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SOLUTION OF A NONLOCAL CAUCHY PROBLEM OF A DIFFERENTIAL EQUATION BY ADOMIAN DECOMPOSITION METHOD

E. A. A. ZIADA

Abstract. In this paper we apply the Adomian decomposition method (ADM) for solving a nonlocal Cauchy problem of nonlinear differential equations. The existence and uniqueness of the solution are proved. The convergence of the series solution and the error analysis are studied.

1. Introduction

The Cauchy problems with multi-point or non-local conditions have been extensively studied by several authors in the last two decades. The interested reader is referred to [[1] -[14]] and the references therein.

Here we are concerned with the nonlocal Cauchy problem of the differential equation

$$\frac{dx\,(t)}{dt} = f\,(t, x\,(t))\,, \qquad t \in (0, T] \tag{1}$$

with the nonlocal condition

$$x(0) + \sum_{k=0}^{n} a_k x\,(t_k) = x_0, \qquad t_k \in (0, T)\,. \tag{2}$$

The existence and uniqueness of the solution $x \in C(J)$,where $C(J)$ is the space of all continuous functions and $J = [0, T]$, $T < \infty$ of the nonlocal problem (1)-(2) will be proved, the integral representation of this solution will be proved and the solution algorithm using ADM will be given.

2. Problem Solving

2.1. **Integral representation.** For the integral representation of the solution of the nonlocal problem (1)-(2) we have the following lemma.

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Lemma 1 If $\left(1+\sum\limits_{k=0}^{n} a_k\right) > 0$, then the nonlocal problem (1)-(2) and the integral equation

$$x(t) = \left(1+\sum_{k=0}^{n} a_k\right)^{-1} \left(x_0 - \sum_{k=0}^{n} a_k \int_0^{t_k} f\left(s, x\left(s\right)\right) ds\right) + \int_0^t f(s, x(s)ds. \quad (3)$$

are equivalent.

Proof. Operating with $I = \int\limits_0^t (.)\, ds$ to both sides of equation (1), we get

$$x\left(t\right) = x(0) + \int_0^t f\left(s, x\left(s\right)\right) ds. \quad (4)$$

Let $t = t_k$ in equation (4), then we get

$$x\left(t_k\right) = x(0) + \int_0^{t_k} f\left(s, x\left(s\right)\right) ds,$$

$$\sum_{k=0}^{n} a_k x\left(t_k\right) = x(0) \sum_{k=0}^{n} a_k + \sum_{k=0}^{n} a_k \int_0^{t_k} f\left(s, x\left(s\right)\right) ds. \quad (5)$$

Substitute from equation (2) into equation (5) we get,

$$x_0 - x(0) = x(0) \sum_{k=0}^{n} a_k + \sum_{k=0}^{n} a_k \int_0^{t_k} f\left(s, x\left(s\right)\right) ds,$$

$$x(0) + x(0) \sum_{k=0}^{n} a_k = x_0 - \sum_{k=0}^{n} a_k \int_0^{t_k} f\left(s, x\left(s\right)\right) ds$$

and

$$x(0) = \left(1+\sum_{k=0}^{n} a_k\right)^{-1} \left(x_0 - \sum_{k=0}^{n} a_k \int_0^{t_k} f\left(s, x\left(s\right)\right) ds\right) \quad (6)$$

Substitute from equation (6) into equation (4) we obtain (3).
To complete the proof, differentiating (3) we obtain (1). Also, let $t = 0$ in (3), then by direct calculations we can get (2).

2.2. **The solution algorithm.** The solution algorithm of equation (3) using ADM is

$$x_0\left(t\right) = x_0 \left(1+\sum_{k=0}^{n} a_k\right)^{-1}, \quad (7)$$

$$x_m\left(t\right) = -\left(1+\sum_{k=0}^{n} a_k\right)^{-1} \left(\sum_{k=0}^{n} a_k \int_0^{t_k} A_{m-1}\left(s\right) ds\right) + \int_0^t A_{m-1}\left(s\right) ds. \quad (8)$$

where A_m are Adomian polynomials of the nonlinear term $f(t, x(t))$ which take the form,

$$A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[f\left(t, \sum_{i=0}^{\infty}\lambda^i x_i\right)\right]_{\lambda=0}$$

Finally, the solution of problem (1)-(2) will be

$$x(t) = \sum_{i=0}^{\infty} x_i(t). \tag{9}$$

3. Convergence Analysis

3.1. **Existence and Uniqueness theorem.** Define the mapping $F : E \to E$ where E is the Banach space $(C(J), \|\cdot\|)$ of all continuous functions on J with the norm $\|x\| = \max_{t\in J}|x(t)|$.

Assume now that the function $f : [0, T] \times R \to R$ is continuous and satisfies the Lipschitz condition

$$|f(t, x) - f(t, y)| \le k|x - y| \tag{10}$$

Theorem 1: *Let f satisfies the Lipschitz condition (10), then the integral equation (3); which equivalent to problem (1)-(2), has a unique solution $x \in C(J)$.*

Proof: The mapping $F : E \to E$ is defined as,

$$Fx = \left(1 + \sum_{k=0}^{n} a_k\right)^{-1}\left(x_0 - \sum_{k=0}^{n} a_k \int_0^{t_k} f(s, x(s))\, ds\right) + \int_0^t f(s, x(s))\, ds$$

Let $x, y \in E$, then

$$\begin{aligned} Fx - Fy &= -\left(1 + \sum_{k=0}^{n} a_k\right)^{-1}\left(\sum_{k=0}^{n} a_k \int_0^{t_k}[f(s, x(s)) - f(s, y(s))]\, ds\right) \\ &\quad + \int_0^t [f(s, x(s)) - f(s, y(s))]\, ds \end{aligned}$$

which implies that

$$\begin{aligned} |Fx - Fy| &= \Bigg|-\left(1 + \sum_{k=0}^{n} a_k\right)^{-1}\left(\sum_{k=0}^{n} a_k \int_0^{t_k}[f(s, x(s)) - f(s, y(s))]\, ds\right) \\ &\quad + \int_0^t [f(s, x(s)) - f(s, y(s))]\, ds\Bigg| \end{aligned}$$

$$
\begin{aligned}
|Fx-Fy| &\leq \left|-\left(1+\sum_{k=0}^{n}a_k\right)^{-1}\left(\sum_{k=0}^{n}a_k\int_0^{t_k}\left[f(s,x(s))-f(s,y(s))\right]ds\right)\right| \\
&\quad+\left|\int_0^{t}\left[f(s,x(s))-f(s,y(s))\right]ds\right| \\
&\leq \left(1+\sum_{k=0}^{n}a_k\right)^{-1}\sum_{k=0}^{n}a_k\int_0^{t_k}\left|f(s,x(s))-f(s,y(s))\right|ds \\
&\quad+\int_0^{t}\left|f(s,x(s))-f(s,y(s))\right|ds \\
&\leq k\left[\left(1+\sum_{k=0}^{n}a_k\right)^{-1}\sum_{k=0}^{n}a_k\int_0^{t_k}|x(s)-y(s)|\,ds+\int_0^{t}|x(s)-y(s)|\,ds\right] \\
\max_{t\in J}|Fx-Fy| &\leq k\left[\left(1+\sum_{k=0}^{n}a_k\right)^{-1}\sum_{k=0}^{n}a_k\max_{t\in J}\int_0^{t_k}|x(s)-y(s)|\,ds\right. \\
&\quad\left.+\max_{t\in J}\int_0^{t}|x(s)-y(s)|\,ds\right]
\end{aligned}
$$

$$
\begin{aligned}
\|Fx-Fy\| &\leq k\left[\left(1+\sum_{k=0}^{n}a_k\right)^{-1}\sum_{k=0}^{n}a_k\int_0^{t_k}ds+\int_0^{t}ds\right]\|x-y\| \\
&\leq k\left[T\left(1+\sum_{k=0}^{n}a_k\right)^{-1}\sum_{k=0}^{n}a_k+T\right]\|x-y\| \\
&\leq kT\left[\left(1+\sum_{k=0}^{n}a_k\right)^{-1}\sum_{k=0}^{n}a_k+1\right]\|x-y\|
\end{aligned}
$$

Now, if $kT\left[\left(1+\sum_{k=0}^{n}a_k\right)^{-1}\sum_{k=0}^{n}a_k+1\right]<1$, then we get

$$\|Fx-Fy\|\leq\|x-y\|,$$

therefore the mapping F is contraction and there exists a unique solution $x\in C(J)$ to the nonlocal Cauchy problem (1)-(2) given by (3), where

$$x(0)=\lim_{t\to 0}x(t)=\left(1+\sum_{k=0}^{n}a_k\right)^{-1}\left(x_0-\sum_{k=0}^{n}a_k\int_0^{t_k}f(s,x(s))\,ds\right)$$

and

$$x(T)=\lim_{t\to T}x(t)=\left(1+\sum_{k=0}^{n}a_k\right)^{-1}\left(x_0-\sum_{k=0}^{n}a_k\int_0^{t_k}f(s,x(s))ds\right)+\int_0^{T}f(s,x(s)ds.$$

This completes the proof. ∎

3.2. **Proof of convergence.** **Theorem 2**: *The series solution (9) of the problem (1)-(2) using ADM converges if* $|x_1(t)| < c, c$ *is a positive constant.*

Proof: Define the sequence $\{S_p\}$ such that, $S_p = \sum_{i=0}^{p} x_i(t)$ is the sequence of partial sums from the series solution $\sum_{i=0}^{\infty} x_i(t)$ since,

$$f(t, x(t)) = \sum_{i=0}^{\infty} A_i,$$

so,

$$f(t, S_p) = \sum_{i=0}^{p} A_i,$$

From equations (8) and (9) we have,

$$\sum_{i=0}^{\infty} x_i = \left(1 + \sum_{k=0}^{n} a_k\right)^{-1} \left(x_0 - \sum_{k=0}^{n} a_k \int_0^{t_k} \sum_{i=0}^{\infty} A_{i-1} ds\right) + \int_0^t \sum_{i=0}^{\infty} A_{i-1}(s)\, ds.$$

Let S_p and S_q be two arbitrary partial sums with $p > q$, then we get,

$$S_p = \sum_{i=0}^{p} x_i = \left(1 + \sum_{k=0}^{n} a_k\right)^{-1} \left(x_0 - \sum_{k=0}^{n} a_k \int_0^{t_k} \sum_{i=0}^{p} A_{i-1} ds\right) + \int_0^t \sum_{i=0}^{p} A_{i-1}(s)\, ds$$

and

$$S_q = \sum_{i=0}^{q} x_i = \left(1 + \sum_{k=0}^{n} a_k\right)^{-1} \left(x_0 - \sum_{k=0}^{n} a_k \int_0^{t_k} \sum_{i=0}^{q} A_{i-1} ds\right) + \int_0^t \sum_{i=0}^{q} A_{i-1}(s)\, ds$$

Now, we are going to prove that $\{S_p\}$ is a Cauchy sequence in this Banach space E.

$$\begin{aligned}
S_p - S_q &= -\left(1 + \sum_{k=0}^{n} a_k\right)^{-1} \left(\sum_{k=0}^{n} a_k \int_0^{t_k} \left[\sum_{i=0}^{p} A_{i-1}(s) - \sum_{i=0}^{q} A_{i-1}(s)\right] ds\right) \\
&\quad + \int_0^t \left[\sum_{i=0}^{p} A_{i-1}(s) - \sum_{i=0}^{q} A_{i-1}(s)\right] ds \\
&= -\left(1 + \sum_{k=0}^{n} a_k\right)^{-1} \left(\sum_{k=0}^{n} a_k \int_0^{t_k} \left[\sum_{i=q+1}^{p} A_{i-1}\right] ds\right) + \int_0^t \left[\sum_{i=q+1}^{p} A_{i-1}\right] ds \\
&= -\left(1 + \sum_{k=0}^{n} a_k\right)^{-1} \left(\sum_{k=0}^{n} a_k \int_0^{t_k} \left[\sum_{i=q}^{p-1} A_i\right] ds\right) + \int_0^t \left[\sum_{i=q}^{p-1} A_i\right] ds
\end{aligned}$$

$$
\begin{aligned}
&= -\left(1+\sum_{k=0}^{n} a_k\right)^{-1} \left(\sum_{k=0}^{n} a_k \int_0^{t_k} \left[f\left(t, S_{p-1}\right) - f\left(t, S_{q-1}\right)\right] ds\right) \\
&\quad + \int_0^{t} \left[f\left(t, S_{p-1}\right) - f\left(t, S_{q-1}\right)\right] ds \\
\left|S_p - S_q\right| &= \left| -\left(1+\sum_{k=0}^{n} a_k\right)^{-1} \left(\sum_{k=0}^{n} a_k \int_0^{t_k} \left[f\left(t, S_{p-1}\right) - f\left(t, S_{q-1}\right)\right] ds\right) \right. \\
&\quad \left. + \int_0^{t} \left[f\left(t, S_{p-1}\right) - f\left(t, S_{q-1}\right)\right] ds \right| \\
&\leq k\left[\left(1+\sum_{k=0}^{n} a_k\right)^{-1} \left(\sum_{k=0}^{n} a_k \int_0^{t_k} \left|S_{p-1} - S_{q-1}\right| ds\right) + \int_0^{t} \left|S_{p-1} - S_{q-1}\right| ds\right] \\
\left\|S_p - S_q\right\| &\leq kT\left[\left(1+\sum_{k=0}^{n} a_k\right)^{-1} \sum_{k=0}^{n} a_k + 1\right] \left\|S_{p-1} - S_{q-1}\right\| \\
&\leq \beta \left\|S_{p-1} - S_{q-1}\right\|
\end{aligned}
$$

Let $p = q + 1$ then,

$$\left\|S_{q+1} - S_q\right\| \leq \beta \left\|S_q - S_{q-1}\right\| \leq \beta^2 \left\|S_{q-1} - S_{q-2}\right\| \leq \cdots \leq \beta^q \left\|S_1 - S_0\right\|$$

From the triangle inequality we have,

$$
\begin{aligned}
\left\|S_p - S_q\right\| &\leq \left\|S_{q+1} - S_q\right\| + \left\|S_{q+2} - S_{q+1}\right\| + \cdots + \left\|S_p - S_{p-1}\right\| \\
&\leq \left[\beta^q + \beta^{q+1} + \cdots + \beta^{p-1}\right] \left\|S_1 - S_0\right\| \\
&\leq \beta^q \left[1 + \beta + \cdots + \beta^{p-q-1}\right] \left\|S_1 - S_0\right\| \\
&\leq \beta^q \left[\frac{1-\beta^{p-q}}{1-\beta}\right] \left\|x_1\right\|
\end{aligned}
$$

Since, $0 < \beta = kT\left[\left(1+\sum_{k=0}^{n} a_k\right)^{-1} \sum_{k=0}^{n} a_k + 1\right] < 1$, and $p > q$ then, $(1-\beta^{p-q}) \leq 1$. Consequently,

$$
\begin{aligned}
\left\|S_p - S_q\right\| &\leq \frac{\beta^q}{1-\beta} \left\|x_1\right\| \\
&\leq \frac{\beta^q}{1-\beta} \max_{t \in J} \left|x_1(t)\right|
\end{aligned}
$$

but, $|x_1(t)| < c$ and as $q \to \infty$ then, $\left\|S_p - S_q\right\| \to 0$ and hence, $\{S_p\}$ is a Cauchy sequence in this Banach space E so, the series $\sum_{i=0}^{\infty} x_i(t)$ converges. ■

3.3. **Error analysis.** **Theorem 3**: *The maximum absolute truncation error of the solution (9) to the problem (1)-(2) is estimated to be,*

$$\left\|x - \sum_{i=0}^{q} x_i\right\| \leq \frac{\beta^q}{1-\beta} \left\|x_1\right\|$$

Proof: From Theorem 2 we have,

$$\|S_p - S_q\| \leq \frac{\beta^q}{1-\beta} \max_{t\in J} |x_1(t)|$$

but, $S_p = \sum_{i=0}^{p} y_i(t)$ as $p \to \infty$ then, $S_p \to y(t)$ so,

$$\|x - S_q\| \leq \frac{\beta^q}{1-\beta} \|x_1\|$$

so, the maximum absolute truncation error in the interval J is,

$$\left\| x - \sum_{i=0}^{q} x_i \right\| \leq \frac{\beta^q}{1-\beta} \|x_1\|$$

and this completes the proof. ■

4. Numerical Examples

Example 1 Let $\alpha > 0$. Consider the following example,

$$\frac{dx}{dt} = \frac{1}{20}x^2 + \frac{1}{10}x, \qquad t \in (0,5), \tag{11}$$

$$x(0) + \alpha x\left(\frac{1}{2}\right) = 1, \tag{12}$$

We prove here, firstly, that as $\alpha \to 0$ the solution of this nonlocal problem continuo to the solution of the usual Cauchy problem (with $\alpha = 0$). This proves the validity of our algorithm.
Using equation (7), problem (11)-(12) will be

$$x(t) = \frac{1}{1+\alpha} - \frac{\alpha}{1+\alpha}\int_0^{1/2}\left[\frac{1}{20}x^2(s) + \frac{1}{10}x(s)\right]ds + \int_0^t \left[\frac{1}{20}x^2(s) + \frac{1}{10}x(s)\right]ds, \tag{13}$$

Applying ADM to equation (13), we have

$$x_0(t) = \frac{1}{1+\alpha}, \tag{14}$$

$$x_i(t) = \frac{-\alpha}{1+\alpha}\int_0^{1/2}\left[\frac{1}{20}A_{i-1}(s) + \frac{1}{10}x_{i-1}(s)\right]ds.$$

$$+ \int_0^t \left[\frac{1}{20}A_{i-1}(s) + \frac{1}{10}x_{i-1}(s)\right]ds, \qquad i \geq 1. \tag{15}$$

From equations (14) and (15), the solution of the problem (11)-(12) is,

$$x(t) = \lim_{m\to\infty}\sum_{i=0}^{m} x_i(t). \tag{16}$$

Figures 1.a - 1.d show ADM solution (when $\alpha = 0.1, 0.001, 0.00001, 0$ respectively, and $m = 5$).

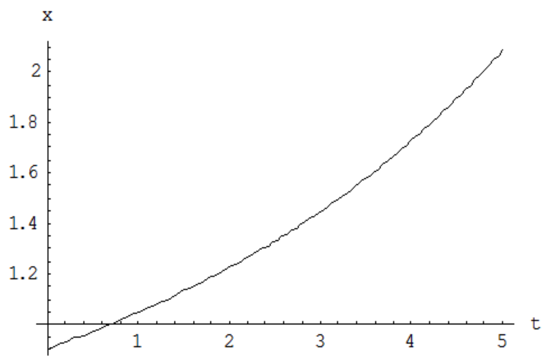

Fig (1-a): ADM solution $[\alpha = 0.1]$

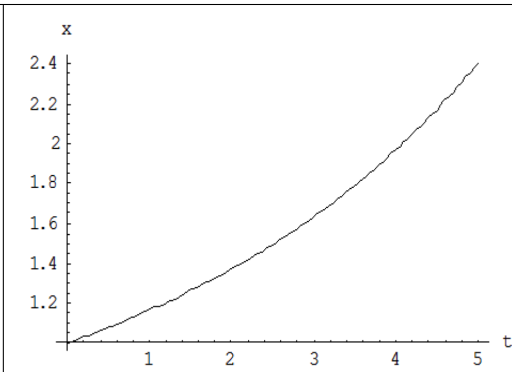

Fig (1-b): ADM solution $[\alpha = 0.001]$

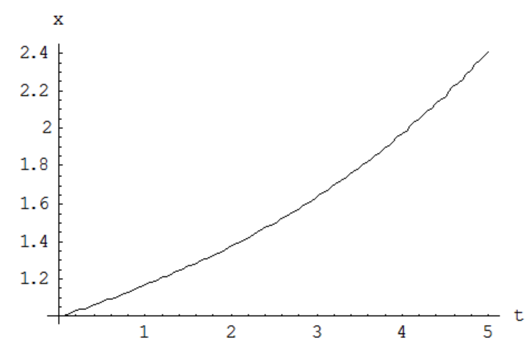

Fig (1-c): ADM solution $[\alpha = 0.00001]$

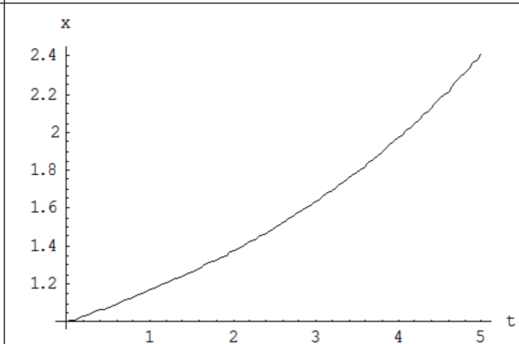

Fig (1-d): ADM solution $[\alpha = 0]$

From this example it is clear that the nonlocal solution tends to the local solution as $\alpha \to 0$.

Example 2 *Consider the following nonlocal DE,*

$$\frac{dx}{dt} = \frac{1}{10}t^2 e^{x^2} - \frac{1}{5}t^3 x, \qquad t \in (0, 3], \tag{17}$$

$$x(0) + \frac{1}{2}x\,(0.1) - \frac{1}{4}x\,(0.2) = \frac{1}{2}, \tag{18}$$

Using equation (7), problem (17)-(18) will be

$$\begin{aligned} x\,(t) \quad &= \quad 0.4 - \frac{2}{5}\int_0^{0.1}\left[\frac{1}{10}s^2 e^{x^2(s)} - \frac{1}{5}s^3 x\,(s)\right] ds + \frac{1}{5}\int_0^{0.2}\left[\frac{1}{10}s^2 e^{x^2(s)} - \frac{1}{5}s^3 x\,(s)\right] ds \\ &\quad + \int_0^{t}\left[\frac{1}{10}s^2 e^{x^2(s)} - \frac{1}{5}s^3 x\,(s)\right] ds, \end{aligned} \tag{19}$$

Applying ADM to equation (19), we have

$$x_0(t) = 0.4, \tag{20}$$

$$x_i(t) = -\frac{2}{5}\int_0^{0.1}\left[\frac{1}{10}s^2A_{i-1}(s) - \frac{1}{5}s^3x_{i-1}(s)\right]ds + \frac{1}{5}\int_0^{0.2}\left[\frac{1}{10}s^2A_{i-1}(s) - \frac{1}{5}s^3x_{i-1}(s)\right]ds$$
$$+\int_0^{t}\left[\frac{1}{10}s^2A_{i-1}(s) - \frac{1}{5}s^3x_{i-1}(s)\right]ds, \qquad i \geq 1. \tag{21}$$

From equations (20) and (21), the solution of the problem (17)-(18) is,

$$x(t) = \lim_{m\to\infty}\sum_{i=0}^{m} x_i(t). \tag{22}$$

Figure 2 shows ADM solution (when $m = 5$).

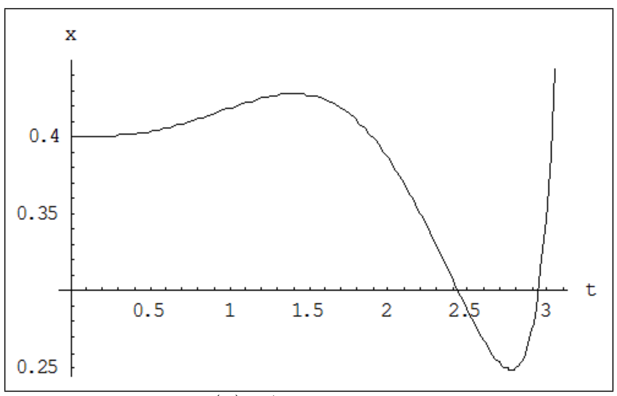

Fig (2): ADM solution.

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E. A. A. Ziada, Faculty of Engineering, Delta University for Science and Technology, Gamasa, Egypt.
E-mail address: eng_emanziada@yahoo.com