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NOTES ON THE FINE SPECTRUM OF THE OPERATOR $\Delta_{a,b}$ OVER THE SEQUENCE SPACE c

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Abstract. The main aim of this paper is to complete and improve the former results for the spectrum and fine spectrum of the generalized difference operator $\Delta_{a,b}$ over the sequence space c which were proved by the authors in [A.M. Akhmedov, S.R. El-Shabrawy, On the fine spectrum of the operator $\Delta_{a,b}$ over the sequence space c, Comput. Math. Appl. 61 (2011) 2994-3002]. The improved results cover a wider class of linear operators which are represented by infinite lower triangular double-band matrices. Illustrative examples showing the advantage of the present results are also given.

1. Introduction and preliminaries

Several authors have studied the spectrum and fine spectrum of linear operators defined by lower and upper triangular matrices over some sequence spaces [1-22].

Throughout this paper, let X be a Banach space. By $R(T), T^*, X^*, B(X), \sigma(T,X)$, $\sigma_p(T,X)$, $\sigma_r(T,X)$ and $\sigma_c(T,X)$, we denote the range of T, the adjoint operator of T, the space of all continuous linear functionals on X, the set of all bounded linear operators on X into itself, the spectrum of T on X, the point spectrum of T on X, the residual spectrum of T on X and the continuous spectrum of T on X, respectively. We shall write c and c_0 for the spaces of all convergent and null sequences, respectively. Also by l_1 we denote the space of all absolutely summable sequences.

We assume here some familiarity with basic concepts of spectral theory and we refer to Kreyszig [23, pp. 370-372] for basic definitions such as resolvent operator, resolvent set, spectrum, point spectrum, residual spectrum and continuous spectrum of a linear operator. Also, we refer to Goldberg [24, pp. 58–71] for Goldberg's classification of spectrum.

In [6], we have defined the operator $\Delta_{a,b}$ on the sequence space c as follows:

$$\Delta_{a,b}x = \Delta_{a,b}(x_k) = (a_k x_k + b_{k-1}x_{k-1})_{k=0}^{\infty} \text{ with } x_{-1} = b_{-1} = 0, \tag{1}$$

where (a_k) and (b_k) are convergent sequences of nonzero real numbers such that $\lim_{k\to\infty} a_k = a$, $\lim_{k\to\infty} b_k = b \neq 0$ and the following condition is satisfied

$$|a - a_k| \neq |b|\,, \quad \text{for all } k \in \mathbb{N}. \tag{2}$$

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It is easy to verify that the operator $\Delta_{a,b}$ can be represented by a lower triangular double-band matrix of the form

$$\Delta_{a,b} = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & \cdots \\ 0 & b_1 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In [6], the following results are obtained:

Result 1: *[6, Corollary 1.2]***.** *The operator $\Delta_{a,b} : c \longrightarrow c$ is a bounded linear operator with the norm $\|\Delta_{a,b}\|_c = \sup_k (|a_k| + |b_{k-1}|)$.*

Result 2: *[6, Theorem 2.2]***.** $\sigma(\Delta_{a,b}, c) = D \cup E$, where $D = \{\lambda \in \mathbb{C} : |\lambda - a| \leq |b|\}$ *and* $E = \{a_k : k \in \mathbb{N}, |a_k - a| > |b|\}$.

Result 3: *[6, Theorem 2.3]***.**

$$\sigma_p(\Delta_{a,b}, c) = \begin{cases} E, & \textit{if there exists } m \in \mathbb{N} : a_i \neq a_j \ \forall i \neq j \geq m, \\ \varnothing, & \textit{otherwise.} \end{cases}$$

Result 4: *[6, Theorem 2.6]***.**

(i) $\{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \cup \{a + b\} \subseteq \sigma_r(\Delta_{a,b}, c)$,

(ii) $\{a_k : k \in \mathbb{N}\} \setminus \sigma_p(\Delta_{a,b}, c) \subseteq \sigma_r(\Delta_{a,b}, c)$,

(iii) $\left\{\lambda \in \mathbb{C} : \sup_k \left|\frac{\lambda - a_k}{b_k}\right| < 1\right\} \subseteq \sigma_r(\Delta_{a,b}, c)$,

(iv) $\sigma_r(\Delta_{a,b}, c) \subseteq \left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{\lambda - a_k}{b_k}\right| < 1\right\} \cup \{a + b\}$,

(v) $\sigma_r(\Delta_{a,b}, c) \subseteq ((D \cup E) \setminus G) \cup \{a + b\}$, *where the set G is defined as $\lambda \in G$ if and only if there exists $k_0 \in \mathbb{N}$ such that $|\lambda - a_k| = |b_k|$, for all $k \geq k_0$.*

Result 5: *[6, Theorem 2.8]***.**

(i) $\sigma_c(\Delta_{a,b}, c) \subseteq (\{\lambda \in \mathbb{C} : |\lambda - a| = |b|\} \cup E) \setminus (\sigma_p(\Delta_{a,b}, c) \cup \{a + b\})$,

(ii) $\sigma_c(\Delta_{a,b}, c) \subseteq \left((D \cup E) \cap \left\{\lambda \in \mathbb{C} : \sup_k \left|\frac{\lambda - a_k}{b_k}\right| \geq 1\right\}\right) \setminus (\sigma_p(\Delta_{a,b}, c) \cup \{a + b\})$,

(iii) $G \setminus \{a + b\} \subseteq \sigma_c(\Delta_{a,b}, c)$,

(iv) $\left(\left\{\lambda \in \mathbb{C} : \inf_k \left|\frac{\lambda - a_k}{b_k}\right| \geq 1\right\} \cap \{\lambda \in \mathbb{C} : |\lambda - a| \leq |b|\}\right) \setminus \{a + b\} \subseteq \sigma_c(\Delta_{a,b}, c)$.

Result 6: *[6, Theorem 2.12]***.** *If $\lambda \in (\{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \setminus \{a_k : k \in \mathbb{N}\}) \cup \{a + b\}$, then $\lambda \in III_2\sigma(\Delta_{a,b}, c)$.*

Result 7: *[6, Theorem 2.13]***.** *If there exists $m \in \mathbb{N}$ such that $a_i \neq a_j$ for all $i, j \geq m$, then $\lambda \in E$ if and only if $\lambda \in III_3\sigma(\Delta_{a,b}, c)$.*

In this paper, we weaken the conditions on the sequences (a_k) and (b_k), assuming only that (a_k) and (b_k) are convergent sequences of real numbers, $b_k \neq 0$ for all $k \in \mathbb{N}$, and that the limit of the sequence (b_k) does not equal zero. We continue to get some new results even from these weaker conditions. Our new theorems give better results while conditions imposed are much weaker than in [6]. Moreover, some examples are given to show the ability and simplicity of applying the new results.

2. Main results and proofs

Throughout this section, (a_k) and (b_k) are assumed to be two convergent sequences of real numbers with

$$\lim_{k\to\infty} a_k = a, \ \lim_{k\to\infty} b_k = b \neq 0 \ \text{ and } \ b_k \neq 0 \text{ for all } \ k \in \mathbb{N}. \tag{3}$$

Note that, the condition (2) is not necessarily satisfied. However, the following two results are still valid.

Theorem 1. *The operator $\Delta_{a,b} : c \longrightarrow c$ is a bounded linear operator with the norm $\|\Delta_{a,b}\|_c = \sup_k (|a_k| + |b_{k-1}|)$.*

Theorem 2. *$\sigma(\Delta_{a,b}, c) = D \cup E$, where $D = \{\lambda \in \mathbb{C} : |\lambda - a| \leq |b|\}$ and $E = \{a_k : k \in \mathbb{N}, |a_k - a| > |b|\}$.*

The following theorem characterizes the set $\sigma_p(\Delta_{a,b}, c)$ completely.

Theorem 3. *$\sigma_p(\Delta_{a,b}, c) = E \cup K$, where*

$$K = \left\{ a_j : j \in \mathbb{N}, |a_j - a| = |b|, \left(\prod_{i=m}^{k} \frac{b_{i-1}}{a_j - a_i} \right) \textit{ is convergent sequence for some } m \in \mathbb{N} \right\}.$$

Proof. Suppose $\Delta_{a,b} x = \lambda x$ for any $x \in c$. Then we obtain

$$(a_0 - \lambda)x_0 = 0 \quad \text{and} \quad b_k x_k + (a_{k+1} - \lambda)x_{k+1} = 0, \quad \text{for all } k \in \mathbb{N}.$$

It is easy to show that $\sigma_p(\Delta_{a,b}, c) \subseteq \{a_k : k \in \mathbb{N}\} \setminus \{a\}$. Now, we will prove that

$$\lambda \in \sigma_p(\Delta_{a,b}, c) \text{ if and only if } \lambda \in E \cup K.$$

If $\lambda \in \sigma_p(\Delta_{a,b}, c)$, then $\lambda = a_j \neq a$ for some $j \in \mathbb{N}$ and there exists $x \in c, \ x \neq \theta$ such that $\Delta_{a,b} x = a_j x$. Then

$$\lim_{k\to\infty} \left| \frac{x_{k+1}}{x_k} \right| = \left| \frac{b}{a - a_j} \right| \leq 1.$$

Then $\lambda = a_j \in E$ or $|a_j - a| = |b|$. In the case when $|a_j - a| = |b|$, we have

$$x_k = x_{m-1} \prod_{i=m}^{k} \frac{b_{i-1}}{a_j - a_i}, \quad k \geq m.$$

Then the sequence $\left(\prod_{i=m}^{k} \frac{b_{i-1}}{a_j - a_i} \right)$ is convergent sequence for some $m \in \mathbb{N}$, since $x \in c$. Therefore $\lambda \in K$ in this case. Thus $\sigma_p(\Delta_{a,b}, c) \subseteq E \cup K$.

Conversily, let $\lambda \in E \cup K$. If $\lambda \in E$, then there exists $i \in \mathbb{N}$ such that $\lambda = a_i \neq a$ and so we can take $x \neq \theta$ such that $\Delta_{a,b} x = a_i x$ and

$$\lim_{k\to\infty} \left| \frac{x_{k+1}}{x_k} \right| = \left| \frac{b}{a - a_i} \right| < 1,$$

that is $x \in c_0 \subset c$. Also, if $\lambda \in K$, then there exists $j \in \mathbb{N}$ such that $\lambda = a_j \neq a$ and $|a_j - a| = |b|$, $\left(\prod_{i=m}^{k} \frac{b_{i-1}}{a_j - a_i} \right)$ is convergent sequence for some $m \in \mathbb{N}$. Then we can take $x \in c$, $x \neq \theta$ such that $\Delta_{a,b} x = a_j x$. Thus $E \cup K \subseteq \sigma_p(\Delta_{a,b}, c)$. This completes the proof. □

Remark 1. *We would like to point out that, under the additional condition (2), Result 3 should be revised as*

$$\sigma_p(\Delta_{a,b}, c) = E,$$

since in this special case we have $K = \varnothing$.

It is known that, for the operator $\Delta_{a,b} : c \to c$, the adjoint operator $\Delta^*_{a,b} \in B(l_1)$ and has a matrix representation of the form

$$\Delta^*_{a,b} = \begin{bmatrix} a+b & 0 \\ 0 & \Delta^t_{a,b} \end{bmatrix}.$$

Theorem 4. $\sigma_p(\Delta^*_{a,b}, c^*) = \{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \cup E \cup H \cup \{a+b\}$, *where*

$$H = \left\{ \lambda \in \mathbb{C} : |\lambda - a| = |b|, \ \sum_{k=0}^{\infty} \left| \prod_{i=0}^{k} \frac{\lambda - a_i}{b_i} \right| < \infty \right\}.$$

Proof. Suppose that $\Delta^*_{a,b} f = \lambda f$ for $f = (f_0, f_1, f_2, ...) \neq \theta$ in $c^* \cong l_1$. Then, we obtain that

$$(a+b) f_0 = \lambda f_0 \quad \text{and} \quad a_{k-2} f_{k-1} + b_{k-2} f_k = \lambda f_{k-1}, \ \ k \geq 2.$$

If $f_0 \neq 0$ then $\lambda = a + b$. So, $\lambda = a + b$ is an eigenvalue with the corresponding eigenvector $f = (f_0, 0, 0, ...)$, that is, $\lambda = a + b \in \sigma_p(\Delta^*_{a,b}, c^*)$. If $\lambda \neq a + b$, then $f_0 = 0$ and therefore, we must take $f_1 \neq 0$ since otherwise we would have $f = \theta$. It is clear that for all $k \in \mathbb{N}$, the vector $f = (0, f_1, f_2, ..., f_{k+1}, 0, 0, ...)$ is an eigenvector of the operator $\Delta^*_{a,b}$ corresponding to the eigenvalue $\lambda = a_k$, where $f_1 \neq 0$ and $f_n = \frac{\lambda - a_{n-2}}{b_{n-2}} f_{n-1}$, for all $n = 2, 3, ..., k+1$. Then, $\{a_k : k \in \mathbb{N}\} \subseteq \sigma_p(\Delta^*_{a,b}, c^*)$. Also, if $\lambda \neq a + b$ and $\lambda \neq a_k$ for all $k \in \mathbb{N}$, then $f_k \neq 0$, for all $k \geq 1$ and $\sum_{k=0}^{\infty} |f_k| < \infty$ if $\lim_{k\to\infty} \left| \frac{f_{k+1}}{f_k} \right| = \left| \frac{\lambda - a}{b} \right| < 1$. Also, if $|\lambda - a| = |b|$, we can easily see that $\sum_{k=0}^{\infty} |f_k| < \infty$ if $\sum_{k=0}^{\infty} \left| \prod_{i=0}^{k} \frac{\lambda - a_i}{b_i} \right| < \infty$, that is, $H \subseteq \sigma_p(\Delta^*_{a,b}, c^*)$. Thus

$$\{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \cup E \cup H \cup \{a+b\} \subseteq \sigma_p(\Delta^*_{a,b}, c^*).$$

The second inclusion can be proved analogously. □

The following lemma is required in the proof of the next theorem.

Lemma 5. *[24, p. 59] T has a dense range if and only if T^* is one to one.*

Theorem 6. $\sigma_r(\Delta_{a,b}, c) = \sigma_p(\Delta^*_{a,b}, c^*) \backslash \sigma_p(\Delta_{a,b}, c)$.

Proof. For $\lambda \in \sigma_p(\Delta^*_{a,b}, c^*) \backslash \sigma_p(\Delta_{a,b}, c)$, the operator $\Delta_{a,b} - \lambda I$ is one to one and hence has an inverse. But $\Delta^*_{a,b} - \lambda I$ is not one to one. Now, Lemma 5 yields the fact that the range of the operator $\Delta_{a,b} - \lambda I$ is not dense in c. This implies that $\lambda \in \sigma_r(\Delta_{a,b}, c)$. The second inclusion can be proved analogously. □

The following theorem is one of our main results, which characterizes the set $\sigma_r(\Delta_{a,b}, c)$ completely.

Theorem 7. $\sigma_r(\Delta_{a,b}, c) = \{\lambda \in \mathbb{C} : |\lambda - a| < |b|\} \cup (H \cup \{a+b\}) \backslash K$.

Proof. The proof follows immediately from Theorems 3, 4 and 6. □

Theorem 8. $\sigma_c(\Delta_{a,b}, c) = \sigma(\Delta_{a,b}, c) \backslash \sigma_p(\Delta^*_{a,b}, c^*)$.

Proof. Since $\sigma(\Delta_{a,b}, c)$ is the disjoint union of the parts $\sigma_p(\Delta_{a,b}, c)$, $\sigma_r(\Delta_{a,b}, c)$ and $\sigma_c(\Delta_{a,b}, c)$ then, by using Theorems 3, 4 and 6, we must have $\sigma_c(\Delta_{a,b}, c) = \sigma(\Delta_{a,b}, c) \setminus \sigma_p(\Delta^*_{a,b}, c^*)$. □

The continuous spectrum of the operator $\Delta_{a,b}$ on the sequence space c is characterized completely from the following theorem.

Theorem 9. $\sigma_c(\Delta_{a,b}, c) = \{ \lambda \in \mathbb{C} : |\lambda - a| = |b|\} \setminus (H \cup \{a+b\})$.

Proof. The proof follows immediately from Theorems 2, 4 and 8. □

Theorem 10. $\lambda \in E \cup K$ *if and only if* $\lambda \in III_3\sigma(\Delta_{a,b}, c)$.

Proof. $\lambda \in E \cup K$ implies that $\lambda \in \sigma_p(\Delta_{a,b}, c)$, and so, $(\Delta_{a,b} - \lambda I)^{-1}$ does not exist. Additionally, $\lambda \in \sigma_p(\Delta^*_{a,b}, c^*)$ implies that $\Delta^*_{a,b} - \lambda I$ is not one to one and hence $\Delta_{a,b} - \lambda I$ has not a dense range. Thus $\lambda \in III_3\sigma(\Delta_{a,b}, c)$. □

Note that Theorem 10 improves Result 7.

Theorem 11. $\lambda \in \sigma_c(\Delta_{a,b}, c)$ *if and only if* $\lambda \in II_2\sigma(\Delta_{a,b}, c)$.

Proof. By Theorem 8, $\Delta^*_{a,b} - \lambda I$ is one to one. By Lemma 5, $\Delta_{a,b} - \lambda I$ has a dense range. Additionally, $\lambda \notin \sigma_p(\Delta_{a,b}, c)$ implies that the operator $\Delta_{a,b} - \lambda I$ has inverse. Therefore, $\lambda \in II_2\sigma(\Delta_{a,b}, c)$ or $\lambda \in I_2\sigma(\Delta_{a,b}, c)$. But $I_2\sigma(\Delta_{a,b}, c) = \varnothing$. Thus $\lambda \in II_2\sigma(\Delta_{a,b}, c)$. □

Finally, we assert that Results 4-6 are still valid.

3. Examples

The advantage of the new results of this paper is that they can be applied to more complex and interesting forms of the operator $\Delta_{a,b}$ as shown in the examples below.

Example 1. *Consider the sequences* (a_k) *and* (b_k), *where*

$$a_0 = 4, \qquad a_1 = 2, \qquad a_k = 1,$$

$$b_0 = b_1 = 1, \qquad b_k = \left(\frac{k}{k+1}\right)^2,$$

for all $k \geq 2$. *Therefore,* $\lim_{k\to\infty} a_k = a = 1$, $\lim_{k\to\infty} b_k = b = 1$, $E = \{4\}$, $K = \{2\}$ *and* $H = \{2\}$. *Then, using Theorems 2, 3, 7 and 9, we have*

$$\sigma(\Delta_{a,b}, c) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\} \cup \{4\},$$

$$\sigma_p(\Delta_{a,b}, c) = \{2, 4\},$$

$$\sigma_r(\Delta_{a,b}, c) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\},$$

$$\sigma_c(\Delta_{a,b}, c) = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\} \setminus \{2\}.$$

Note that, in this example, we have $E \neq \varnothing, K \neq \varnothing$ *and* $H \neq \varnothing$.

Example 2. *Consider the sequences* (a_k) *and* (b_k)*, where*

$$a_0 = 4, \qquad a_1 = 2, \qquad a_k = 1,$$
$$b_0 = b_1 = 1, \qquad b_k = \left(\frac{k+1}{k}\right)^2,$$

for all $k \geq 2$*. Therefore,* $\lim_{k\to\infty} a_k = a = 1$*,* $\lim_{k\to\infty} b_k = b = 1$*,* $E = \{4\}$*,* $K = \varnothing$ *and* $H = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\}$*. Then*

$$\sigma(\Delta_{a,b}, c) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\} \cup \{4\},$$
$$\sigma_p(\Delta_{a,b}, c) = \{4\},$$
$$\sigma_r(\Delta_{a,b}, c) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\},$$
$$\sigma_c(\Delta_{a,b}, c) = \varnothing.$$

Example 3. *Let* $a_k = \frac{k+1}{k+2}$ *and* $b_k = \frac{k+1}{k+3}$ *for all* $k \in \mathbb{N}$*. Then,* $\lim_{k\to\infty} a_k = a = 1$ *and* $\lim_{k\to\infty} b_k = b = 1$*. Similarly, we can prove that* $E = K = H = \varnothing$*. Then*

$$\sigma(\Delta_{a,b}, c) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leq 1\},$$
$$\sigma_p(\Delta_{a,b}, c) = \varnothing,$$
$$\sigma_r(\Delta_{a,b}, c) = \{\lambda \in \mathbb{C} : |\lambda - 1| < 1\} \cup \{2\},$$
$$\sigma_c(\Delta_{a,b}, c) = \{\lambda \in \mathbb{C} : |\lambda - 1| = 1\} \setminus \{2\}.$$

Note that, in Example 3, one can easily prove that for all $\lambda \in \mathbb{C}$ with $|\lambda - 1| = 1$, we have $\left|\frac{\lambda - a_i}{b_i}\right| \geq 1$ for all $i \in \mathbb{N}$ and so $H = \varnothing$.

4. Conclusion

In this paper we have improved on some results of our recent paper [6] concerning the fine spectrum of the generalized difference operator $\Delta_{a,b}$ which is represented by a lower triangular double-band matrix whose entries are the elements of two sequences (a_k) and (b_k). An important point is that there is no additional restriction on the sequences (a_k) and (b_k) beside the requirement that (a_k) and (b_k) are convergent sequences, $b_k \neq 0$ for all $k \in \mathbb{N}$, and that the limit of the sequence (b_k) does not equal zero. The results of this paper generalize the fine spectrum of all earlier lower triangular double-band matrices as operators on the sequence space c. Illustrative examples are also given.

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