), thereby allowing for ease of computation.

ii)- Convenient for computer programming, because all old polynomials are saved and used in the calculation of the current one, which saving in memory usage and consequently execution time on the processor.

iii)- The solution using formula (11) converges faster than the solution using formula (9). This is easily verified if, for example, we take $f(u) = u^2$ then the polynomials using the traditional formula (9) are:

$H_0 = u_0^2,$
$H_1 = 2u_0u_1,$
$H_2 = u_1^2 + 2u_0u_2,$
$H_3 = 2u_1u_2 + 2u_0u_3,$
$H_4 = u_2^2 + 2u_1u_3 + 2u_0u_4,$
$H_5 = 2u_2u_3 + 2u_1u_4 + 2u_0u_5,$
$\vdots$

while the polynomials using the accelerated formulas (11) are:

$\tilde{H}_0 = u_0^2,$
$\tilde{H}_1 = 2u_0u_1 + u_1^2,$
$\tilde{H}_2 = 2u_0u_2 + 2u_1u_2 + u_2^2,$
$\tilde{H}_3 = 2u_0u_3 + 2u_1u_3 + 2u_2u_3 + u_3^2,$
$\tilde{H}_4 = 2u_0u_4 + +2u_1u_4 + 2u_2u_4 + 2u_3u_4 + u_4^2,$
$\tilde{H}_5 = 2u_0u_5 + 2u_1u_5 + 2u_2u_5 + 2u_3u_5 + 2u_4u_5 + u_5^2,$
$\vdots$

Clearly, the first five polynomials computed using the accelerated formulas (11) include the first five polynomials computed using the traditional formula (9) in addition to other terms which should appear in $H_6, H_7, H_8, \ldots$using the traditional formula (9). Thus, the solution using the accelerated formula (11) advances additional terms to be entered earlier in the calculation process, thus yielding a faster rate of convergence.

3. Numerical Examples

In order to verify the high efficiency of AHPM, consider the following simple examples

Example 1 consider the following functional equation

$$y(x) = g(x) + y^2(x). \tag{13}$$

According to HPM we have:

$y_0 = g$
$y_1 = H_0 = y_0^2 = g^2,$
$y_2 = H_1 = 2y_0y_1 = 2g^3,$

$\vdots$

and according to AHPM we have:

$y_0 = g$

$y_1 = \tilde{H}_0 = y_0^2 = g^2,$

$y_2 = \tilde{H}_1 = 2y_0y_1 + y_1^2 = 2g^3 + g^4,$

$\vdots$

Using the MATHEMATICA package, S_{10} is computed by HPM to be:

$S_{10} = \sum_{n=0}^{10} y_n = g + g^2 + 2g^3 + 5g^4 + 14g^5 + 26g^6 + 44g^7 + 69g^8 + 94g^9 + 114g^{10} + 116g^{11}$,

while, only S_4 is computed by the AHPM to be:

$S_4 = \sum_{n=0}^{4} y_n = g + g^2 + 2g^3 + 5g^4 + 14g^5 + 26g^6 + 44g^7 + 69g^8 + 94g^9 + 114g^{10} + 116g^{11} + 94g^{12} + 60g^{13} + 28g^{14} + 8g^{15} + g^{16}.$

Under the condition that the series solution converge, it is clear that S_4 using AHPM includes S_{10} using HPM in addition to other terms which should appear in $S_{11}, S_{12}, S_{13}, S_{14}, S_{15}$. So, we conclude that the AHPM converges faster than the classical HPM.

Example 2 consider the nonlinear integral equation [16]

$$y(x) = \sin(\pi x) + \frac{1}{5}\int_0^1 \cos(\pi x)\sin(\pi t)\, y^3(t)\, dt, 0 \le x \le 1 \tag{14}$$

with exact solution $y(x) = \sin(\pi x) + \frac{1}{3}\left(20 - \sqrt{391}\right)\cos(\pi x)$. In this problem we define the homotopy

$$H(u,p) = u - g - p\frac{1}{5}\int_0^1 \cos(\pi x)\sin(\pi t)\, u^3(t)\, dt = 0, \tag{15}$$

where, $g(x) = \sin(\pi x)$. This example is solved using HPM and AHPM expressing the nonlinear term y^3 in terms of H and $\tilde{H}$ respectively. Using MATHEMATICA, table 1 shows the relative absolute error (RAE) for the same partial sum S_5 at different values of x.

Table 1 RAE of example 2

x	RAE using HPM	RAE using AHPM
0.0	2.8020211×10^{-5}	1.0110403×10^{-9}
0.2	9.7020904×10^{-5}	7.3077101×10^{-9}
0.4	1.0052306×10^{-4}	1.0057214×10^{-8}
0.6	5.5270200×10^{-4}	4.0920200×10^{-8}
0.8	1.1057004×10^{-4}	2.0010984×10^{-8}
1.0	3.3051016×10^{-5}	5.2010900×10^{-9}

Example 3 consider the homogeneous nonlinear integral equation

$$y(x) = \int_0^1 \exp(x - 2t)\;\; y^2(t)\, dt \tag{16}$$

with exact solution $y(x) = \exp(x)$. It is difficult to solve this example using classical ADM; since, we have a problem in choosing the initial guess, but HPM or AHPM still work if we construct the homotopy

$$H(u,p) = u(x) - \int_0^1 \exp(x - 2t)\;\; \exp\left[p\left(\ln\left(u^2(t)\right)\right)\right] dt = 0. \tag{17}$$

It is clear that, homotopy (17) still satisfy (3) and in this case the initial guess is $u_0(x) = \int_0^1 \exp(x-2t)\,dt$. Using MATHEMATICA, table 2 shows the RAE for the same partial sum S_9 at different values of x.

Table 2 RAE of example 3

x	RAE using HPM	RAE using AHPM
0.0	9.0020421×10^{-6}	$5.7000433 \times 10^{-11}$
0.2	7.0100004×10^{-5}	$1.1012101 \times 10^{-10}$
0.4	5.5552306×10^{-5}	$7.1000210 \times 10^{-10}$
0.6	9.5070220×10^{-4}	1.2029200×10^{-9}
0.8	3.8630503×10^{-5}	$8.0110904 \times 10^{-10}$
1.0	6.2947565×10^{-6}	$2.7030986 \times 10^{-11}$

Example 4 consider the fractional Riccati equation [17]

$$D_x^{\alpha} y(x) + y^2(x) = 1, 0 < \alpha \leq 1, x > 0 \tag{18}$$

subjected to the initial condition $y(0) = 0$; where, D_*^{α} is the well known Caputo fractional derivative of order α. The exact solution, when $\alpha = 1$, is $y(x) = \frac{\exp(2x)-1}{\exp(2x)+1}$. Applying the fractional integral operator of order α, (18) will be reduced to its equivalent fractional integral equation

$$y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\,dt - \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y^2(t)\,dt. \tag{19}$$

In (19), define the homotopy

$$H(u,p) = u - g + \frac{p}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y^2(t)\,dt = 0, \tag{20}$$

where, $g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\,dt$. This example is solved using HPM and AHPM expressing the nonlinear term y^2 in terms of H and $\tilde{H}$ respectively. Using MATHEMATICA, table 3 shows the RAE for the same partial sum S_4 at $\alpha = 1$ for different values of x.

Table 3 RAE of example 4

x	RAE using HPM	RAE using AHPM
0.0	8.6101322×10^{-6}	7.3110803×10^{-9}
0.2	2.7610157×10^{-5}	1.0110903×10^{-8}
0.4	9.8710110×10^{-5}	7.3119003×10^{-8}
0.6	3.3911034×10^{-4}	2.8110903×10^{-7}
0.8	8.6101550×10^{-4}	9.0191001×10^{-7}
1.0	5.7019823×10^{-3}	2.5310900×10^{-6}

Now we have the following theorem

4. Conclusion

The proposed AHPM converges faster than the classical HPM. This approach can be generalized for solving different types of nonlinear equation. Using AHPM, we can solve problems which can not be solved easily using the ADM.

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Ibrahim L. El-Kalla, Physics and Engineering Mathematics Department, Faculty of Engineering, Mansoura University, PO 35516, Mansoura, Egypt.

E-mail address: al_kalla@mans.edu.eg