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CHEBYSHEV FINITE DIFFERENCE METHOD FOR SOLVING PROBLEMS IN CALCULUS OF VARIATIONS COMPARING WITH VARIATIONAL ITERATION METHOD

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Abstract. In this article, an accurate Chebyshev finite difference method (ChFDM) for solving problems in calculus of variations is presented. The main objective is to find the numerical solution of ODEs which arise from the variational problems. The useful properties of the Chebyshev polynomials and finite difference method are utilized to reduce the computation of the problem to a set of linear or non-linear algebraic equations. Some examples are given to verify and illustrate the efficiency and simplicity of the proposed method. We compared our numerical results against the variational iteration method (VIM). Special attention is given to study the convergence analysis of VIM. The results indicate that the presented method yields more accurate results than those obtained by other methods. Also, from the presented examples, we found that the proposed method can be applied to wide class of problems in calculus of variations.

1. Introduction

Chebyshev polynomials are examples of eigenfunctions of singular Sturm-Liouville problems. Chebyshev polynomials have been used widely in the numerical solutions of the boundary value problems [1] and in computational fluid dynamics ([11], [27]). The existence of a fast Fourier transform for Chebyshev polynomials to efficiently compute matrix-vector products has meant that they have been more widely used than other sets of orthogonal polynomials. Chebyshev polynomials are well known family of orthogonal polynomials on the interval $[-1, 1]$ that have many applications ([14], [18]-[21]). They are widely used because of their good properties in the approximation of functions. One of the advantages of using Chebyshev polynomials $T_n(x)$ as expansion functions is the good representation of smooth functions by finite Chebyshev expansions, provided that the function $u(x)$ is infinitely differentiable. This method is used for solving second and fourth-order elliptic equations [20]. also this method is adopted for solving fractional order integro-differential equations [22] and for obtaining the numerical solution of ODEs with non-analytic solution [3].

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The finite difference methods have been used extensively for solving numerically more of ODEs and PDEs ([2], [4], [10]).

The present work deals with application Chebyshev finite difference method to compute the numerical solution of the resulted ODEs from problem in calculus of variations. This approach requires the definition of a grid as the finite difference and elements techniques also it is applied to satisfy the differential equation and the boundary conditions at the grid points. It can be regarded as a non-uniform finite difference scheme. The derivatives of the function $u(x)$ at a point x_k is linear combination from the values of a function $u(x)$ at the Gauss-Lobatto points $x_k = -\cos(\frac{k\pi}{N})$, where $k = 0, 1, 2, ..., N$, and k is an integer, $0 \leq k \leq N$. The suggested method is more accurate in comparison to the finite difference and finite elements methods because the approximation of the derivatives are defined over the whole domain.

Over the last decades several analytical and approximate methods have been developed to solve the nonlinear ODEs. Among them the variational iteration method which is proposed by J. H. He [16] as a modification of the general Lagrange multiplier method. This method is based on the use of restricted variations and correction functionals which has found a wide applications for the solution of nonlinear differential equations ([17], [23]-[26]). This method does not require the presence of small parameters in the differential equation, and does not require that the nonlinearities be differentiable with respect to the dependent variable and its derivatives. This technique provides a sequence of functions which converges to the exact solution of the problem. This procedure is a powerful tool for solving various kinds of problems, for example, VIM is used to solve the one dimensional system of nonlinear equations in thermo-elasticity [23] and the two dimensional Maxwell equations [26]. This technique solves the problem without any need to discretization of the variables, therefore, in some problems, it is not affected by computation round off errors and one is not faced with necessity of large computer memory and time.

Although, these advantages for VIM, however, it has some drawbacks, for example, this method invalid when we applied it to solve some problems or it is slowly convergent, especially, in problems which is presented by differential equations with non-homogeneous term is complicated function.

In the large number of problems arising in analysis, mechanics, geometry, and so forth, it is necessary to determine the maximal and minimal of a certain functional. Because of the important role of this subject in science and engineering, considerable attention has been received on this kind of problems. Such problems are called variational problems.

There are more problems that have an important role in the development of the calculus of variations ([8], [13]).

The most known of them is the problem of **brachistochrone** which proposed in 1696 by Johann Bernoulli to find the line connecting two certain points A and B that do not lie on a vectorial line and possessing the property that a moving particle slides down this line from A to B in the shortest time. This problem was solved by Johann Bernoulli, Jacob Bernoulli, Leibnitz, Newton, and L'Hospital. It is shown that the solution of this problem is a cycloid [8].

For more details about the historical comments for the variational problems, see ([13], [15]).

The simplest form of a variational problem can be considered as:

$$v[u(t)] = \int_{t_0}^{t_1} F(t, u(t), u'(t))dt, \tag{1}$$

where v is the functional that its extremum must be found. To find the extreme value of v, the boundary points of the admissible curves are known in the following form:

$$u(t_0) = \epsilon_0, \qquad u(t_1) = \epsilon_1. \tag{2}$$

One of the popular methods for solving variational problems are direct methods. In these methods the variational problem is regarded as a limiting case of a finite number of variables. This extremum problem of a function of a finite number of variables is solved by ordinary methods, then a passage of limit yields the solution of the appropriate variational problem [13]. The direct method of Ritz and Galerkin has been investigated for solving variational problems in ([13], [15]). Using Walsh series method, a piecewise constant solution is obtained for variational methods [6]. Some orthogonal polynomials are applied on variational problems to find continuous solutions for these problems ([5], [18]). Also Fourier series and Taylor series are applied to variational problems in [21], to find a continuous solution for this kind of problems.

The necessary condition for the solution of the problem (1) is to satisfy the Euler-Lagrange equation [8]:

$$F_u - \frac{d}{dt}F_{u'} = 0, \tag{3}$$

with the boundary conditions given in (2). The boundary value problem (3) does not always has a solution and if the solution exists, it may not be unique. Note that in many variational problems the existence of a solution is obvious from the physical or geometrical meaning of the problem, and if the solution of Euler's equation satisfies the boundary conditions, it is unique, then this unique extremal will be the solution of the given variational problem [13]. Thus another approach for solving variational problem (1) is finding the solution of the ordinary differential equation (3) which satisfies boundary conditions (2).

The general form of the variational problem (1) is:

$$v[u_1, u_2, ..., u_n] = \int_{t_0}^{t_1} F(t, u_1, u_2, ..., u_n, u_1', u_2', ..., u_n')dt, \tag{4}$$

with the given boundary conditions for all functions:

$$\begin{aligned} &u_1(t_0) = \epsilon_{10}, \qquad u_2(t_0) = \epsilon_{20}, \quad ..., \qquad u_n(t_0) = \epsilon_{n0}, \\ &u_1(t_1) = \epsilon_{11}, \qquad u_2(t_1) = \epsilon_{21}, \quad ..., \qquad u_n(t_1) = \epsilon_{n1}. \end{aligned} \tag{5}$$

Here the necessary condition for the extremum of the functional (4) is to satisfy the following system of second-order differential equations:

$$F_{u_i} - \frac{d}{dt}F_{u_i'} = 0, \qquad i = 1, 2, ..., n, \tag{6}$$

with boundary conditions given in (5). In the present work, we find the solution of variational problem by applying the ChFDM and VIM on the Euler-Lagrange equations.

Also it is possible to define the variational problem for functionals dependent on higher-order derivatives in the following form [13]:

$$v[u(t)] = \int_{t_0}^{t_1} F(t, u(t), u'(t), ..., u^{(n)}(t))dt, \tag{7}$$

with the given boundary conditions:

$$\begin{aligned} &u(t_0) = \epsilon_0, \qquad u'(t_0) = \epsilon_1, \quad ..., \qquad u^{(n-1)}(t_0) = \epsilon_{n-1}, \\ &u(t_1) = \theta_0, \qquad u'(t_1) = \theta_1, \quad ..., \qquad u^{(n-1)}(t_1) = \theta_{n-1}. \end{aligned} \tag{8}$$

The function $u(t)$ which extermizes the functional (7) must satisfy the Euler-Poisson equation:

$$F_u - \frac{d}{dt}F_{u'} + \frac{d^2}{dt^2}F_{u''} + ... + (-1)^n \frac{d^n}{dt^n}F_{u^{(n)}} = 0, \tag{9}$$

which is an ordinary differential equation of order 2n, with boundary conditions given in (8).

The rest of this paper is organized as follows: Section 2 is assigned to the analysis of the standard VIM. In section 3, the convergence study of VIM is given. In section 4, some test problems have been solved by the Chebyshev finite difference method and variational iteration method, to illustrate the efficiency of the proposed method. And the conclusions will appear in section 5.

2. Analysis of the Variational Iteration Method

To illustrate the analysis of VIM, we limit ourselves to consider the following nonlinear differential equation in the type:

$$Lu + R\,u + N(u) = 0, \tag{10}$$

with suitable conditions, where L and R are linear bounded operators, i.e., it is possible to find numbers m_1, $m_2 > 0$ such that $||Lu|| \leq m_1||u||$, $||R\,u|| \leq m_2||u||$. The nonlinear term $N(u)$ is Lipschitz continuous with $|N(u) - N(v)| \leq m\,|u - v|, \forall\, t \in J = [0, T]$, for constant $m > 0$.

The VIM gives the possibility to write the solution of Eq.(10) with the aid of the correction functional:

$$u_p = u_{p-1} + \int_0^t \lambda(\tau)[\,Lu_{p-1} + R\,\tilde{u}_{p-1} + N(\tilde{u}_{p-1})\,]d\tau, \qquad p \geq 1. \tag{11}$$

It is obvious that the successive approximations u_p, $p \geq 0$ can be established by determining, the general Lagrange multiplier, λ, which can be identified optimally via the variational theory. The function $\tilde{u}_p$ is a restricted variation, which means that $\delta\tilde{u}_p = 0$ [13]. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations u_p, $p \geq 1$, of the solution u will be readily obtained upon using the Lagrange multiplier obtained and by using any selective function u_0. The initial values of the solution are usually used for selecting the zeroth approximation u_0. With λ determined, then several approximations u_p, $p \geq 1$, follow immediately. Consequently, the exact solution may be obtained by using:

$$u(t) = \lim_{p\to\infty} u_p. \tag{12}$$

Now, to illustrate how to find the value of the Lagrange multiplier λ, we will consider the following case, which depends on the order of the operator L in Eq.(10), we

study the case of the operator $L = \frac{\partial}{\partial t}$ (without lose of generality).
Making the above correction functional stationary, and noticing that $\delta\tilde{u}_p = 0$, we obtain:

$$\begin{aligned}\delta u_p &= \delta u_{p-1} + \delta \int_0^t \lambda(\tau)[\,\frac{\partial u_{p-1}}{\partial \tau} + R\,\tilde{u}_{p-1} + N(\tilde{u}_{p-1})\,]d\tau \\ &= \delta u_{p-1} + [\lambda(\tau)\,\delta u_{p-1}]_{\tau=t} - \int_0^t \dot{\lambda}(\tau)[\,\delta u_{p-1}\,]d\tau = 0,\end{aligned}$$

where $\delta\tilde{u}_p$ is considered as a restricted variation i.e., $\delta\tilde{u}_p = 0$, yields the following stationary conditions:

$$\dot{\lambda}(\tau) = 0, \qquad 1 + \lambda(\tau)|_{\tau=t} = 0 \tag{13}$$

Eq.(13) is called Lagrange-Euler equation with its boundary condition. The Lagrange multiplier can be identified by solving this equation as: $\lambda(\tau) = -1$.
Now, the following variational iteration formula can be obtained:

$$u_p(t) = u_{p-1}(t) - \int_0^t [\,Lu_{p-1} + R\,u_{p-1} + N(u_{p-1})\,]d\tau. \tag{14}$$

We start with an initial approximation, and by using the above iteration formula (14), we can obtain directly the other components of the solution.

3. Convergence Analysis of VIM

In this section, the sufficient conditions are presented to guarantee the convergence of VIM, when applied to solve the differential equations, where the main point is that we prove the convergence of the recurrence sequence, which is generated by using VIM.

Lemma 1. Let $A : U \to V$ be a bounded linear operator and let $\{u_p\}$ be a convergent sequence in U with limit u, then $u_p \to u$ in U implies that $A(u_p) \to A(u)$ in V.

Now, to prove the convergence of the variational iteration method, we rewrite Eq.(14) in the operator form as follows [13]:

$$u_p = A[u_{p-1}], \tag{15}$$

where the operator A takes the following form:

$$A\,[u] = u - \int_0^t [\,Lu + R\,u + N(u)\,]\,d\tau. \tag{16}$$

Theorem 1. Assume that X be a Banach space and $A : X \to X$ is a nonlinear mapping, and suppose that

$$||\,A[u] - A[v]\,|| \le \gamma\,||u - v||, \quad \forall\ u, v \in X, \tag{17}$$

for some constant $0 < \gamma < 1$ where $\gamma = (1 + m + m_1 + m_2)T$. Then A has a unique fixed point. Furthermore, the sequence (15) using VIM with an arbitrary choice of $u_0 \in X$, converges to the fixed point of A and

$$||u_p - u_q|| \le \frac{\gamma^q}{1-\gamma}\,||u_1 - u_0||. \tag{18}$$

Proof. See [24].
In the following theorem we introduce an estimation of the absolute error of the approximate solution of problem (10).

Theorem 2. The maximum absolute error of the approximate solution u_p to problem (10) is estimated to be:

$$\max_{t \in J} |u_{exact} - u_p| \leq \beta, \tag{19}$$

where $\beta = \dfrac{\gamma^q \, T \, [\, (1 + m_1 + m_2) \, ||u_0|| + k \,]}{1 - \gamma}, \qquad k = \max_{t \in J} |N(u_0)|.$

Proof. From Theorem 1 inequality (18) we have:

$$||u_p - u_q|| \leq \frac{\gamma^q}{1 - \gamma} ||u_1 - u_0||,$$

as $p \to \infty$ then $u_p \to u_{exact}$ and:

$$\begin{aligned} ||u_1 - u_0|| &= \max_{t \in J} \left| u_0 - \int_0^t [\, L u_0 + R\, u_0 + N(u_0) \,] \, d\tau \right| \\ &\leq \max_{t \in J} \left(|u_0| + \int_0^t [\, |L u_0| + |R\, u_0| + |N(u_0)| \,] \, d\tau \right) \\ &\leq T \, [\, (1 + m_1 + m_2) \, ||u_0|| + k \,], \end{aligned}$$

so, the maximum absolute error in the interval J is:

$$||u_{exact} - u_p|| = \max_{t \in J} |u_{exact} - u_p| \leq \beta.$$

This completes the proof.

4. Applications and Numerical Results

In this section, we introduce two variational problems. We find the numerical solution of these problems using ChFDM and VIM and plot the curves of these solutions. These examples are chosen such that there exist analytical solutions for them to give an obvious overview and show the efficiency of the proposed method and VIM. Note that we have computed the numerical results using Mathematica programming.

Problem 4.1:

Consider the following variational problem:

$$min\, v = \int_0^1 \left(u(t) + u'(t) - 4e^{3t} \right)^2 dt, \tag{20}$$

under the following boundary conditions:

$$u(0) = 1, \qquad u(1) = e^3. \tag{21}$$

The corresponding Euler-Lagrange equation is:

$$u''(t) - u(t) - 8e^{3t} = 0, \tag{22}$$

with boundary conditions (21). The exact solution of this problem is $u(t) = e^{3t}$.

1.I: Procedure solution using Chebyshev finite difference method

To solve the linear ODE of the form (22) with the given boundary conditions (21) by using Chebyshev finite difference method, we use the transformation $t = \frac{1}{2}(\eta+1)$ to reduce the interval $[0,1]$ to $[-1,1]$. In this case Eq.(22) will take the following form:

$$u''(\eta) - \frac{1}{4}u(\eta) - 2e^{\frac{3}{2}(\eta+1)} = 0, \qquad -1 < \eta < 1. \tag{23}$$

The transformed boundary conditions are given by:

$$u(-1) = 1, \qquad u(1) = e^3, \tag{24}$$

where $u(\eta)$ is an unknown function from $C^m[-1,1]$. Where the differentiation in Eq.(23) will be with respect to the new variable η. The procedure of the solution will be as follows:
We approximate the unknown solution $u(\eta)$, in the following form [7]:

$$u(\eta) = \sum_{n=0}^{N}{}'' a_n T_n(\eta), \tag{25}$$

where $a_n = \frac{2}{N}\sum_{j=0}^{N}{}'' u(\eta_j)T_n(\eta_j)$ and the summation symbol with double primes denotes a sum with both the first and last terms halved.
The first and the second derivatives of the Chebyshev functions are formed as following:

$$T_n'(\eta) = \sum_{\substack{k=0 \\ (n+k)\,\text{odd}}}^{n-1} \frac{2n}{c_k}T_k(\eta), \quad T_n''(\eta) = \sum_{\substack{k=0 \\ (n+k)\,\text{even}}}^{n-2} \frac{n}{c_k}(n^2-k^2)T_k(\eta), \tag{26}$$

where $c_0 = 2$ and $c_i = 1$ for $i \geq 1$. From Eq.(26) and by differentiated the series in Eq.(25) term by term, we get:

$$u'(\eta) = \frac{4}{N}\sum_{n=0}^{N}{}''\sum_{j=0}^{N}{}''\sum_{\substack{k=0 \\ (n+k)\,\text{odd}}}^{n-1} \frac{n}{c_k}u(\eta_j)T_n(\eta_j)T_k(\eta), \tag{27}$$

$$u''(\eta) = \frac{2}{N}\sum_{n=0}^{N}{}''\sum_{j=0}^{N}{}''\sum_{\substack{k=0 \\ (n+k)\,\text{even}}}^{n-2} \frac{n}{c_k}(n^2-k^2)u(\eta_j)T_n(\eta_j)T_k(\eta). \tag{28}$$

In [12] Elbarbary and El-Sayed proved the error estimate of the first and second derivatives (27)-(28). From Eqs.(27)-(28), we can define the elements of the matrices D_n, $n = 1,2$ which are defined in the following relations:

$$[u^{(n)}] = D_n[u], \quad n = 1,2,$$

where $D_n = [d_{i,j}^{(n)}]$ is a square matrix of order $N+1$ and the elements of the column matrix $[u^{(n)}]$ are given by $u_i^{(n)} = u^{(n)}(\eta_i)$, $i = 0,1,...,N,\ n = 0,1,2$. The derivatives of the function $u(\eta)$ at the points η_k are given by:

$$u^{(n)}(\eta_k) = \sum_{j=0}^{N} d_{k,j}^{(n)}u(\eta_j), \quad n = 1,2, \tag{29}$$

where $d_{k,j}^{(n)},\ j=0,1,...,N$ are the elements of the kth row of the matrix D_n. They are given as follows:

$$d_{k,j}^{(1)}=\frac{4\theta_j}{N}\sum_{n=0}^{N}\sum_{\substack{\ell=0\\(n+\ell)\,\text{odd}}}^{n-1}\frac{n\theta_n}{c_\ell}T_n(\eta_j)T_\ell(\eta_k),\quad k,j=0,1,...,N,$$

$$d_{k,j}^{(2)}=\frac{2\theta_j}{N}\sum_{n=0}^{N}\sum_{n=0}^{N}\sum_{\substack{\ell=0\\(n+\ell)\,\text{even}}}^{n-2}\frac{n\theta_n}{c_\ell}(n^2-\ell^2)T_n(\eta_j)T_\ell(\eta_k),\quad k,j=0,1,...,N,$$

where $\theta_0=\theta_N=\frac{1}{2},\ \theta_1=1$ for $j=1,2,...,N-1$.
By applying the ChFDM to solve Eq.(23), we obtain a system of linear algebraic equations for the unknowns $u(\xi_i)$, with, $\xi_i=-\cos(\frac{i\pi}{N})$, $i=0,1,2,...,N$:

$$\sum_{j=0}^{N}d_{k,j}^{(2)}u(\xi_j)-\frac{1}{4}u(\xi_j)-2e^{\frac{3}{2}(\xi_j+1)}=0,\qquad k=0,1,2,...,N, \tag{30}$$

which is given in the matrix form as follows

$$D_2[u]-\frac{1}{4}[u]-2I[f]=0, \tag{31}$$

where $[f]=e^{\frac{3}{2}(\xi_j+1)}$ and I is the identity matrix. The resulting linear system of $N+1$ of algebraic equations is solved by conjugate gradient method.

1.II: Procedure solution using VIM

The VIM gives the possibility to write the solution of Eq.(22) with the aid of the correction functionals:

$$u_{n+1}(t)=u_n(t)+\int_0^t\lambda(\tau)[\,u_n''-\tilde{u}_n(\tau)-8e^{3\tau}\,]d\tau,\qquad n\geq 0, \tag{32}$$

where λ is general Lagrange multiplier. Making the above correction functional stationary:

$$\begin{aligned}\delta u_{n+1}(t)&=\delta u_n(t)+\delta\int_0^t\lambda(\tau)[\,u_n''-\tilde{u}_n(\tau)-8e^{3\tau}\,]d\tau\\&=\delta u_n(t)+[\lambda(\tau)\,\delta u_n'-\lambda'\,\delta u_n]_{\tau=t}+\int_0^t[\lambda''(\tau)\,\delta u_n]d\tau=0,\end{aligned} \tag{33}$$

where $\delta\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n=0$, yields the following stationary conditions (by comparison the two sides in the above equation):

$$\lambda''(\tau)=0,\qquad \lambda(\tau)|_{\tau=t}=0,\qquad 1-\lambda'(\tau)|_{\tau=t}=0. \tag{34}$$

The equations in (34) are called Lagrange-Euler equation and the natural boundary conditions respectively, the Lagrange multiplier, therefore

$$\lambda(\tau)=\tau-t. \tag{35}$$

Now, by substituting from (35) in (32), the following variational iteration formula can be obtained:

$$u_{n+1}(t)=u_n(t)+\int_0^t(\tau-t)[\,u_n''-u_n(\tau)-8e^{3\tau}\,]d\tau,\qquad n\geq 0. \tag{36}$$

We start with initial approximation $u_0(t) = 1 + at$, for arbitrary constant a, and by using the above iteration formula (36), we can directly obtain the components of the solution.
Now, the first three components of the solution $u(t)$ by using (36) of Eq.(22) are:

$$u_0(t) = 1 + at,$$

$$u_1(t) = 1 + at + \frac{1}{18}(-16 + 16e^{3t} - 48t + 9t^2 + 3at^3),$$

$$u_2(t) = 1 + at + \frac{1}{18}(-16 + 16e^{3t} - 48t + 9t^2 + 3at^3) + \frac{1}{3240}(-320 + 320e^{3t} + 3t(-320 + 3t(-160 + t(-160 + 3t(5 + at))))).$$

Now, to find the constant a, we impose the boundary condition $u(1) = e^3$ on the n-term approximation $u_3(t)$, we obtain $a = 3.00028$.

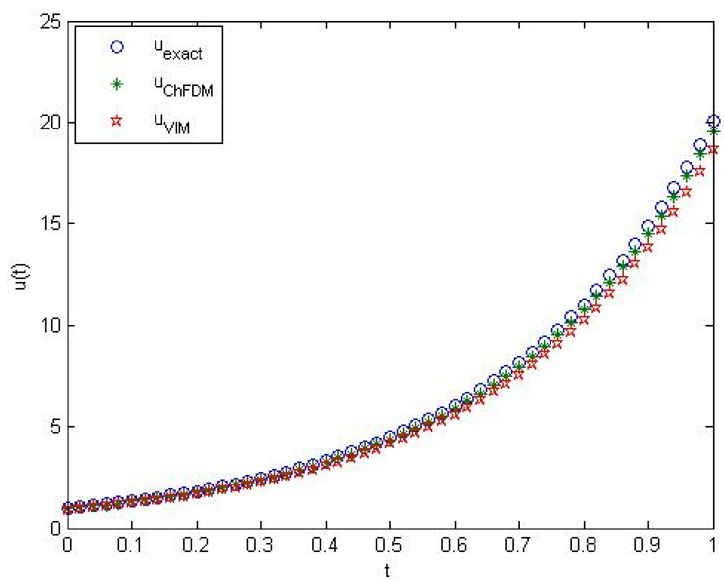

Figure 1: The behavior of numerical solution using ChFDM, u_{ChFDM}, the approximate solution using VIM, u_{VIM} and the exact solution, u_{exact}.

The behavior of the numerical solutions using Chebyshev finite difference method, u_{ChFDM}, with $N = 12$, compared with the approximate solution using VIM, u_{VIM}, with three components ($n = 3$) are presented in figure 1.

Problem 4.2: Consider the following brachistochrone problem [9]:

$$min\, v = \int_0^1 \left[\frac{1 + u'^2(t)}{1 - u(t)}\right]^{1/2} dt, \tag{37}$$

with the given boundary conditions:

$$u(0) = 0, \qquad u(1) = -0.5. \tag{38}$$

The corresponding Euler-Lagrange equation of problem (37) takes the following form:

$$u'' = -\frac{1+u'^2}{2(u-1)}. \tag{39}$$

2.I: Procedure solution using Chebyshev finite difference method

To solve the non-linear ODE of the form (39) with the given boundary conditions (38)by using Chebyshev finite difference method, we use the transformation $t = \frac{1}{2}(\eta+1)$ to reduce the interval $[0,1]$ to $[-1,1]$. In this case Eq.(39) will take the following form:

$$4u''(\eta) + \frac{1+4u'^2}{2(u-1)} = 0, \qquad -1 < \eta < 1. \tag{40}$$

The transformed boundary conditions are given by:

$$u(-1) = 0, \qquad u(1) = -0.5, \tag{41}$$

where $u(\eta)$ is an unknown function from $C^m[-1,1]$. Where the differentiation in Eq.(40) will be with respect to the new variable η.
By the same procedure follows in the previous example, we can apply the proposed ChFDM to solve this example. The resulting system of non-linear algebraic equations for the unknowns $u(\xi_i)$, with, $\xi_i = -\cos(\frac{i\pi}{N})$, $i = 0, 1, 2, ..., N$:

$$4\sum_{j=0}^{N} d_{k,j}^{(2)} u(\xi_j) + 0.5(u(\xi_j)-1)^{-1}\left(1 + 4(\sum_{j=0}^{N} d_{k,j}^{(1)} u(\xi_j))^2\right) = 0, \quad k = 0, 1, 2, ..., N, \tag{42}$$

The resulting non-linear system of $N+1$ algebraic equations is solved by Newton's method.

2.II: Procedure solution using VIM

The VIM gives the possibility to write the solution of (39) with the aid of the correction functionals:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\left[u_n'' + \frac{1+\tilde{u}_n'^{\,2}}{2(\tilde{u}_n-1)}\right]d\tau, \qquad n \geq 0, \tag{43}$$

where λ is a general Lagrange multiplier. Making the above correction functional stationary:

$$\begin{aligned} \delta u_{n+1}(t) &= \delta u_n(t) + \delta\int_0^t \lambda(\tau)\left[u_n'' + \frac{1+\tilde{u}_n'^{\,2}}{2(\tilde{u}_n-1)}\right]d\tau \\ &= \delta u_n(t) + [\lambda(\tau)\,\delta u_n' - \lambda'\,\delta u_n]_{\tau=t} + \int_0^t [\lambda''(\tau)\,\delta u_n]d\tau = 0. \end{aligned} \tag{44}$$

By the same way we can obtain the Lagrange multiplier $\lambda(\tau) = \tau - t$.

Now, by substituting in (43), the following variational iteration formula can be obtained:

$$u_{n+1}(t) = u_n(t) + \int_0^t (\tau - t)\left[u_n'' + \frac{1+u_n'^{\,2}}{2(u_n-1)}\right]d\tau, \qquad n \geq 0. \tag{45}$$

We start with initial approximation $u_0(t) = at$ for arbitrary constant a, and by using the above iteration formula (45), we can directly obtain the components of the solution.

Now, the first three components of the solution $u(t)$ by using (45) of Eq.(39) are:

$u_0(t) = at,$

$u_1(t) = at + \frac{1}{4}t^2 + \frac{1}{4}a^2t^2,$

$u_2(t) = at + \frac{1}{4}t^2 + \frac{1}{4}a^2t^2 + \frac{1}{6}a^3t^3 + 0.0208333\, t^4 + \frac{1}{24}a^2\, t^4 + 0.0208333\, a^4t^4.$

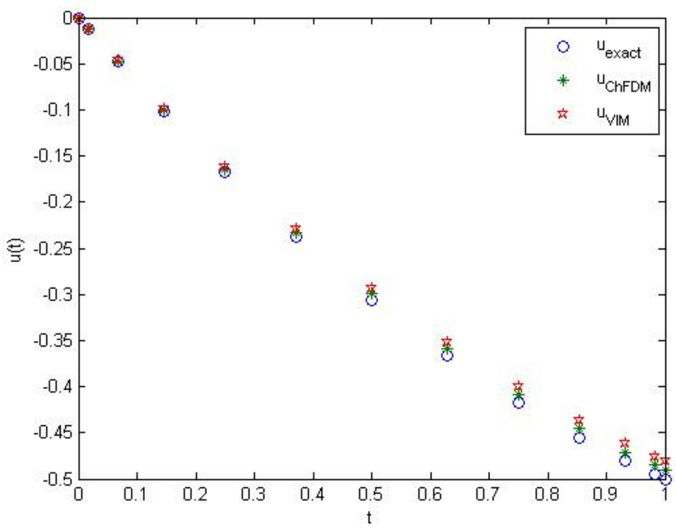

Figure 2: The behavior of numerical solution using ChFDM, u_{ChFDM}, the approximate solution using VIM, u_{VIM} and the exact solution, u_{exact}.

Now, to find the constant a, we impose the boundary condition at $t = 1$ on the n-term approximation $u_3(t)$, we obtain $a = -0.8079390$.
From the numerical results in figure 2, we can see that the proposed ChFDM is in excellent agreement with the exact solution and better than VIM.

5. Conclusion and remarks

Since, as it is known that the problems in calculus of variations reduce to linear or non-linear ODEs and it is also known very difficult to find the analytical solutions of higher-order non-linear ODEs, so, we interest in this article with using high accuracy ChFD method to solve numerically such these equations. Since, we know that the Chebyshev polynomial approximation method is valid in the interval $[-1, 1]$, so, we used the transformation $t = \frac{a}{2}(\eta+1)$ to change the interval $[0, a]$. The proposed method reduces the considered non-linear differential equation to a non-linear system of algebraic equations, which solved using the well known method, namely, Newton iteration method. Also, by using VIM the solutions may take the closed form of the exact solution. In general since the VIM solves the problems on a few steps later of iteration satisfying the desired precision, it does not need more calculation in order to solve the differential equation. Special attention is given to study the convergence of VIM and satisfy this theoretical study in view the introduced numerical examples. In the end, from our numerical results using the proposed method we can conclude that, the solutions are in excellent agreement with the exact solution in most cases. Also, the obtained results demonstrate reliability and efficiency of the proposed method.

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