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INTRODUCTION TO PIECEWISE ANALYTIC METHOD

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Abstract. The purpose of this paper is introducing the piecewise analytic method (PAM). PAM is a new method for solving differential equations. PAM gives an approximate analytic solution which is very accurate. The error in PAM approaches to zero as the interval size approaches zero or the convergence order goes to infinity. The piecewise analytic method can be made to be of any order of accuracy. Comparison between PAM and Runge-Kutta enhances the use of PAM.

1. Introduction

Mathematical modeling of many engineering and physical systems leads to nonlinear ordinary and partial differential equations. In general, exact solutions of such equations are unknown and thus numerical integration, perturbation techniques or geometrical methods have been applied to obtain approximate solutions. However, the ability to solve nonlinear equations by analytical methods is important because linearization changes the problem being analyzed to a different problem, perturbation methods are only reasonable when nonlinear effects are very small, and the numerical methods need a substantial amount of computations but only lead to limited information. Such procedures change the actual problem to make it tractable by the conventional methods, lead to loss of most important information. These approaches sometimes change the solution seriously ([1], [2] and [3]).

The drawback in the old methods is the lake of adequate control of the accuracy of the results and the local error due to truncation. The reason for this lack of accuracy in standard numerical methods is the short lengths of the equivalent power series. For example, a fourth order Runge-Kutta method is equivalent to a Taylor series with only five terms obtained by piecewise analytic method (PAM) [4]. A comparison between PAM and Runge-Kutta method is shown in one of the following sections. The Runge-Kutta method is one of the most famous and popular method, which is used for solving ordinary differential equations. The Runge-Kutta method is named for its' creators Carl Runge(1856-1927) and Wilhelm Kutta (1867-1944). The Runge-Kutta formulas are available from order 2 up to order 10. It should

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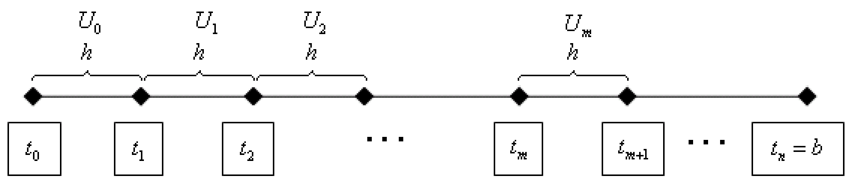


FIGURE 1.

be noted that no Runge-Kutta formula of order 11 is available at present (see [5]-[11]).

The piecewise analytic method (PAM) introduces a new treatment for solving nonlinear differential equations [12]. The PAM is based on dividing the solution interval into subintervals. Next, we obtain an approximate analytic solution which is very accurate and can be applied to each subinterval successively. The approximate analytic solution is based on truncated Taylor series ([13] and [14]) or Padé approximants ([15], [16] and [17]). In PAM, the solution accuracy can be controlled as needed. The PAM gives the exact solution in some special cases.

In the following sections, I'll explain the piecewise analytic method and then apply the method to two case-studies which show how much the piecewise analytic method is effective. Error estimation is presented in section 4 and a comparison between PAM and Runge-Kutta method is shown in section 5.

2. Piecewise Analytic Method

Consider the general first order differential equation:

$$u' = \phi(t, u), \qquad u(t_0) = f_0,\ qquadt_0 \leq t \leq b. \tag{1}$$

For solving (1) using piecewise analytic method, the interval $t_0 \leq t \leq b$ is divided into n equal parts, each of length h, by the points $t_m = mh, \ \ m = 0, 1, 2, ..., n$ The value $h = \frac{b-t_0}{n}$ is called the subinterval length. The points t_m are called interval points see Fig 1.

U_m denotes to the approximate analytic solution in the m^{th} subinterval $[t_m, t_{m+1}]$. U_m can be applied to any subinterval m $(t \in [t_m, t_{m+1}], \ \ m = 0, 1, 2, ..., (n-1))$.

For calculating U_m , Eq. 1 is written in the form.

$$\frac{dU_m}{dt} = \phi(t, U_m),$$
$$U_m(t_m) = f_m, \qquad t \in [t_m, t_{m+1}], \qquad m = 0, 1, 2, ..., (n-1). \tag{2}$$

Then using any symbolic mathematical program like Mathematica for obtaining the approximate solution U_m. I have two forms of approximate solutions, one is the truncated Taylor series solution and the other is the Padé approximants solution.

In the case of truncated Taylor solution [13], U_m is defined according to the needed accuracy. If we need accuracy $O(h^s)$, U_m will take the form

$$U_m(t) = \sum_{n=0}^{s-1} c_n (t - t_m)^n = \sum_{n=0}^{s-1} \frac{(t - t_m)^n}{n!} \left(\frac{dU_m}{dt}\right)_{t=t_m} \qquad t \in [t_m, t_{m+1}] \tag{3}$$

In the case of Padé approximants solution([15],[16],and [17]), will take the form

$$U_m(t) = \frac{P_l}{Q_k} = \frac{\sum_{n=0}^{l} p_n(t-t_m)^n}{\sum_{n=0}^{k} q_n(t-t_m)^n} \quad \text{where } l+k = s-1, \qquad t \in [t_m, t_{m+1}] \tag{4}$$

if we need the accuracy to be of $O(h^s)$.

Another approximate solution is under study. The final step is using the approximate analytic solution formula U_m and apply it to each subinterval successively with the initial value $f_m = U_{m-1}(t_m)$, $U_{-1}(t_0) = f_0$.

Notes:

- We have two methods for calculating(3). The first method is the substitution by $U_m(t) = \sum_{n=0}^{s-1} c_n(t-t_m)^n$ and its derivatives into the ODE, then equating the coefficients of each power of $(t-t_m)^n$sum to zero to get a recurrence relation. The recurrence relation expresses a coefficient c_n in terms of the coefficients c_m where $m < n$. The second method is calculating $U_m(t) = \sum_{n=0}^{s-1} \frac{(t-t_m)^n}{n!} \left(\frac{dU_m}{dt}\right)_{t=t_m}$ which is based on the manipulation of the function formula by classical differential calculus techniques. The results are constants that represent the value of derivatives at the point of evaluation.
- The PAM gives the exact solution in two cases:
  1. If the exact solution is a polynomial with order w and the truncated series approximation (3) is used with $s-1 \geq w$.
  2. If the exact solution is a rational function $\frac{\sum_{n=0}^{z} p_n(t-t_m)^n}{\sum_{n=0}^{w} q_n(t-t_m)^n}$ and the Padé approximants (4) is used with $l \geq z$ and $k \geq w$.
- The truncated series (3) is suitable if the solution has no poles and bounded, if not, the Padé approximants (4) is more suitable than truncated series (3).
- I don't know the best form for Padé approximants (4) but by experience I prefer $l = k = even$.
- The Padé approximants coefficients $p_n(n = 0, 1, 2, ..., l)$ and $q_n(n = 0, 1, 2, ..., k)$ are determined by ([15], [16] and [17])

$$\sum_{n=0}^{s \geq l+k} c_n(t-t_m)^n - \frac{\sum_{n=0}^{l} p_n(t-t_m)^n}{\sum_{n=0}^{k} q_n(t-t_m)^n} = O\left((t-t_m)^{l+k+1}\right) \tag{5}$$

setting $q_0 = 1$and multiply (5) by $\sum_{n=0}^{k} q_n(t-t_m)^n$, which linearizes the equations coefficient. It can be written out in more detail as:

$$\begin{gathered} c_{i+1} + c_i q_1 + ... + c_{i-k+1} q_k = 0 \\ c_{i+2} + c_{i+1} q_1 + ... + c_{i-k+2} q_k = 0 \\ \vdots \\ c_{i+k} + c_{i+k-1} q_1 + ... + c_i q_k = 0 \end{gathered} \tag{6}$$

$$\begin{gathered} c_0 = p_0 \\ c_1 + c_0 q_1 = p_1 \\ c_2 + c_1 q_1 + c_0 q_2 = p_2 \\ \vdots \\ c_i + c_{i-1} q_1 + c_0 q_i = p_i \end{gathered} \tag{7}$$

Once, the $q's$ are known from equations (6), equations (7) can be solved easily. If equations (6) and (7) are nonsingular, then they can be solved directly as follows;

$$U_m(t) = \frac{\det \begin{vmatrix} c_{i-k+1} & c_{i-k+2} & \cdots & c_{i+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_i & c_{i+1} & \cdots & c_{i+k} \\ \sum_{j=k}^{i} c_{i-k} t^j & \sum_{j=k-1}^{i} c_{i-k+1} t^j & \vdots & \sum_{j=0}^{i} c_j t^j \end{vmatrix}}{\begin{vmatrix} c_{i-k+1} & c_{i-k+2} & \cdots & c_{i+1} \\ \cdots & \cdots & \ddots & \cdots \\ c_i & c_{i+1} & \cdots & c_{i+k} \\ t^k & t^{k-1} & \cdots & 1 \end{vmatrix}} \tag{8}$$

For $l = 1$ and $k = 1$

$$\begin{array}{cc} p_0 = c_0, & q_0 = 1 \\ p_1 = \frac{c_1^2 - c_0 c_2}{c_1}, & q_1 = -\frac{c_2}{c_1}. \end{array} \tag{9}$$

For $l = 1$ and $k = 2$

$$\begin{array}{cc} p_0 = c_0, & q_0 = 1 \\ p_1 = \frac{c_1^3 - 2c_0 c_1 c_2 + c_0^2 c_3}{c_1^2 - c_0 c_2}, & q_1 = \frac{-c_1 c_2 + c_0 c_3}{c_1^2 - c_0 c_2}, \\ & q_2 = \frac{-c_2^2 + c_1 c_3}{-c_1^2 + c_0 c_2}. \end{array} \tag{10}$$

For $l = 2$ and $k = 1$

$$\begin{array}{cc} p_0 = c_0, & \\ p_1 = \frac{c_1 c_2 - c_0 c_3}{c_2}, & q_0 = 1 \\ p_2 = \frac{c_2^2 - c_1 c_3}{c_2}, & q_1 = -\frac{c_3}{c_2}. \end{array} \tag{11}$$

For $l = 2$ and $k = 2$

$$\begin{array}{cc} p_0 = c_0, & q_0 = 1 \\ p_1 = \frac{-c_1 c_2^2 + c_1^2 c_3 + c_0 c_2 c_3 - c_0 c_1 c_4}{-c_2^2 + c_1 c_3}, & q_1 = \frac{-c_2 c_3 + c_1 c_4}{c_2^2 - c_1 c_3}, \\ p_2 = \frac{c_2^3 - 2c_1 c_2 c_3 + c_0 c_3^2 + c_1^2 c_4 - c_0 c_2 c_4}{c_2^2 - c_1 c_3}, & q_2 = \frac{-c_3^2 + c_2 c_4}{-c_2^2 + c_1 c_3}. \end{array} \tag{12}$$

For $l = 3$ and $k = 3$

$$
\begin{gathered}
p_0 = c_0, \\
p_1 = \frac{\begin{pmatrix} -c_1c_3^3 + 2c_1c_2c_3c_4 + c_0c_3^2c_4 - c_1^2c_4^2 - c_0c_2c_4^2 - c_1c_2^2c_5 + \\ c_1^2c_3c_5 - c_0c_2c_3c_5 + c_0c_1c_4c_5 + c_0c_2^2c_6 - c_0c_1c_3c_6 \end{pmatrix}}{\left(-c_3^3 + 2c_2c_3c_4 - c_1c_4^2 - c_2^2c_5 + c_1c_3c_5\right)}, \\
p_2 = \frac{\begin{pmatrix} c_2c_3^3 - 2c_2^2c_3c_4 - c_1c_3^2c_4 + 2c_1c_2c_4^2 + c_0c_3c_4^2 + c_2^3c_5 - c_0c_3^2c_5 - \\ c_1^2c_4c_5 - c_0c_2c_4c_5 + c_0c_1c_5^2 - c_1c_2^2c_6 + c_1^2c_3c_6 + c_0c_2c_3c_6 - \\ c_0c_1c_4c_6 \end{pmatrix}}{\left(c_3^3 - 2c_2c_3c_4 + c_1c_4^2 + c_2^2c_5 - c_1c_3c_5\right)}, \\
p_3 = \frac{\begin{pmatrix} c_3^4 - 3c_2c_3^2c_4 + c_2^2c_4^2 + 2c_1c_3c_4^2 - c_0c_4^3 + 2c_2^2c_3c_5 - 2c_1c_3^2c_5 - \\ 2c_1c_2c_4c_5 + 2c_0c_3c_4c_5 + c_1^2c_5^2 - c_0c_2c_5^2 - c_2^3c_6 + 2c_1c_2c_3c_6 - \\ c_0c_3^2c_6 - c_1^2c_4c_6 + c_0c_2c_4c_6 \end{pmatrix}}{\left(c_3^3 - 2c_2c_3c_4 + c_1c_4^2 + c_2^2c_5 - c_1c_3c_5\right)}, \\
q_0 = 1 \\
q_1 = \frac{-c_3^2c_4 + c_2c_4^2 + c_2c_3c_5 - c_1c_4c_5 - c_2^2c_6 + c_1c_3c_6}{c_3^3 - 2c_2c_3c_4 + c_1c_4^2 + c_2^2c_5 - c_1c_3c_5}, \\
q_2 = \frac{c_3c_4^2 - c_3^2c_5 - c_2c_4c_5 + c_1c_5^2 + c_2c_3c_6 - c_1c_4c_6}{c_3^3 - 2c_2c_3c_4 + c_1c_4^2 + c_2^2c_5 - c_1c_3c_5}, \\
q_3 = \frac{-c_4^3 + 2c_3c_4c_5 - c_2c_5^2 - c_3^2c_6 + c_2c_4c_6}{c_3^3 - 2c_2c_3c_4 + c_1c_4^2 + c_2^2c_5 - c_1c_3c_5}.
\end{gathered} \tag{13}
$$

And so on, where we can calculate any desired Padé approximants by using any symbolic mathematical program for any series $U_m(t) = \sum_{n=0}^{s} c_n(t - t_m)^n$ and then all what we will do is only substituting by $c_n's$ in the suitable $p's$ and $q's$ for obtaining the desired Padé approximants.

3. Case-Studies

In the following case-studies, I show how PAM can be used for solving ordinary differential equation (linear and non-linear).

3.1. **Case-Study 1.** Consider the differential equation

$$
u'(t) = -u\tan(t) - \frac{1}{\cos(t)}, \qquad u(0) = 1. \tag{14}
$$

This is a linear problem which has the exact solution

$$
u(t) = \cos(t) - \sin(t). \tag{15}
$$

Defining a differential equation for each subinterval m from (14)

$$
\frac{dU_m}{dt} = -U_m\tan(t) - \frac{1}{\cos(t)}, \qquad U_m(t_m) = f_m, \qquad t \in [t_m, t_{m+1}]. \tag{16}
$$

where $f_m = U_{m-1}(t_m)$, $U_{-1}(t_0) = u(0) = 1$, $m = 0, 1, 2, ..., n - 1$.
Substituting by $U_m(t) = \sum_{n=0}^{s} c_n(t - t_m)^n$ and its derivatives into (16) leads to

$$
\begin{gathered}
\left(\sum_{n=0}^{s} \left(\frac{d^n}{dt^n}\cos(t)\right)_{t=t_m} (t - t_m)^n\right) \sum_{n=1}^{s} nc_n(t - t_m)^{n-1} = \\
-\left(\sum_{n=0}^{s} \left(\frac{d^n}{dt^n}\sin(t)\right)_{t=t_m} (t - t_m)^n\right) \sum_{n=0}^{s} c_n(t - t_m)^n - 1, \\
c_0 = f_m, t \in [t_m, t_{m+1}].
\end{gathered} \tag{17}
$$

solving (17) leads to:

$$\begin{gathered} c_0 = f_m, \\ c_1 = -\sec(t_m)(1 + f_m \sin(t_m)), \\ c_2 = -\frac{f_m}{2}, \\ c_3 = \frac{\sec(t)}{6} + \frac{1}{6} f_m \tan(t_m), \\ c_4 = \frac{f_m}{24}, \\ c_5 = -\frac{\sec(t)}{120} - \frac{1}{120} f_m \tan(t_m), \\ c_6 = -\frac{f_m}{720}, \\ c_7 = \frac{\sec(t)}{5040} + \frac{f_m \tan(t_m)}{5040}, \\ c_8 = \frac{f_m}{40320}, \\ \vdots \end{gathered} \tag{18}$$

Substituting by (18) into(3) for obtaining the needed approximate analytic Taylor series and substituting by (18) into one of (9)-(13) for obtaining the appropriate $p's$ and $q's$ which are used to obtain Padé approximants. The PAM truncated series solution $O(h^5)$ is

$$\begin{gathered} U_m(t) \simeq f_m - \sec(t_m)(1 + f_m \sin(t_m))(t - t_m) - \frac{f_m}{2}(t - t_m)^2 + \\ \frac{1}{6} \sec(t_m)(1 + f_m \sin(t_m))(t - t_m)^3 + \frac{f_m}{24}(t - t_m)^4, \qquad t \in [t_m, t_{m+1}]. \end{gathered} \tag{19}$$

The PAM Padé solution $O(h^5)$ is

$$\begin{gathered} U_m(t) \simeq (-24(t - t_m)\sec(t_m)^3 + 24\sec(t_m)^2 f_m - 14(t - t_m)^2 \sec(t_m)^2 f_m - \\ 30(t - t_m)\sec(t_m) f_m^2 + 36 f_m^3 - 15(t - t_m)^2 f_m^3 - 72(t - t_m)\sec(t_m)^2 f_m \tan(t_m) + \\ 48\sec(t_m) f_m)^2 \tan(t_m) + 24 f_m^3 \tan(t_m)^2 - 28(t - t_m)^2 \sec(t_m) f_m^2 \tan(t_m) - \\ 30(t - t_m)\tan(t_m) f_m^3 + 24 f_m^3 \tan(t_m)^2 - 14(t - t_m)^2 \tan(t_m)^2 f_m^3 \\ 24(t - t_m)\tan(t_m)^3 f_m^3 - 72(t - t_m)\sec(t_m)\tan(t_m)^2 f_m^2)/(24\sec(t_m)^2 + \\ 4(t - t_m)^2 \sec(t_m)^2 + 6(t - t_m)\sec(t_m) f_m + 36 f_m^2 + 3(t - t_m)^2 f_m^2 + \\ 48\sec(t_m) f_m \tan(t_m) + 8((t - t_m)^2 \sec(t_m) f_m \tan(t_m) + 6(t - t_m) f_m^2 \tan(t_m) + \\ 24\tan(t_m)^2 f_m^2 + 4(t - t_m)^2 f_m^2 \tan(t_m)^2), \qquad t \in [t_m, t_{m+1}]. \end{gathered} \tag{20}$$

The PAM truncated series solution $O(h^9)$ is

$$\begin{gathered} U_m(t) \simeq f_m - \sec(t_m)(1 + f_m \sin(t_m))(t - t_m) - \frac{f_m}{2}(t - t_m)^2 + \\ \frac{1}{6} \sec(t_m)(1 + f_m \sin(t_m))(t - t_m)^3 + \frac{f_m}{24}(t - t_m)^4 - \\ (\frac{\sec(t)}{120} - \frac{1}{120} f_m \tan(t_m))(t - t_m)^5 - (\frac{f_m}{720})(t - t_m)^6 + (\frac{\sec(t)}{5040} + \\ \frac{f_m \tan(t_m)}{5040})(t - t_m)^7 + (\frac{f_m}{40320})(t - t_m)^8, \qquad t \in [t_m, t_{m+1}]. \end{gathered} \tag{21}$$

It is massive to write the obtained PAM Padé solution $O(h^9)$ but the results are summarized in the following. Figure 2 shows the exact solution of (14). Table 1 shows the absolute error between (14) exact solution and Padé approximants and Taylor series for different values of h . Table 2 shows the absolute error between (14) exact solution and different forms of Padé approximants and Taylor series for $h = 0.1$. As seen in Table 1 and 2, the truncated series is better than Padé. In this case, the system is an oscillatory system so we can use alternative technique for Padé, where we can apply Laplace transformation to the series obtained by PAM, and then converted the transformed series into a meromorphic function by forming its Padé approximant, and then inverted the approximant, which yields a better solution that is periodic (for more details see [18]-[20]).

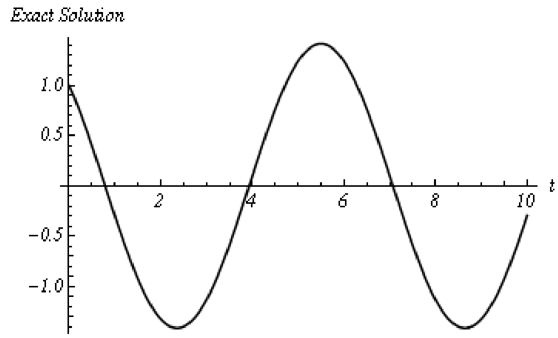


FIGURE 2. The exact solution of 14.

Table 1: The absolute error between (14) exact solution and Padé approximants and Taylor series for different values of h.

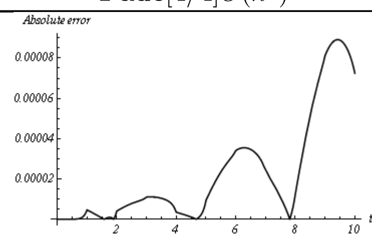
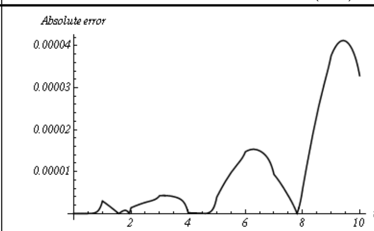
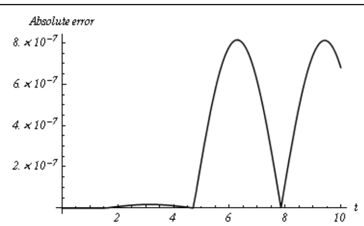
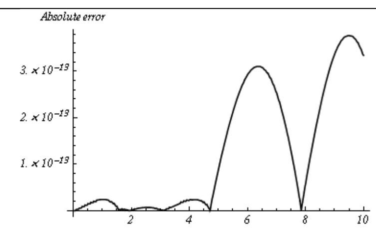
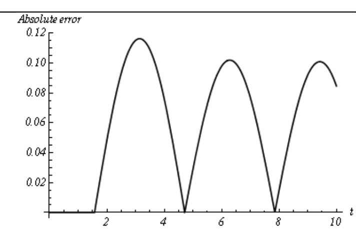
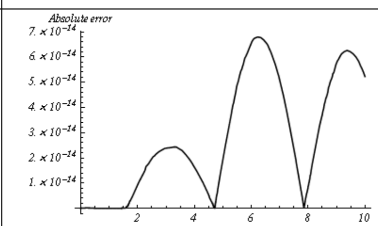
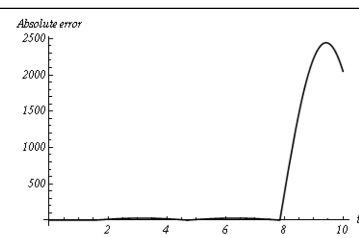
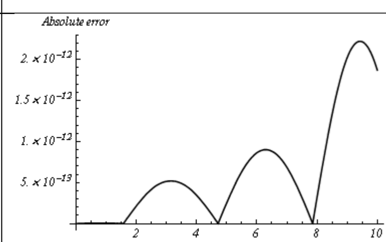
h	**Padé**$[4/4]O(h^9)$	**Truncated Series** $O(h^9)$
1		
0.1		
0.01		
0.001		

Table 2: The absolute error between (14) exact solution and different forms of Padé approximants and Taylor series for $h = 0.1$.

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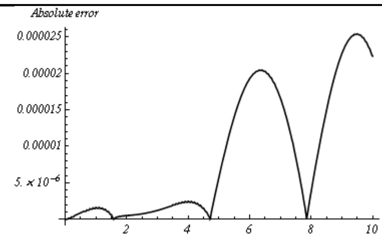
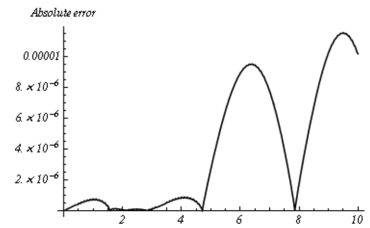
<tr><td>5</td><td>
</td><td>
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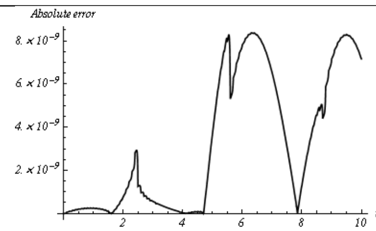
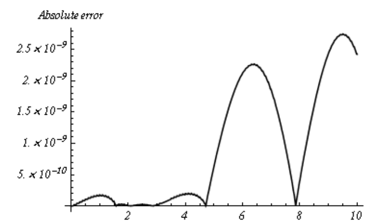
<tr><td>7</td><td>
</td><td>
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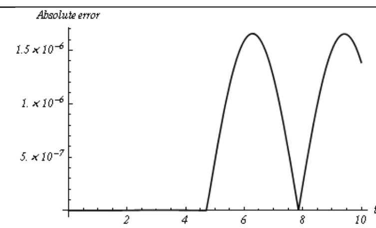
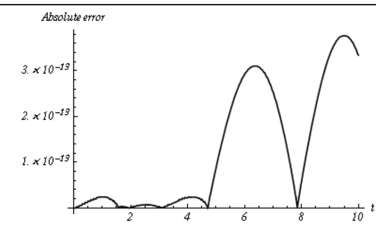
<tr><td>9</td><td>
</td><td>
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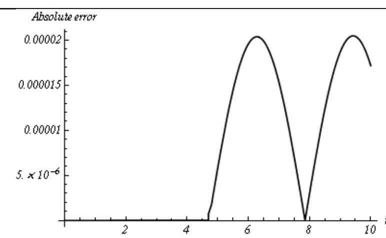
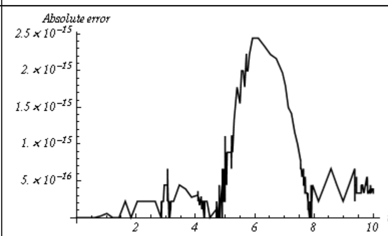
<tr><td>11</td><td>
</td><td>
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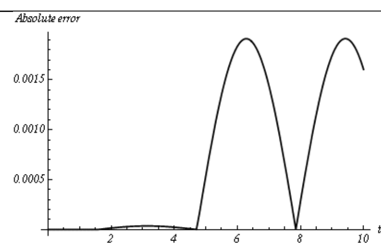
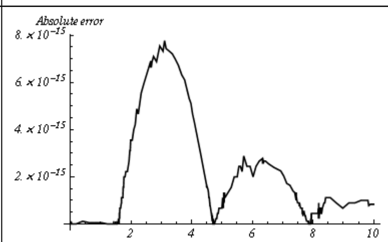
<tr><td>13</td><td>
</td><td>
</td></tr>
</table>

3.2. **Case-Study 2.** Consider the nonlinear differential equation

$$u'(t) = (3 - 2t)u^2, \qquad u(0) = \frac{1}{2}. \tag{22}$$

This is a nonlinear problem which has the exact solution

$$u(t) = \frac{1}{(t^2 - 3t + 2)}. \tag{23}$$

Defining a differential equation for each subinterval m from (22)

$$\frac{dU_m}{dt} = (3 - 2t)\left(U_m\right)^2, \qquad U_m(t_m) = f_m, \qquad t \in [t_m, t_{m+1}]. \tag{24}$$

where $f_m = U_{m-1}(t_m)$, $U_{-1}(t_0) = u(0) = \frac{1}{2}$, $m = 0, 1, 2, ..., n-1$.
Substituting by $U_m(t) = \sum_{n=0}^{s} c_n (t - t_m)^n$ and its derivatives into (24) leads to

$$\begin{array}{c} \sum_{n=1}^{s} n c_n (t - t_m)^{n-1} = (3 - 2(t_m + (t - t_m)))\left(\sum_{n=0}^{s} c_n (t - t_m)^n\right)^2, \\ \text{where} \qquad c_0 = f_m, t \in [t_m, t_{m+1}]. \end{array} \tag{25}$$

solving (25) leads to:

$$\begin{array}{c} c_0 = f_m, \\ c_1 = 3f_m^2 - 2t_m f_m^2, \\ c_2 = f_m^2(-1 + 9f_m - 12t_m f_m + 4t_m^2 f_m), \\ c_3 = -f_m^3(6 - 4t_m - 27f_m + 54t_m f_m - 36t_m^2 f_m + 8t_m^3 f_m), \\ c_4 = f_m^3(1 - 27f_m + 36t_m f_m - 12t_m^2 f_m + 81f_m^2 - 216t_m^2 f_m^2 - 96t_m^3 f_m^2 + 16t_m^4 f_m^4), \\ c_5 = -f_m^4(-9 + 6t_m + 108f_m - 216t_m f_m + 144t_m^2 f_m - 32t_m^3 f_m - 243f_m^2 + \\ 810t_m f_m^2 - 1080t_m^2 f_m^2 + 720t_m^3 f_m^2 + t_m^5 f_m^2 \\ \vdots \end{array} \tag{26}$$

Substituting by (26) into(3) for obtaining the needed approximate analytic Taylor series and substituting by (26) into one of (9)-(13) for obtaining the appropriate $p's$ and $q's$ which are used to obtain Padé approximants.
The PAM truncated series solution $O(h^5)$ is

$$\begin{array}{c} U_m(t) \simeq f_m + (3f_m^2 - 2t_m f_m^2)(t - t_m) + f_m^2(-1 + 9f_m - 12t_m f_m + 4t_m^2 f_m) \\ (t - t_m)^2 - f_m^3(6 - 4t_m - 27f_m + 54t_m f_m - 36t_m^2 f_m + 8t_m^3 f_m)(t - t_m)^3 + \\ f_m^3(1 - 27f_m + 36t_m f_m - 12t_m^2 f_m + 81f_m^2 - 216t_m^2 f_m^2 - 96t_m^3 f_m^2 + \\ 16t_m^4 f_m^4)(t - t_m)^4, \qquad t \in [t_m - t_{m+1}]. \end{array} \tag{27}$$

The PAM Padé solution $O(h^5)$ is

$$U_m(t) \simeq \frac{f_m}{1 + (-3f_m + 2t_m f_m)(t - t_m) + (t - t_m)^2 f_m}, \qquad t \in [t_m - t_{m+1}]. \tag{28}$$

The PAM Padé solution and order $O(h^5)$ above are identical with the exact solution if there is no roundoff error. The PAM truncated series is not accepted at all because of the poles of the solution. The results are summarized in the following figures. Figure 3 shows the exact solution of (22) which is identical with the obtained PAM Padé. Table 3 shows the absolute error between (22) exact solution and Taylor series for different values of . Table 4 shows the absolute error between (22) exact solution and different forms of Taylor series for $h = 0.1$.

4. Error Estimation

In fact, in the limit as h approaches zero, PAM solution is exact, since the error bound is then zero. Of course, it does not make sense to apply a zero interval size to PAM, but the point is that we can make the error as small as we wish by selecting h sufficiently small or the order of accuracy sufficiently high. In this

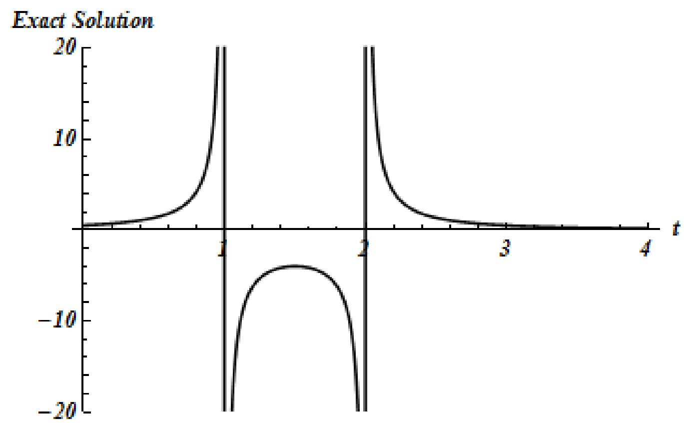


FIGURE 3. The exact solution of 22.

case the truncation error in computing $Um(t)$ using PAM is bounded in direct proportion with respect to the interval size h and order of accuracy which takes us to PAM is convergent.

Turning now to the computation of the PAM solution as described by the algorithm, we first specify values for the parameters of the problem and the initial data. Because, in any practical computing device, the number of digits allocated to a number is limited, it will probably be necessary to chop or round these numbers before they are stored. The error committed by doing this is called inherent roundoff. Also, during the computation, arithmetic operations are performed that produce results with more digits than the operands, and these results must be chopped or rounded before they are stored. This error is called arithmetic roundoff [21].

In practice, what will we do if we solve problems and don't know its exact solution or needs to change its parameters or initial conditions? If one wishes arbitrarily high accuracy, one need only choose h sufficiently small or large order of accuracy. If one has a prescribed accuracy, it is often estimated in an a posteriori manner as follows. One calculates for both h and smaller h and takes those figures which are in agreement for the two calculations. For example, if at a point and for $h = 0.1$ one finds $U = 0.876532$ while for $h = 0.01$ one finds at the same point that $U = 0.876513$, then one assumes that the result $U = 0.8765$ is an accurate result.

In the following, I'll use the notation $u_m^{[h,m]}(t)$ for denoting the PAM solution with step size h and order of accuracy $o(h^m)$.

If we take case study1 for example, Table 5 shows the difference between two PAM solutions for two different values of h, fixing the accuracy order, which indicates that the accuracy is increased as the step size h is reduced. Table 6 and Table 7 show the difference between two PAM solutions for two different order of accuracy, fixing the step size, which indicates that the accuracy is increased as the order is increased.

Table 3: The absolute error between 22 exact solution and PAM Taylor series for different values of h.

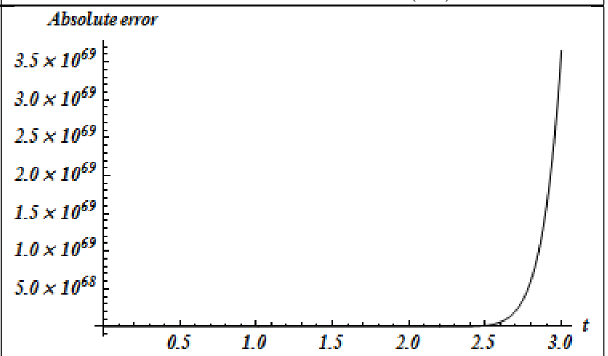
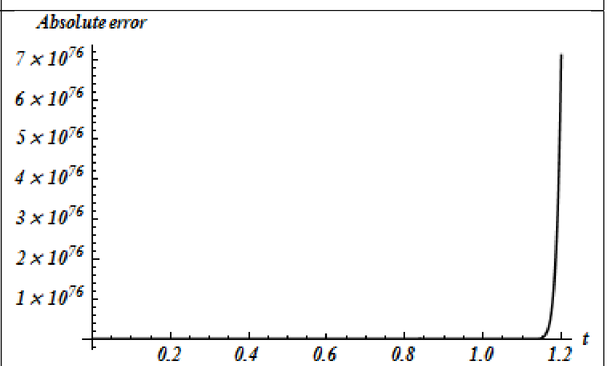
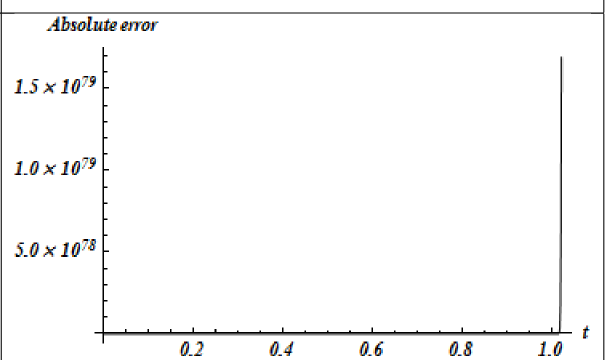
h	**Truncated Series** $O(h^9)$
1	
0.1	
0.01	
	Continued on next page

Table 3 – continued from previous page

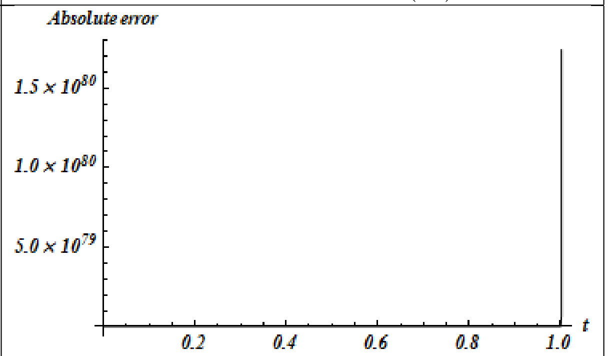
h	Truncated Series $O(h^9)$
0.001	

Table 4: The absolute error between 22 exact solution and different forms of PAM Taylor series for($h = 0.1$).

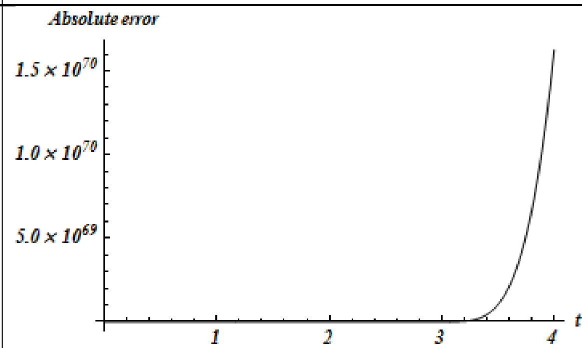
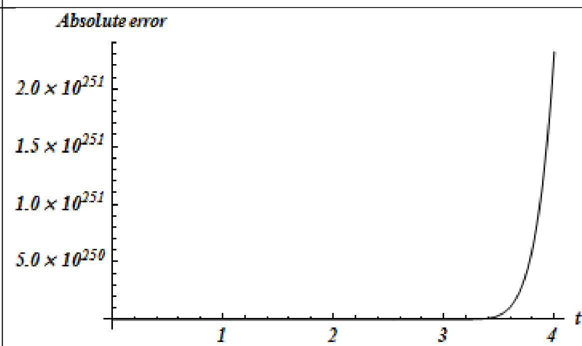
order of h	Truncated Series
5	
7	

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Table 4 – continued from previous page

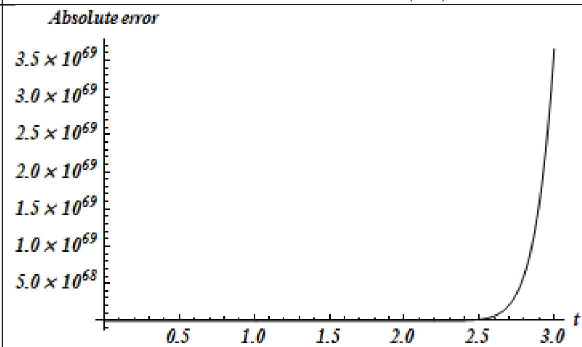
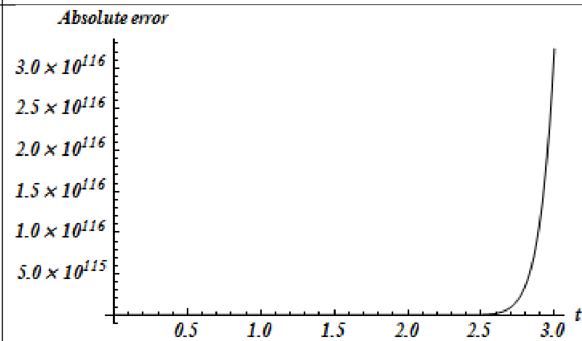
<table>
<tr><th>h</th><th>Truncated Series $O(h^9)$</th></tr>
<tr><td>9</td><td>
</td></tr>
<tr><td>11</td><td>
</td></tr>
</table>

Table 5: The absolute difference between two PAM solutions as h is changed.

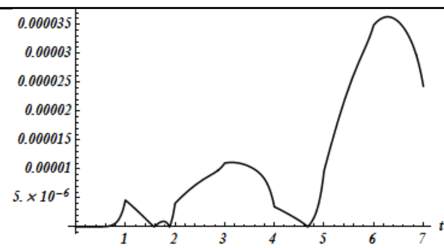
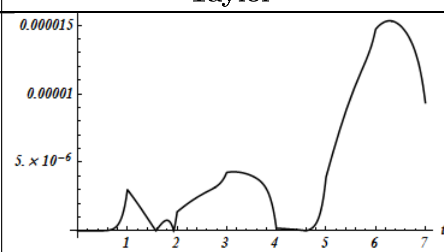
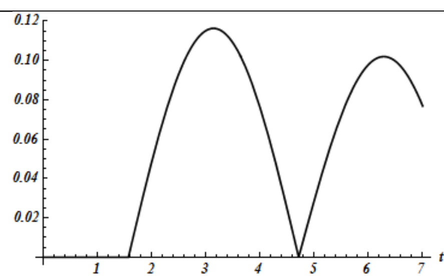
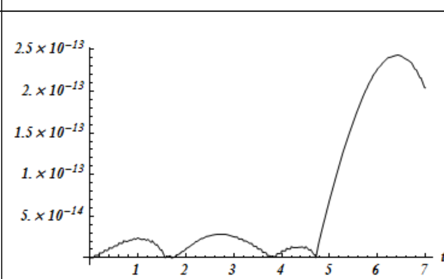
<table>
<tr><th></th><th>Padé</th><th>Taylor</th></tr>
<tr><td>$|U_m^{(1,9)} - U_m^{(0.1,9)}|$</td><td>
</td><td>
</td></tr>
<tr><td>$|U_m^{(0.1,9)} - U_m^{(0.01,9)}|$</td><td>
</td><td>
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Table 5 – continued from previous page

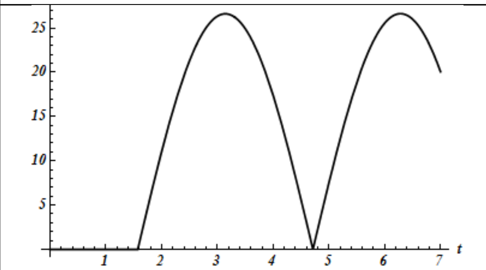
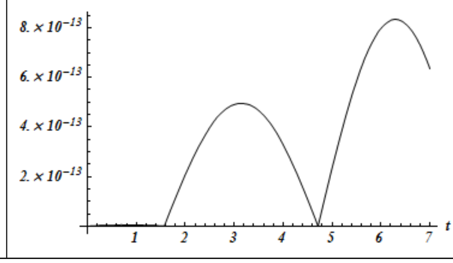
	Padé	**Taylor**
$\lvert U_m^{(0.01,9)} - U_m^{(0.001,9)} \rvert$		

Table 6: The absolute difference between two PAM Padé solutions as order of accuracy is changed.

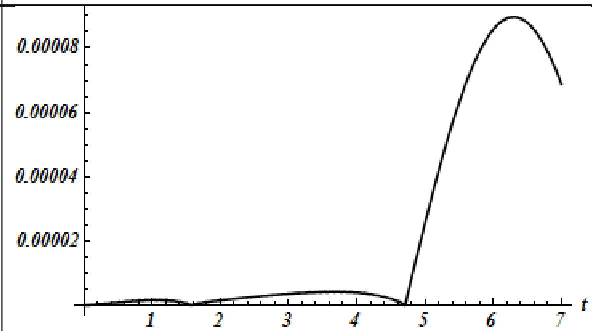
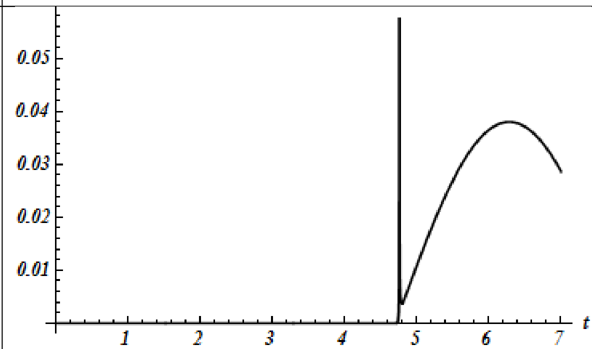
	Padé
$\lvert U_m^{(0.1,5)} - U_m^{(0.1,7)} \rvert$	
$\lvert U_m^{(0.1,7)} - U_m^{(0.1,9)} \rvert$	
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Table 6 – continued from previous page

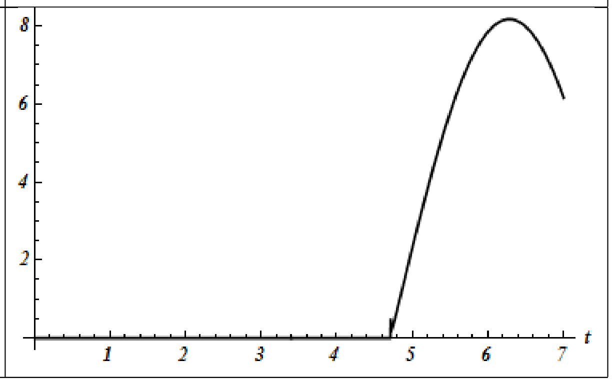
	Padé
$\lvert U_m^{(0.1,9)} - U_m^{(0.1,11)} \rvert$	

Table 7: The absolute difference between two PAM Taylor series solution as order of accuracy is changed.

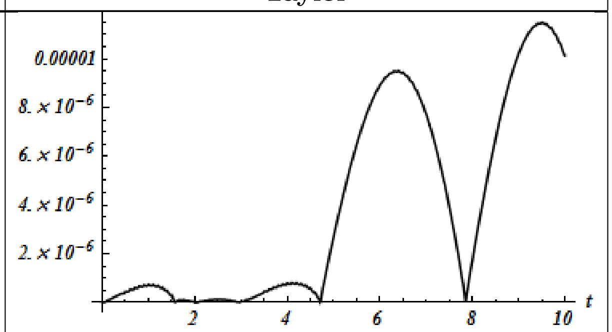
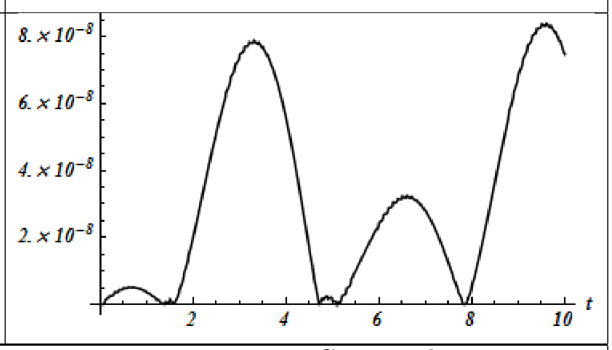
	Taylor
$\lvert U_m^{(0.1,5)} - U_m^{(0.1,6)} \rvert$	
$\lvert U_m^{(0.1,6)} - U_m^{(0.1,7)} \rvert$	

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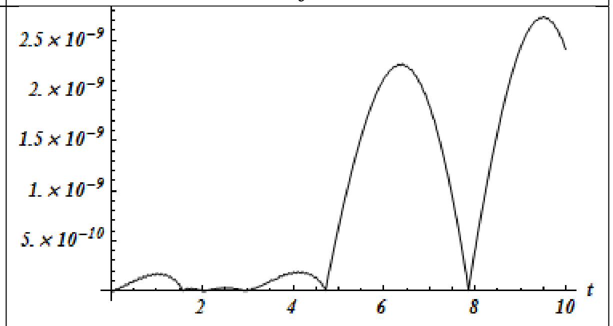
Table 7 – continued from previous page

<table>
<tr><th></th><th>Taylor</th></tr>
<tr><td>$\left|U_m^{(0.1,7)} - U_m^{(0.1,8)}\right|$</td><td>
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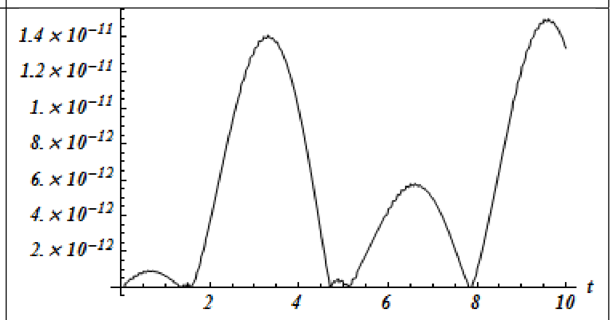
<tr><td>$\left|U_m^{(0.1,8)} - U_m^{(0.1,9)}\right|$</td><td>
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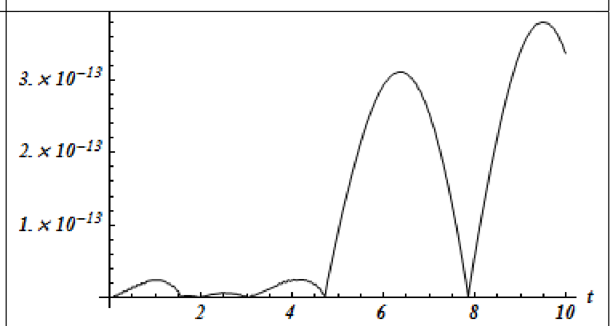
<tr><td>$\left|U_m^{(0.1,9)} - U_m^{(0.1,10)}\right|$</td><td>
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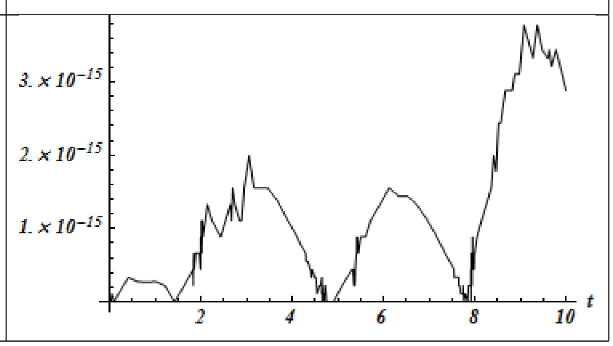
<tr><td>$\left|U_m^{(0.1,10)} - U_m^{(0.1,11)}\right|$</td><td>
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5. Comparing the PAM with Runge-Kutta Method

If we try to solve the same case studies with Runge-Kutta, it is founded that Runge-Kutta doesn't give accepted results in case study 2 because of the poles in the solution. Case study 1 can be solved by Runge-Kutta. Table 8 shows figures of the absolute error using Runge-Kutta and PAM with different order of accuracy.

Comparison between PAM and Runge-Kutta enhances the use of PAM. PAM has no order limit. PAM gives an analytic solution form which can be used for analytic differentiation and integration. In the other side, Runge-Kutta gives only numerical values at limited points of the interval.

Table 8: The absolute error between the exact solution and different methods with different order of accuracy ($h = 0.01$).

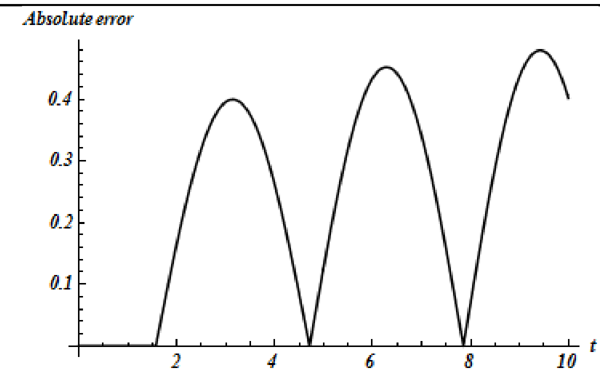
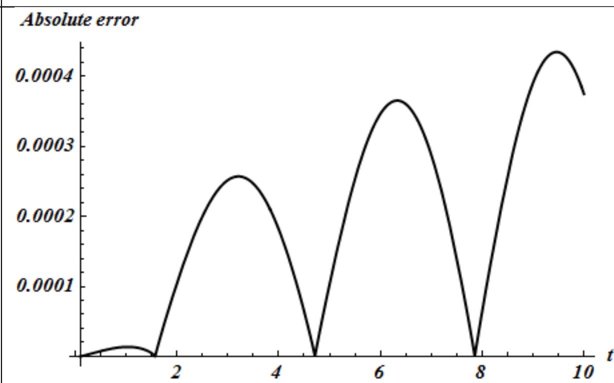
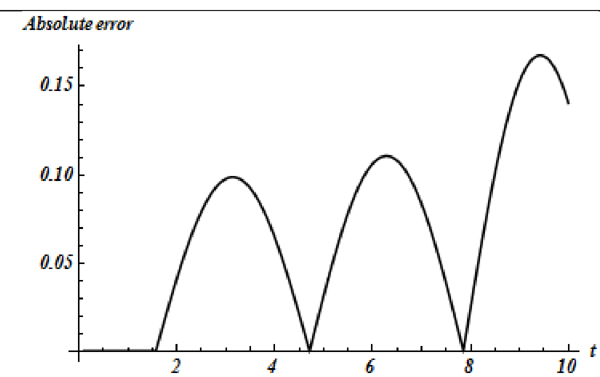
Method	**Absolute Error**
RK2-Euler-Cauchy	
PAM series $o(h^2)$	
RK4-Classical	

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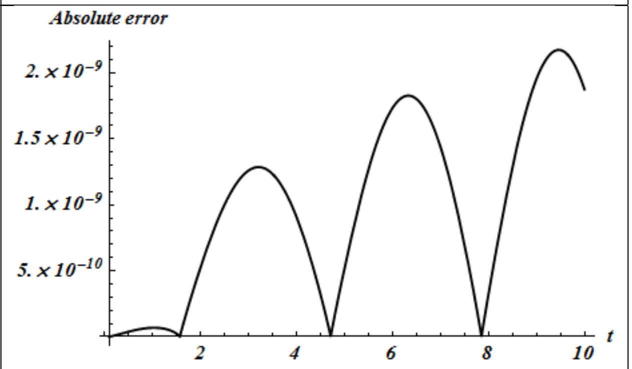
Table 8 – continued from previous page

Method	Absolute Error
PAM series $o(h^4)$	

6. Conclusion

The piecewise analytic method is nontraditional approximating method and can be promising approximating method. It can solve strongly nonlinear differential equation - something not possible through purely analytical techniques. We can control the order of accuracy as needed. In PAM, the Padé approximants can be used when the solution contains singular points and it is suitable for large interval. The truncated series solution is faster than Padé approximants and more suitable if the system doesn't have singular points. Comparison between PAM and Runge-Kutta enhances the use of PAM.

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