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DEFORMATION OF AN INFINITELY LONG, ELASTIC CYLINDER WITH CASSINI CURVE CROSS SECTION BY A BOUNDARY INTEGRAL METHOD. A NUMERICAL SOLUTION.

A. R. EL DHABA

Abstract. We use a boundary integral method to obtain the numerical solution of the first fundamental problem of elasticity for a long cylinder with Cassini curve cross-section under a uniformly distributed pressure on the boundary.

1. Introduction

The boundary integral method to be used here was suggested by Abou-Dina and Ashour [1] to solve a problem of current sheets in Electrodynamics. Abou-Dina and Helal [2] used the same method to solve a problem of nonlinear gravity wave propagation in ideal fluids. Abou-Dina and Ghaleb [3] introduced some modifications to the method in order to deal with the plane problem of linear elasticity for isotropic, homogeneous media occupying simply connected regions analytically. They also presented the numerical aspect of the method [4] for problems with non simple shape of the boundary.

In this work, we use the above mentioned boundary integral method in numerical form to solve the first fundamental problem of elasticity for a long cylinder of an isotropic material with Cassini curve cross section, subjected to a uniformly distributed pressure on the boundary.

2. Description of the physical problem

It is required to find the deformation occurring in a long cylinder of an isotropic material, with cross- section in the form of Cassini curve, subjected to an external pressure (first fundamental problem) by the boundary integral method in numerical form. The cross section of the cylinder is shown on the figure.

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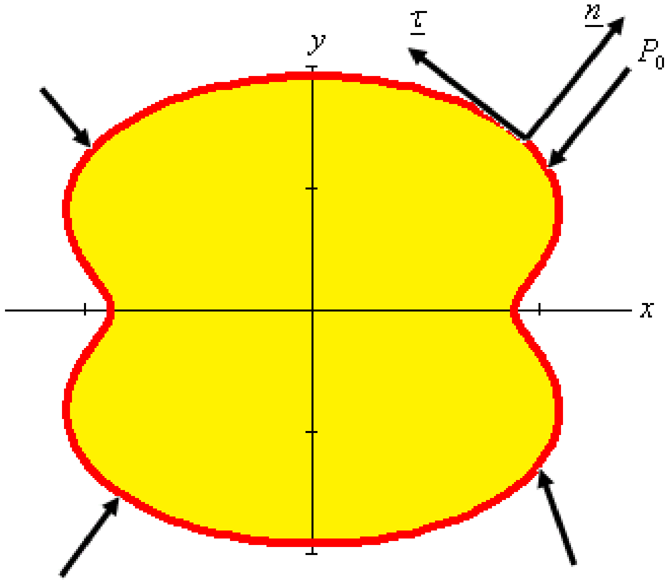

Fig. 1: Cassini curve cross-section subjected to a uniform pressure

With respect to a system of orthogonal Cartesian coordinates (x, y) with center O inside the domain D, the parametric representation for Cassini curve is

$$\frac{x}{c} = \frac{\cosh\xi\cos\theta}{\sinh^2\xi + \cos^2\theta}, \qquad \frac{y}{c} = \frac{\sinh\xi\sin\theta}{\sinh^2\xi + \cos^2\theta}. \tag{1}$$

The normal and the tangent unit vectors to the contour are

$$\boldsymbol{\tau} = \left(\frac{\dot{x}}{\omega}, \frac{\dot{y}}{\omega}\right) \qquad \mathbf{n} = \left(\frac{\dot{y}}{\omega}, -\frac{\dot{x}}{\omega}\right), \tag{2}$$

with $\omega = \sqrt{\dot{x}^2 + \dot{y}^2}$.

3. The governing equations

We solve the biharmonic equation

$$\nabla^4 U = 0,$$

inside the domain bounded externally by Cassini curve. For this, we use Airy's stress function representation in terms of two real harmonic functions:

$$U = x\Phi + y\Phi^c + \Psi, \tag{3}$$

where Φ and Ψ are harmonic functions belonging to the class of functions $C^2(D)\cap C^1(\overline{D})$, where $\overline{D}$ denotes the closure of D, and superscript "c" denotes the harmonic conjugate. These harmonic functions satisfy the following well-known integral representations of Potential Theory written as in [1, 2, 3]:

$$\Phi(x_0,y_0)=\frac{1}{2\pi}\oint_C\left(\Phi(x,y)\frac{\partial\ln R}{\partial n}+\Phi^c(x,y)\frac{\partial\ln R}{\partial\tau}\right)ds, \tag{4}$$

$$\Phi^c(x_0,y_0)=\frac{1}{2\pi}\oint_C\left(\Phi^c(x,y)\frac{\partial\ln R}{\partial n}-\Phi(x,y)\frac{\partial\ln R}{\partial\tau}\right)ds, \tag{5}$$

$$\Psi(x_0,y_0)=\frac{1}{2\pi}\oint_C\left(\Psi(x,y)\frac{\partial\ln R}{\partial n}+\Psi^c(x,y)\frac{\partial\ln R}{\partial\tau}\right)ds, \tag{6}$$

$$\Psi^c(x_0,y_0)=\frac{1}{2\pi}\oint_C\left(\Psi^c(x,y)\frac{\partial\ln R}{\partial n}-\Psi(x,y)\frac{\partial\ln R}{\partial\tau}\right)ds, \tag{7}$$

and

$$R=\sqrt{(x-x_0)^2+(y-y_0)^2}, \tag{8}$$

where (x_0,y_0) is a field point inside the domain D, and the point (x,y) belongs to the boundary. When the point (x_0,y_0) tends towards a boundary point, the expressions for the harmonic functions are

$$\Phi(x_0,y_0)=\frac{1}{\pi}\oint_C\left(\Phi(x,y)\frac{\partial\ln R}{\partial n}+\Phi^c(x,y)\frac{\partial\ln R}{\partial\tau}\right)ds, \tag{9}$$

$$\Phi^c(x_0,y_0)=\frac{1}{\pi}\oint_C\left(\Phi^c(x,y)\frac{\partial\ln R}{\partial n}-\Phi(x,y)\frac{\partial\ln R}{\partial\tau}\right)ds, \tag{10}$$

$$\Psi(x_0,y_0)=\frac{1}{\pi}\oint_C\left(\Psi(x,y)\frac{\partial\ln R}{\partial n}+\Psi^c(x,y)\frac{\partial\ln R}{\partial\tau}\right)ds, \tag{11}$$

$$\Psi^c(x_0,y_0)=\frac{1}{\pi}\oint_C\left(\Psi^c(x,y)\frac{\partial\ln R}{\partial n}-\Psi(x,y)\frac{\partial\ln R}{\partial\tau}\right)ds. \tag{12}$$

3.1. **The boundary conditions.** The force distribution prescribed on the boundary C of the domain D is given as:

$$\mathbf{f}=f_x\,\mathbf{i}+f_y\,\mathbf{j}=f_\tau\,\boldsymbol{\tau}+f_n\,\mathbf{n},$$

where $\mathbf{f}$ denotes the external force per unit length of the boundary. Then, at a general boundary point Q the stress vector satisfies:

$$\boldsymbol{\sigma}_n=\mathbf{f},\qquad \mathbf{f}=-P_0\mathbf{n},$$

where P_0 is the pressure acting on the boundary, where the Airy's stress function is:

$$\boldsymbol{\sigma}_{xx}=\frac{\partial^2U}{\partial y^2},\qquad \boldsymbol{\sigma}_{yy}=\frac{\partial^2U}{\partial x^2},\qquad \boldsymbol{\sigma}_{xy}=-\frac{\partial^2U}{\partial x\partial y}.$$

The boundary conditions are [3]

$$\left(x\Phi_{yy}+2\Phi^c_y+y\Phi^c_{yy}+\Psi_{yy}\right)\dot{y}+\left(x\Phi_{xy}+y\Phi^c_{xy}+\Psi_{xy}\right)\dot{x}=\omega f_x, \tag{13a}$$

$$-\left(x\Phi_{xy}+y\Phi^c_{xy}+\Psi_{xy}\right)\dot{y}-\left(2\Phi_x+x\Phi_{xx}+y\Phi^c_{xx}+\Psi_{xx}\right)\dot{x}=\omega f_y, \tag{13b}$$

with

$$f_x = -P_0 \frac{\dot{y}}{\omega}, \qquad f_y = P_0 \frac{\dot{x}}{\omega}, \tag{14}$$

where the dot over a symbol denotes derivative with respect to the parameter s.

3.2. **Conditions for eliminating the rigid body translation.** These are two conditions. Following [5], we require that the displacement at the point O vanishes, i.e.

$$u\,(0,0) = 0, \qquad v\,(0,0) = 0.$$

These two conditions may be rewritten respectively as

$$(3-4\nu)\,\Phi\,(0,0) - \frac{\partial \Psi}{\partial x}\,(0,0) = 0, \tag{15}$$

$$(3-4\nu)\,\Phi^c\,(0,0) - \frac{\partial \Psi}{\partial y}\,(0,0) = 0. \tag{16}$$

In terms of the boundary values of the unknown harmonic functions, conditions (15) and (16) become

$$\oint_C \left\{ (3-4\nu) \left(\Phi\,(s') \frac{\partial \ln R_0}{\partial n'} + \Phi^c\,(s') \frac{\partial \ln R_0}{\partial \tau'} \right) + \left(\Psi\,(s') \frac{\partial}{\partial n'} \frac{x\,(s')}{R_0^2} + \Psi^c\,(s') \frac{\partial}{\partial \tau'} \frac{x\,(s')}{R_0^2} \right) \right\} ds', \tag{17a}$$

$$\oint_C \left\{ (3-4\nu) \left(\Phi^c\,(s') \frac{\partial \ln R_0}{\partial n'} - \Phi\,(s') \frac{\partial \ln R_0}{\partial \tau'} \right) + \left(\Psi\,(s') \frac{\partial}{\partial n'} \frac{y\,(s')}{R_0^2} + \Psi^c\,(s') \frac{\partial}{\partial \tau'} \frac{y\,(s')}{R_0^2} \right) \right\} ds' = 0. \tag{17b}$$

3.3. **Condition for eliminating the rigid body rotation.** This condition is applied in the form:

$$\frac{\partial u}{\partial y}\,(0,0) - \frac{\partial v}{\partial x}\,(0,0) = 0, \qquad \text{or} \qquad 4\,(1-\nu)\,\frac{\partial \Phi}{\partial y}\,(0,0) = 0. \tag{18}$$

In terms of the boundary values of the unknown harmonic functions, conditions (18) become

$$4\,(1-\nu) \oint_C \left(\Phi\,(s') \frac{\partial}{\partial n'} \frac{y\,(s')}{R_0^2} + \Phi^c\,(s') \frac{\partial}{\partial s'} \frac{y\,(s')}{R_0^2} \right) ds' = 0. \tag{19}$$

3.4. **Additional simplifying conditions.** We shall require the following supplementary conditions to be satisfied at the point Q_0 $(s=0)$ of the boundary, in order to determine the totality of the arbitrary integration constants appearing throughout the solution process. These additional conditions have no physical implication on the solution of the problem:

$$x(0)\,\Phi(0)+y(0)\,\Phi^c(0)+\Psi(0)=0, \tag{20a}$$

$$x(0)\,\Phi^c(0)-y(0)\,\Phi(0)+\Psi^c(0)=0, \tag{20b}$$

$$x(0)\,\frac{\partial\Phi}{\partial x}(0)+\Phi(0)+y(0)\,\frac{\partial\Phi^c}{\partial x}(0)+\frac{\partial\Psi}{\partial x}(0)=0, \tag{20c}$$

$$x(0)\,\frac{\partial\Phi}{\partial y}(0)+\Phi^c(0)+y(0)\,\frac{\partial\Phi^c}{\partial y}(0)+\frac{\partial\Psi}{\partial y}(0)=0. \tag{20d}$$

This last condition amounts to determining the value of Ψ^c at Q_0 and is chosen for the uniformity of presentation as in [6].

4. The displacement vector

The mechanical displacement components may be expressed in terms of the harmonic functions Φ, Φ^c and Ψ as [3]:

$$\frac{E}{1+\nu}u = (3-4\nu)\Phi - x\frac{\partial\Phi}{\partial x} - y\frac{\partial\Phi^c}{\partial x} - \frac{\partial\Psi}{\partial x}, \tag{21}$$

$$\frac{E}{1+\nu}v = (3-4\nu)\Phi^c - x\frac{\partial\Phi}{\partial y} - y\frac{\partial\Phi^c}{\partial y} - \frac{\partial\Psi}{\partial y}. \tag{22}$$

We can get the values of displacement vector at the boundary of the domain by substituting the harmonic functions from equations (9, 10 and 11) into equations (21 and 22). The values of the displacement vector inside the domain at any point may be obtained by substituting the harmonic functions from equations (4, 5 and 6) into equations (21 and 22).

5. The numerical method

The numerical method depends on partitioning the contour of the domain by means of a properly chosen set of M points (nodes) and applying the system of equations which consists of the boundary representations of four harmonic functions (9, 10, 11 and 12), two boundary conditions (13a and 13b) and the seven conditions (17a, 17b,19, 20a, 20b, 20c and 20d) at each point of the contour to get a rectangular system of linear algebraic equations

$$\mathbf{AX}=\mathbf{B}, \tag{23}$$

where $\mathbf{A}$ is the coefficients matrix of type $6M+7\times 4M$, $\mathbf{X}$ is the vector of unknown boundary values of the harmonic functions $[\Phi_i,\ \Phi_i^c,\ \Psi_i,\ \Psi_i^c]^T$ of $4M\times 1$ and $\mathbf{B}$ is the right hand side vector of type $6M+7\times 1$, and $i=1...M$.

5.1. **Discretization of the harmonic representations of harmonic functions.** For any function f harmonic inside the domain, this is carried out using equations (9, 10, 11, and 12) as in [4] and [7]:

$$\oint_C f\,ds=\sum_{i=1}^{M} f_i\ \Delta s_i,$$

then the value of the harmonic function at the $i-th$ node may be written as:

$$f_i = \frac{1}{\pi}\sum_{j=1}^{M}\left(f_j\frac{\partial \ln R_{ij}}{\partial n_j} + \left(f_j^c - f_i^c\right)\frac{\partial \ln R_{ij}}{\partial \tau_j}\right)\triangle s_j,$$

with

$$\begin{aligned} R_{ij} &= \sqrt{\left(x_j - x_i\right)^2 + \left(y_j - y_i\right)^2}, \\ \frac{\partial \ln R_{ij}}{\partial \tau_j} &= \frac{1}{\omega_j}\frac{\left(x_j - x_i\right)\dot{x_j} + \left(y_j - y_i\right)\dot{y_j}}{\left(x_j - x_i\right)^2 + \left(y_j - y_i\right)^2}, \\ \frac{\partial \ln R_{ij}}{\partial n_j} &= \frac{1}{\omega_j}\frac{\left(x_j - x_i\right)\dot{y_j} - \left(y_j - y_i\right)\dot{x_j}}{\left(x_j - x_i\right)^2 + \left(y_j - y_i\right)^2}. \end{aligned}$$

These equations include singularities. To remove them, one may use the results of [7]:

$$\begin{aligned} \lim_{j\to i}\left(f_j\frac{\partial \ln R_{ij}}{\partial n_j}\right) &= \frac{\dot{x_i}\ddot{y_i} - \ddot{x_i}\dot{y_i}}{2\omega_i^3}f_i, \\ \lim_{j\to i}\left(\left(f_j^c - f_i^c\right)\frac{\partial \ln R_{ij}}{\partial \tau_j}\right) &= \frac{1}{\omega_i}\dot{f_i^c}, \qquad \triangle s_j = \omega_j\ \triangle\theta. \end{aligned}$$

to get

$$\beta_i f_i + \sum_{j=1,j\neq i}^{M}\alpha_{ji}\ f_j + \sum_{j=1,j\neq i}^{M}\gamma_{ji}\ f_j^c - \sum_{j=1,j\neq i}^{M}\gamma_{ji}\ f_i^c + \dot{f_i^c}\triangle\theta = 0, \tag{24}$$

with

$$\begin{aligned} \beta_i &= \frac{\ddot{y_i}\dot{x_i} - \ddot{x_i}\dot{y_i}}{2\omega_i^2}\triangle\theta - \pi, \\ \alpha_{ji} &= \frac{\left(x_j - x_i\right)\dot{y_j} - \left(y_j - y_i\right)\dot{x_j}}{\left(x_j - x_i\right)^2 + \left(y_j - y_i\right)^2}\triangle\theta, \\ \gamma_{ji} &= \frac{\left(x_j - x_i\right)\dot{x_j} + \left(y_j - y_i\right)\dot{y_j}}{\left(x_j - x_i\right)^2 + \left(y_j - y_i\right)^2}\triangle\theta. \end{aligned}$$

The harmonic functions (equations 9, 10, 11, and 12) can be written in matrix form as follows

$$[\mathbf{A}]\ \ [\mathbf{\Phi}] + [\mathbf{B}]\ \ [\mathbf{\Phi}^c] + [\dot{\Phi}^{c}] + [\mathbf{0}]\,[\mathbf{\Psi}] + + [\mathbf{0}]\,[\mathbf{\Psi}^c] = 0, \tag{25}$$

$$-[\mathbf{B}]\ \ [\mathbf{\Phi}] - [\ \dot{\mathbf{\Phi}}\ ] + [\mathbf{A}]\ \ [\mathbf{\Phi}^c] + [\mathbf{0}]\,[\mathbf{\Psi}] + + [\mathbf{0}]\,[\mathbf{\Psi}^c] = 0, \tag{26}$$

$$[\mathbf{0}]\,[\mathbf{\Phi}] + [\mathbf{0}]\,[\mathbf{\Phi}^c] + [\mathbf{A}]\ \ [\mathbf{\Psi}] + [\mathbf{B}]\ \ [\mathbf{\Psi}^c] + [\dot{\Psi}^{c}] = 0, \tag{27}$$

$$[\mathbf{0}]\,[\mathbf{\Phi}] + [\mathbf{0}]\,[\mathbf{\Phi}^c] + -[\mathbf{B}]\ \ [\mathbf{\Psi}] - [\dot{\mathbf{\Psi}}] + [\mathbf{A}]\ \ [\mathbf{\Psi}^c] = 0, \tag{28}$$

where

$$[\mathbf{A}] = \begin{cases} \alpha_{ij}, & i > j \\ \beta_i, & i = j \\ \alpha_{ij}, & i < j \end{cases}, \quad [\mathbf{B}] = \begin{cases} \gamma_{ij}, & i > j \\ -\lambda_i, & i = j \\ \gamma_{ij}, & i < j \end{cases}, \quad \lambda_i = \sum_{j=1, j\neq i}^{M} \gamma_{ij},$$

The derivatives along the contour must be evaluated carefully. This is carried out by using 15 points on each side of the point where the derivative is to be evaluated and Taylor's expansion as:

$$\dot{f}_i = \frac{1}{\triangle\theta} \sum_{j=1}^{15} (-1)^{j+1} jC_j \left(f_{i+j} - f_{i-j}\right), \qquad i = 1, 2, ..M. \tag{29}$$

Constants C_j are defined in Appendix, Also

$$[\mathbf{\Phi}] = [\Phi_i], \qquad [\mathbf{\Phi}^c] = [\Phi_i^c], \qquad [\mathbf{\Psi}] = [\Psi_i], \qquad [\mathbf{\Psi}^c] = [\Psi_i^c], \qquad i = 1, 2, ..M.$$

The dot over the symbol f refers to differentiation relative to θ.

5.2. **Two boundary conditions.** The two boundary conditions are

$$[\mathbf{A}^*][\mathbf{\Phi}] + [\mathbf{B}^*][\mathbf{\Phi}^c] + [\mathbf{C}^*][\mathbf{\Psi}] + [\mathbf{D}^*][\mathbf{\Psi}^c] = [f_x], \tag{31}$$

$$[\mathbf{A}^{**}][\mathbf{\Phi}] + [\mathbf{B}^{**}][\mathbf{\Phi}^c] + [\mathbf{C}^{**}][\mathbf{\Psi}] + [\mathbf{D}^{**}][\mathbf{\Psi}^c] = [f_y], \tag{32}$$

where

$$[f_x] = [\omega_i (f_x)_i], \qquad [f_y] = \left[\omega_i (f_y)_i\right], \qquad i = 1, 2, ...M,$$

and

$$\mathbf{A}^* = \begin{cases} A_i^{(1)} & i = k \\ B_{i,k-i}^{(1)} & i+1 \le k \le i+15 \\ C_{i,i-k}^{(1)} & i-1 \le k \le i-15 \end{cases},$$

$$\mathbf{B}^* = \begin{cases} D_i^{(1)} & i = k \\ E_{i,k-i}^{(1)} & i+1 \le k \le i+15 \\ F_{i,i-k}^{(1)} & i-1 \le k \le i-15 \end{cases},$$

$$\mathbf{C}^* = \begin{cases} G_i^{(1)}, & i = k \\ H_{i,k-i}^{(1)} & i+1 \le k \le i+15 \\ K_{i,i-k}^{(1)} & i-1 \le k \le i-15 \end{cases},$$

$$\mathbf{D}^* = \begin{cases} L_i^{(1)} & i = k \\ N_{i,k-i}^{(1)} & i+1 \le k \le i+15 \\ M_{i,i-k}^{(1)} & i-1 \le k \le i-15 \end{cases},$$

and

$$\mathbf{A}^{**}=\begin{cases} A_i^{(2)} & i=k \\ B_{i,k-i}^{(2)} & i+1\le k\le i+15 \\ C_{i,i-k}^{(2)} & i-1\le k\le i-15 \end{cases},$$

$$\mathbf{B}^{**}=\begin{cases} D_i^{(2)}, & i=k \\ E_{i,k-i}^{(2)} & i+1\le k\le i+15 \\ F_{i,i-k}^{(2)} & i-1\le k\le i-15 \end{cases},$$

$$\mathbf{C}^{**}=\begin{cases} G_i^{(2)}, & i=k \\ H_{i,k-i}^{(2)} & i+1\le k\le i+15 \\ K_{i,i-k}^{(2)} & i-1\le k\le i-15 \end{cases},$$

$$\mathbf{D}^{**}=\begin{cases} L_i^{(2)}, & i=k \\ N_{i,k-i}^{(2)} & i+1\le k\le i+15 \\ M_{i,i-k}^{(2)} & i-1\le k\le i-15 \end{cases}.$$

5.3. **Seven additional conditions.** The seven additional conditions yield:

$$\sum_{k=1}^{M} K_k^{(1)}\ \Phi_k+\sum_{k=1}^{M} K_k^{(2)}\ \Phi_k^c+\sum_{k=1}^{M} K_k^{(3)}\ \Psi_k+\sum_{k=1}^{M} K_k^{(4)}\ \Psi_k^c=0, \tag{33a}$$

$$-\sum_{k=1}^{M} K_k^{(2)}\ \Phi_k+\sum_{k=1}^{M} K_k^{(1)}\ \Phi_k^c+\sum_{k=1}^{M} K_k^{(4)}\ \Psi_k-\sum_{k=1}^{M} K_k^{(3)}\ \Psi_k^c=0, \tag{33b}$$

$$\sum_{k=1}^{M} K_k^{(4)}\ \Phi_k-\sum_{k=1}^{M} K_k^{(3)}\ \Phi_k^c=0, \tag{33c}$$

$$x_1\Phi_1+\Psi_1=0, \tag{33d}$$

$$x_1\Phi_1^c+\Psi_1^c=0, \tag{33e}$$

$$\begin{aligned}\Phi_1&+\sum_{k=2}^{16}\mathcal{I}_k^{(1)}\Phi_k+\sum_{k=M-14}^{M}\mathcal{I}_k^{(2)}\Phi_k+\sum_{k=2}^{16}\mathcal{I}_k^{(3)}\Phi_k^c+\sum_{k=M-14}^{M}\mathcal{I}_k^{(4)}\Phi_k^c\\&+\sum_{k=2}^{16}\mathcal{I}_k^{(5)}\Psi_k+\sum_{k=M-14}^{M}\mathcal{I}_k^{(6)}\Psi_k+\sum_{k=2}^{16}\mathcal{I}_k^{(7)}\Psi_k^c+\sum_{k=M-14}^{M}\mathcal{I}_k^{(8)}\Psi_k^c=0,\end{aligned} \tag{33f}$$

$$\begin{aligned}&\sum_{k=2}^{16}\mathcal{I}_k^{(3)}\Phi_k+\sum_{k=M-14}^{M}\mathcal{I}_k^{(4)}\Phi_k+\Phi_1^c-\sum_{k=2}^{16}\mathcal{I}_k^{(1)}\Phi_k^c-\sum_{k=M-14}^{M}\mathcal{I}_k^{(2)}\Phi_k^c\\&+\sum_{k=2}^{16}\mathcal{I}_k^{(7)}\Psi_k+\sum_{k=M-14}^{M}\mathcal{I}_k^{(8)}\Psi_k-\sum_{k=2}^{16}\mathcal{I}_k^{(5)}\Psi_k^c-\sum_{k=M-14}^{M}\mathcal{I}_k^{(6)}\Psi_k^c=0,\end{aligned} \tag{33g}$$

6. The solving method

Equations (25, 26, 27, 28, 31, 32 ,33a, 33b, 33c, 33d, 33e ,33f and 33g) written at each nodal point of the contour, yield a rectangular system of linear algebraic equations in the unknown functions $[\Phi_1,\Phi_2,...\Phi_M]^T$, $[\Phi_1^c,\Phi_2^c,...\Phi_M^c]^T$, $[\Psi_1,\Psi_2,...\Psi_M]^T$ and $[\Psi_1^c,\Psi_2^c,...\Psi_M^c]^T$. Solving this system of equations by the Least Square Method or by Gauss Elimination or by any other method yields the values of harmonic functions on the boundary.

7. Discretization of the displacement vector

7.1. **Numerical treatment of the partial derivatives of harmonic functions.** From equations (4 and 5 or 6 and 7), the integral representations for the derivatives with respect to the coordinates of a field point $(x_0,\ y_0)$ are

$$\frac{\partial f}{\partial x_0} = \frac{1}{2\pi}\oint_C \left(f(x,y)\frac{\partial}{\partial x_0}\left(\frac{\partial \ln R}{\partial n}\right) + f^c(x,y)\frac{\partial}{\partial x_0}\left(\frac{\partial \ln R}{\partial \tau}\right)\right) ds,$$
$$\frac{\partial f}{\partial y_0} = \frac{1}{2\pi}\oint_C f\left((x,y)\frac{\partial}{\partial y_0}\left(\frac{\partial \ln R}{\partial n}\right) + f^c(x,y)\frac{\partial}{\partial y_0}\left(\frac{\partial \ln R}{\partial \tau}\right)\right) ds,$$
$$\frac{\partial f^c}{\partial x_0} = \frac{1}{2\pi}\oint_C \left(f^c(x,y)\frac{\partial}{\partial x_0}\left(\frac{\partial \ln R}{\partial n}\right) - f(x,y)\frac{\partial}{\partial x_0}\left(\frac{\partial \ln R}{\partial \tau}\right)\right) ds,$$
$$\frac{\partial f^c}{\partial y_0} = \frac{1}{2\pi}\oint_C \left(f^c(x,y)\frac{\partial}{\partial y_0}\left(\frac{\partial \ln R}{\partial n}\right) - f(x,y)\frac{\partial}{\partial y_0}\left(\frac{\partial \ln R}{\partial \tau}\right)\right) ds,$$

but

$$\frac{\partial}{\partial x_0}\left(\frac{\partial \ln R}{\partial \tau}\right) = \frac{\left((x-x_0)^2-(y-y_0)^2\right)\dot{x}+2(x-x_0)(y-y_0)\dot{y}}{\omega\left((x-x_0)^2+(y-y_0)^2\right)^2},$$

$$\frac{\partial}{\partial x_0}\left(\frac{\partial \ln R}{\partial n}\right) = \frac{\left((x-x_0)^2-(y-y_0)^2\right)\dot{y}-2(x-x_0)(y-y_0)\dot{x}}{\omega\left((x-x_0)^2+(y-y_0)^2\right)^2},$$

$$\frac{\partial}{\partial y_0}\left(\frac{\partial \ln R}{\partial \tau}\right) = \frac{2(x-x_0)(y-y_0)\dot{x}+\left((y-y_0)^2-(x-x_0)^2\right)\dot{y}}{\omega\left((x-x_0)^2+(y-y_0)^2\right)^2},$$

$$\frac{\partial}{\partial y_0}\left(\frac{\partial \ln R}{\partial n'}\right) = \frac{2(x-x_0)(y-y_0)\dot{y}-\left((y-y_0)^2-(x-x_0)^2\right)\dot{x}}{\omega\left((x-x_0)^2+(y-y_0)^2\right)^2},$$

then

$$\frac{\partial f}{\partial x_0} = \frac{1}{2\pi}\oint_C \left(\frac{\left((x-x_0)^2-(y-y_0)^2\right)\dot{y}-2(x-x_0)(y-y_0)\dot{x}}{\omega\left((x-x_0)^2+(y-y_0)^2\right)^2} f(x,y) + \frac{\left((x-x_0)^2-(y-y_0)^2\right)\dot{x}+2(x-x_0)(y-y_0)\dot{y}}{\omega\left((x-x_0)^2+(y-y_0)^2\right)^2} f^c(x,y)\right) ds, \tag{34}$$

$$\frac{\partial f}{\partial y_0} = \frac{1}{2\pi}\oint_C \left(\frac{2(x-x_0)(y-y_0)\dot{y}-\left((y-y_0)^2-(x-x_0)^2\right)\dot{x}}{\omega\left((x-x_0)^2+(y-y_0)^2\right)^2} f(x,y) + \frac{2(x-x_0)(y-y_0)\dot{x}+\left((y-y_0)^2-(x-x_0)^2\right)\dot{y}}{\omega\left((x-x_0)^2+(y-y_0)^2\right)^2} f^c(x,y)\right) ds, \tag{35}$$

and

$$\frac{\partial f^c}{\partial x_0}=\frac{1}{2\pi}\oint_C\left(\frac{\left((x-x_0)^2-(y-y_0)^2\right)\dot{y}-2(x-x_0)(y-y_0)\dot{x}}{\omega\left((x-x_0)^2+(y-y_0)^2\right)^2}f^c(x,y)\right.$$
$$\left.-\frac{\left((x-x_0)^2-(y-y_0)^2\right)\dot{x}+2(x-x_0)(y-y_0)\dot{y}}{\omega\left((x-x_0)^2+(y-y_0)^2\right)^2}f(x,y)\right)ds, \tag{36}$$

$$\frac{\partial f^c}{\partial y_0}=\frac{1}{2\pi}\oint_C\left(\frac{2(x-x_0)(y-y_0)\dot{y}-\left((y-y_0)^2-(x-x_0)^2\right)\dot{x}}{\omega\left((x-x_0)^2+(y-y_0)^2\right)^2}f^c(x,y)\right.$$
$$\left.-\frac{2(x-x_0)(y-y_0)\dot{x}+\left((y-y_0)^2-(x-x_0)^2\right)\dot{y}}{\omega\left((x-x_0)^2+(y-y_0)^2\right)^2}f(x,y)\right)ds. \tag{37}$$

Numerically

$$\left(\frac{\partial f}{\partial x_0}\right)_{j,k}=\sum_{k=1,k\neq j}^{M}\left(\alpha'_{j,k}f_k+\gamma'_{j,k}f_k^c\right), \tag{38a}$$
$$\left(\frac{\partial f}{\partial y_0}\right)_{j,k}=\sum_{k=1,k\neq j}^{M}\left(\gamma'_{j,k}f_k-\alpha'_{j,k}f_k^c\right), \tag{38b}$$
$$\left(\frac{\partial f^c}{\partial x_0}\right)_{j,k}=\sum_{k=1,k\neq j}^{M}\left(\alpha'_{j,k}f_k^c-\gamma'_{j,k}f_k\right), \tag{38c}$$
$$\left(\frac{\partial f^c}{\partial y_0}\right)_{j,k}=\sum_{k=1,k\neq j}^{M}\left(\gamma'_{j,k}f_k^c+\alpha'_{j,k}f_k\right), \tag{38d}$$

with

$$\alpha'_{j,k}=\frac{1}{2\pi}\frac{\left((x_j-x_{j,k})^2-(y_j-y_{j,k})^2\right)\dot{y_j}-2(x_j-x_{j,k})(y_j-y_{j,k})\dot{x_j}}{\left((x_j-x_{j,k})^2+(y_j-y_{j,k})^2\right)^2}\triangle\theta, \qquad j\neq k,$$

$$\gamma'_{j,k}=\frac{1}{2\pi}\frac{\left((x_j-x_{j,k})^2-(y_j-y_{j,k})^2\right)\dot{x_j}+2(x_j-x_{j,k})(y_j-y_{j,k})\dot{y_j}}{\left((x_j-x_{j,k})^2+(y_j-y_{j,k})^2\right)^2}\triangle\theta, \qquad j\neq k,$$

where $(x_{j,k},y_{j,k})$ is a field point and (x_j,y_j) is a boundary point. The notation $f_{j,k}$ means the value of the function f taken at the node $(j,\ k)$ for the coordinates $(\xi,\ \theta)$.

7.2. **The displacement vector inside the domain.** From equations (21 and 22) the displacement vector inside the domain numerically written as

$$\frac{E}{1+\nu}u_i = (3-4\nu)\,\Phi_i-x_i\left(\frac{\partial\Phi}{\partial x}\right)_i-y_i\left(\frac{\partial\Phi^c}{\partial x}\right)_i-\left(\frac{\partial\Psi}{\partial x}\right)_i,$$
$$\frac{E}{1+\nu}v_i = (3-4\nu)\,\Phi_i^c-x_i\left(\frac{\partial\Phi}{\partial y}\right)_i-y_i\left(\frac{\partial\Phi^c}{\partial y}\right)_i-\left(\frac{\partial\Psi}{\partial y}\right)_i,$$

using equations (38a and 38b) into the last equations, then

$$\frac{E}{1+\nu}u_{j,p} = (3-4\nu)\,\Phi_{j,p} - x_{j,p}\sum_{k=1,k\neq j}^{M}\left(\alpha'_{j,k}\Phi_k + \gamma'_{j,k}\Phi^c_k\right)$$
$$- y_{j,p}\sum_{k=1,k\neq j}^{M}\left(\alpha'_{j,k}\Phi^c_k - \gamma'_{j,k}\Phi_k\right) - \sum_{k=1,k\neq j}^{M}\left(\alpha'_{j,k}\Psi_k + \gamma'_{j,k}\Psi^c_k\right), \quad (40a)$$

and

$$\frac{E}{1+\nu}v_{j,p} = (3-4\nu)\,\Phi^c_{j,p} - x_{j,p}\sum_{k=1,k\neq j}^{M}\left(\gamma'_{j,k}\Phi_k - \alpha'_{j,k}\Phi^c_k\right)$$
$$- y_{j,p}\sum_{k=1,k\neq j}^{M}\left(\gamma'_{j,k}\Phi^c_k + \alpha'_{j,k}\Phi_k\right) - \sum_{k=1,k\neq j}^{M}\left(\gamma'_{j,k}\Psi_k - \alpha'_{j,k}\Psi^c_k\right). \quad (40b)$$

7.3. **The displacement vector on the boundary.** The displacement vector on the boundary take the form:

$$\frac{E}{1+\nu}u_i = (3-4\nu)\,\Phi_i - \frac{\Sigma_i}{\omega_i^2}\dot{\Phi}_i - \frac{\Lambda_i}{\omega_i^2}\dot{\Phi}^c_i - \frac{\dot{x}_i}{\omega_i^2}\dot{\Psi}_i - \frac{\dot{y}_i}{\omega_i^2}\dot{\Psi}_i^{\,c}, \quad (41a)$$

$$\frac{E}{1+\nu}v_i = (3-4\nu)\,\Phi^c_i - \frac{\Lambda_i}{\omega_i^2}\dot{\Phi}_i + \frac{\Sigma_i}{\omega_i^2}\dot{\Phi}^c_i - \frac{\dot{y}_i}{\omega_i^2}\dot{\Psi}_i + \frac{\dot{x}_i}{\omega_i^2}\dot{\Psi}_i^{\,c}, \quad (41b)$$

8. Numerical example

In equation (1) to get the equation of Cassini curve set $\xi = \xi_0$, then

$$\frac{x}{c} = \frac{\cosh\xi_0\cos\theta}{\sinh^2\xi_0 + \cos^2\theta}, \qquad \frac{y}{c} = \frac{\sinh\xi_0\sin\theta}{\sinh^2\xi_0 + \cos^2\theta}, \quad (42)$$

use

$$\sinh^2\xi = \frac{1}{2}\left(\cosh 2\xi - 1\right) \qquad \cos^2\theta = \frac{1}{2}\left(\cos 2\theta + 1\right),$$

into equation (42) to get

$$\frac{x}{c} = \frac{2\cosh\xi_0}{\cosh 2\xi_0}\frac{\cos\theta}{\left(1+\frac{1}{\cosh 2\xi_0}\cos 2\theta\right)}, \qquad \frac{y}{c} = \frac{2\sinh\xi_0}{\cosh 2\xi_0}\frac{\sin\theta}{\left(1+\frac{1}{\cosh 2\xi_0}\cos 2\theta\right)}.$$

Let

$$k = \frac{1}{\cosh 2\xi_0}, \qquad a = \frac{2\cosh\xi_0}{\cosh 2\xi_0}, \qquad b = \frac{2\sinh\xi_0}{\cosh 2\xi_0},$$

then

$$\frac{x}{c} = \frac{a\cos\theta}{1+k\cos 2\theta}, \qquad \frac{y}{c} = \frac{b\sin\theta}{1+k\cos 2\theta}. \quad (43)$$

Equation (42) gives an interior point $(\xi,\ \theta)$ and boundary point with $(\xi_0,\ \theta)$, where ξ_0 is constant.

Equation (43) gives all the points on the boundary. From equation (43), if we choose $k = 0$, then equation (43) gives the region outside an ellipse (see figure)

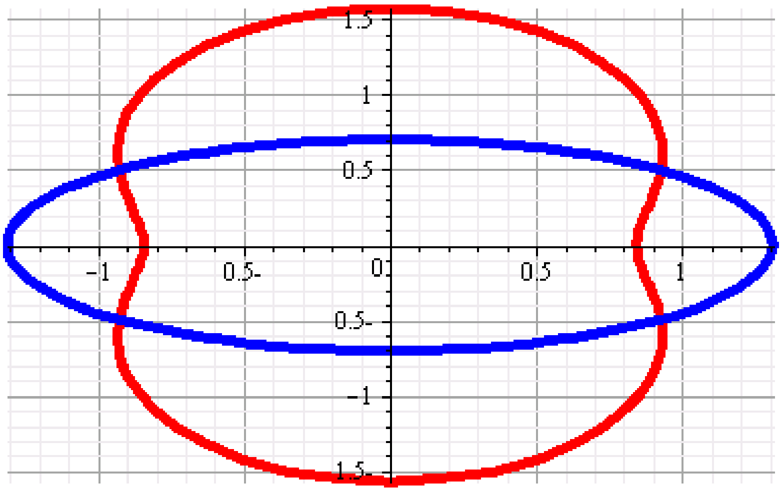

Fig. 2: an ellipse and Cassini curve with $\xi_0 = 0.6$

Let us choose

$$\begin{aligned} P_0 &= 0.1,\ 0.3,\ 0.5, \qquad \nu = 0.25, \qquad \xi_0 = 0.6, \\ M &= 100, \qquad d\theta = \frac{2\pi}{M}, \qquad \theta_i = (i-1)\,\frac{2\pi}{M}, \qquad i = 1,\ 2,\ ...M. \end{aligned}$$

Solving the system of equations (23) one gets the values of harmonic functions $[\Phi_1, \Phi_2, ...\Phi_M]^T$, $[\Phi_1^c, \Phi_2^c, ...\Phi_M^c]^T$, $[\Psi_1, \Psi_2, ...\Psi_M]^T$ and $[\Psi_1^c, \Psi_2^c, ...\Psi_M^c]^T$ on the boundary.

8.1. **The harmonic functions on the boundary.** Plotting the data for the harmonic functions obtained from the system of linear equations with three different values for the pressure yields the following figures:

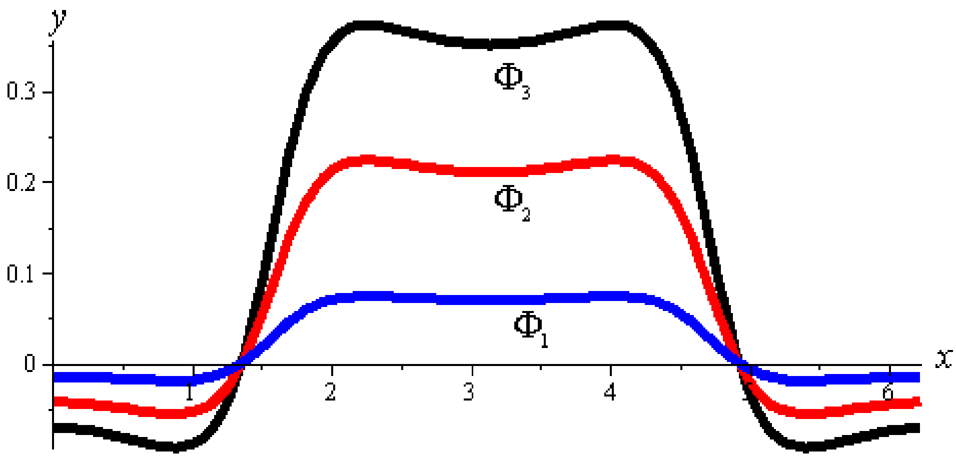

Fig. 3: The harmonic function Φ at the boundary with changing the pressure

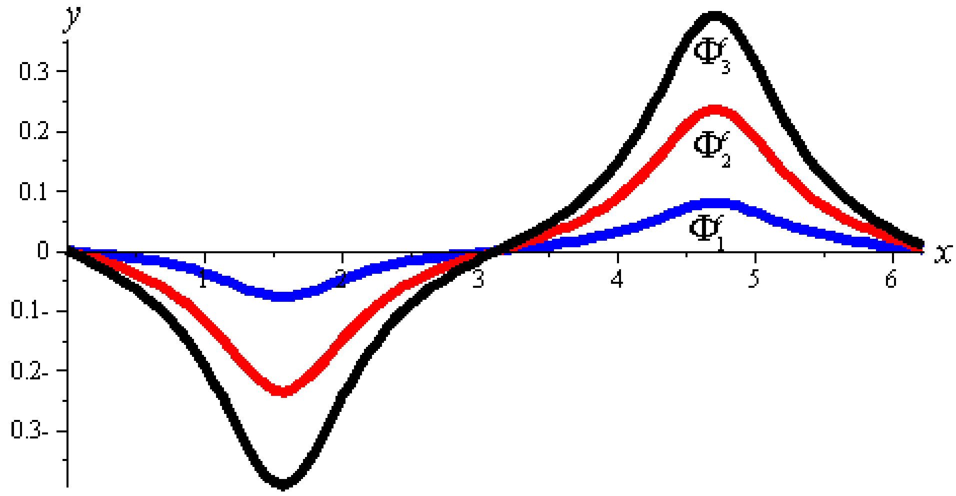

Fig. 4: The harmonic function Φ^c at the boundary with changing the pressure

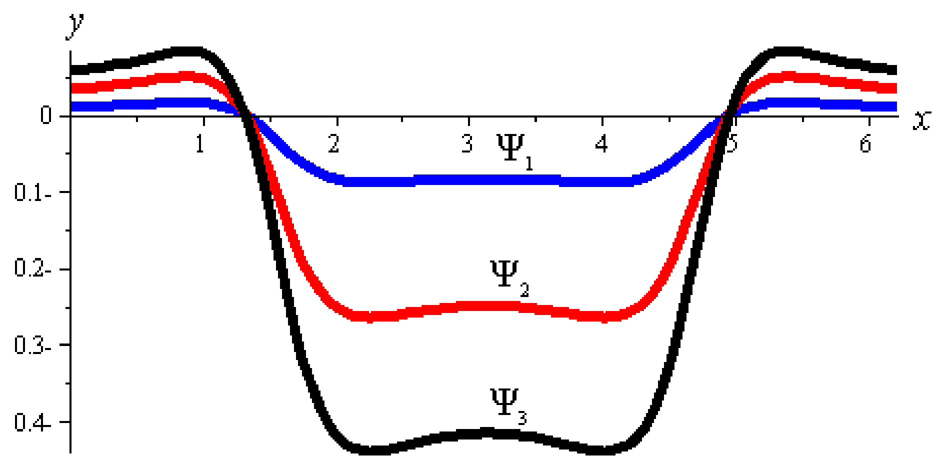

Fig. 5: The harmonic function Ψ at the boundary with changing the pressure

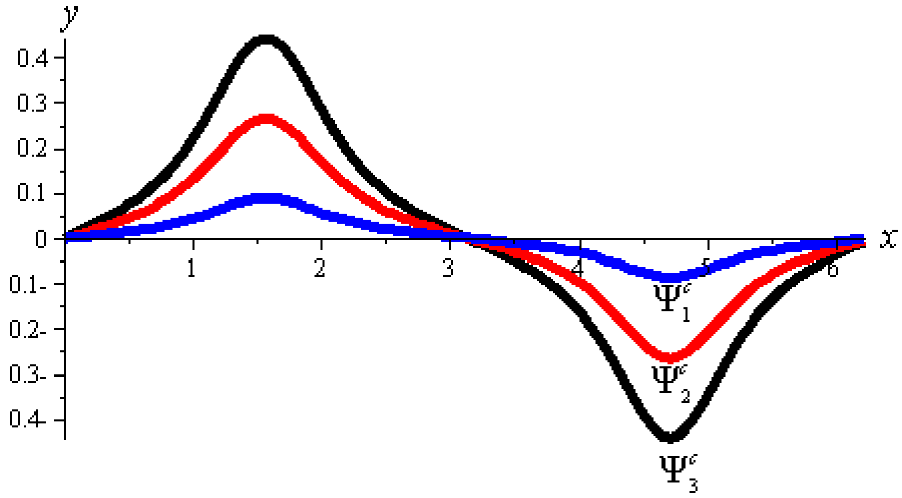

Fig. 6: The harmonic function Ψ^c at the boundary with changing the pressure

8.2. **The displacement vector at the boundary.** Substituting the data for the harmonic functions obtained from the system of linear equations into equations (41a and 41b) with three different values for the pressure yields the following figures:

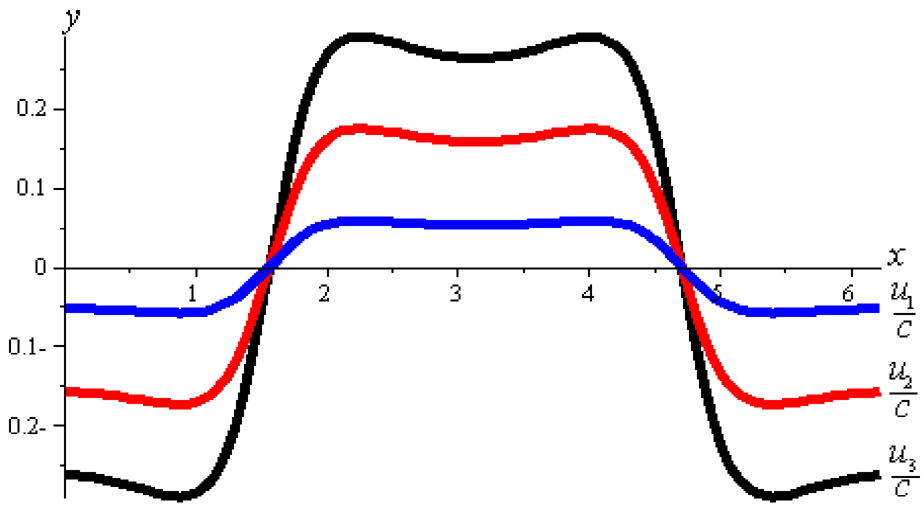

Fig. 7: The harmonic function $\frac{u}{c}$ at the boundary with changing the pressure

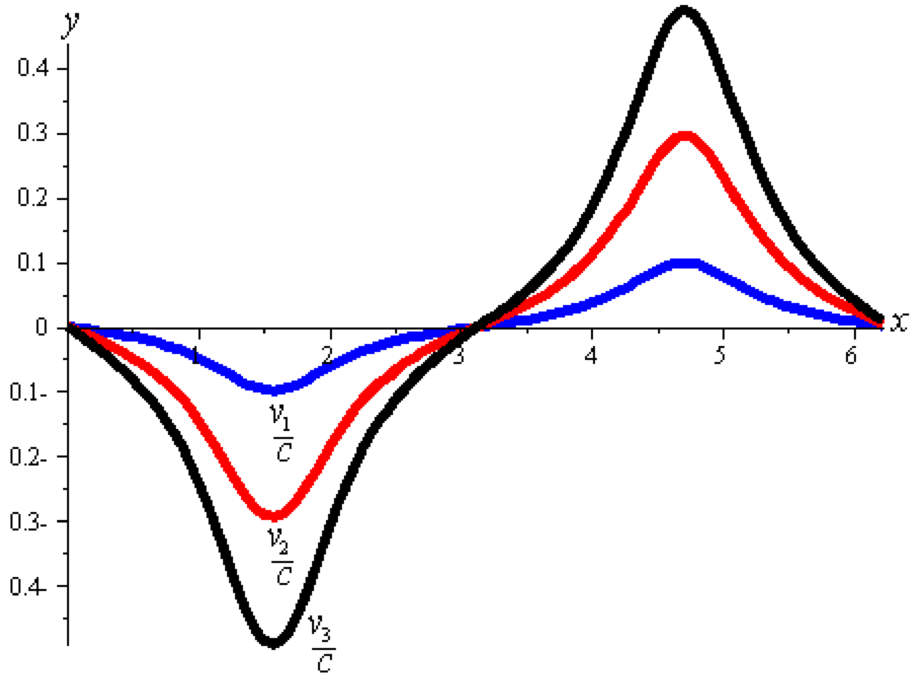

Fig. 8: The harmonic function $\frac{v}{c}$ at the boundary with changing the pressure

8.3. **The harmonic functions inside the domain.** To plot the harmonic function inside the domain, we use the data from the system of linear equations into equations (40a, 40b). For this, consider the domain as composed of a multitude (20) of Cassini curves each characterized by a constant value of ξ_0 as shown on the figure:

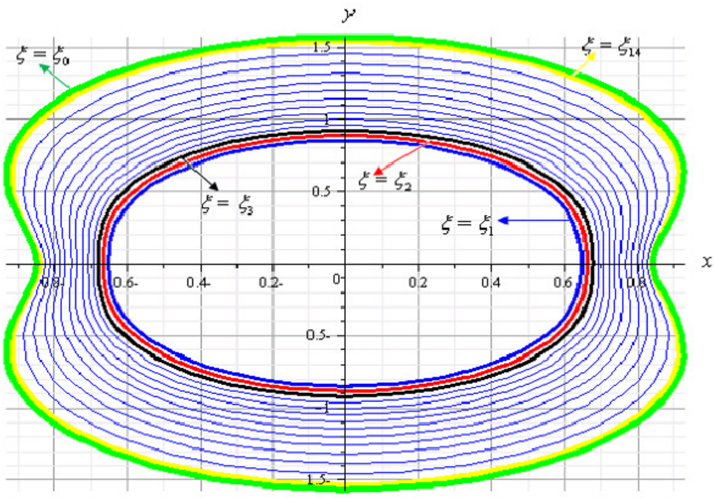

Fig: 9 :- the region of inside the domain.

then we plotted the harmonic functions on each of the arising curves:

$$N = 20, \qquad \xi_0 = 0.6, \qquad \xi_1 = 1, \qquad \xi_{j+1} = \xi_j - \frac{\xi_0}{N}, \qquad j = 1,\ 2,\ ...N - 6.$$

The following three-dimensional plots show the distributions of the harmonic functions inside the domain:

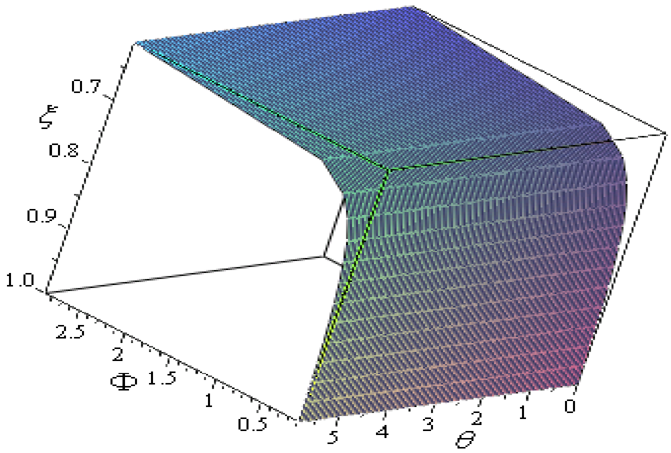

Fig. 10:- The harmonic function Φ inside the domain

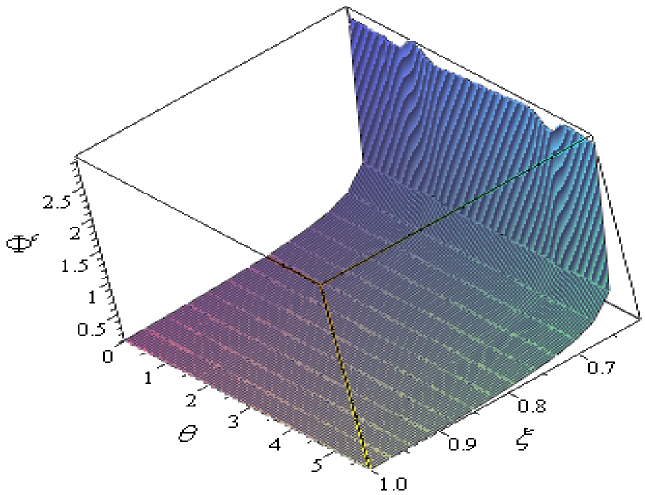

Fig. 11:- The harmonic function Φ^c inside the domain

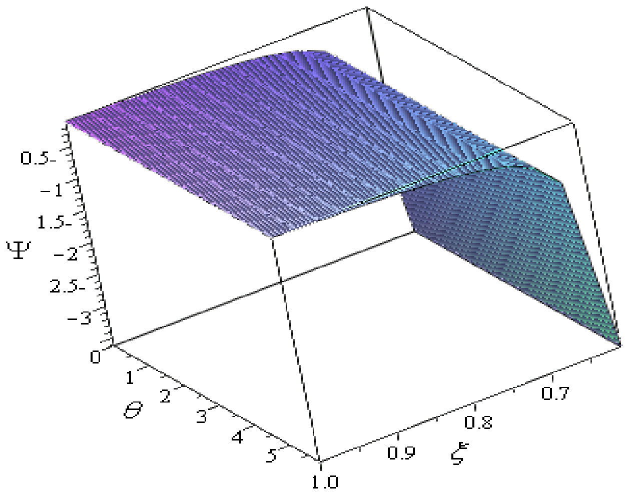

Fig. 12:- The harmonic function Ψ inside the domain

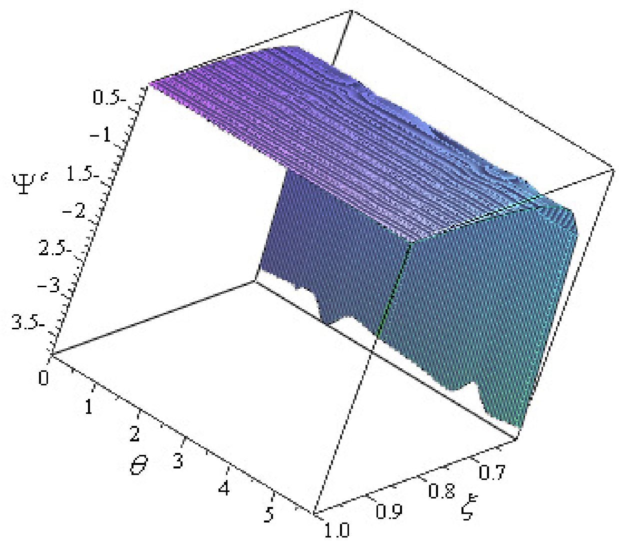

Fig. 13:- The harmonic function Ψ^c inside the domain

8.4. **The displacement vector inside the domain.** The displacement vector inside the domain can be obtained from the harmonic function distributions inside the domain. They are shown on the following three-dimensional plots:

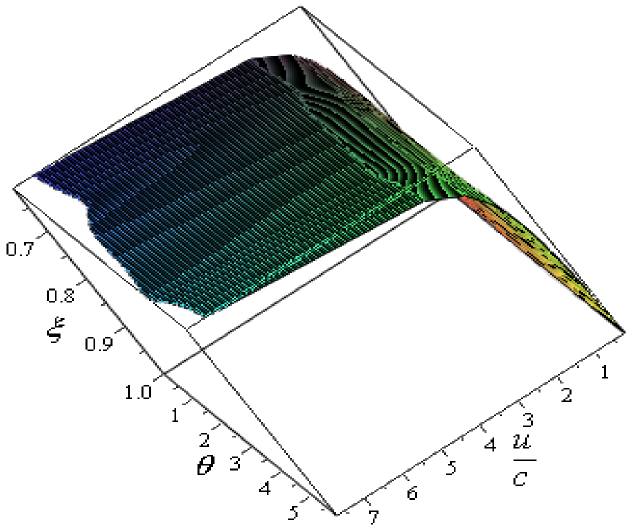

Fig. 14:- The displacement $\frac{u}{c}$

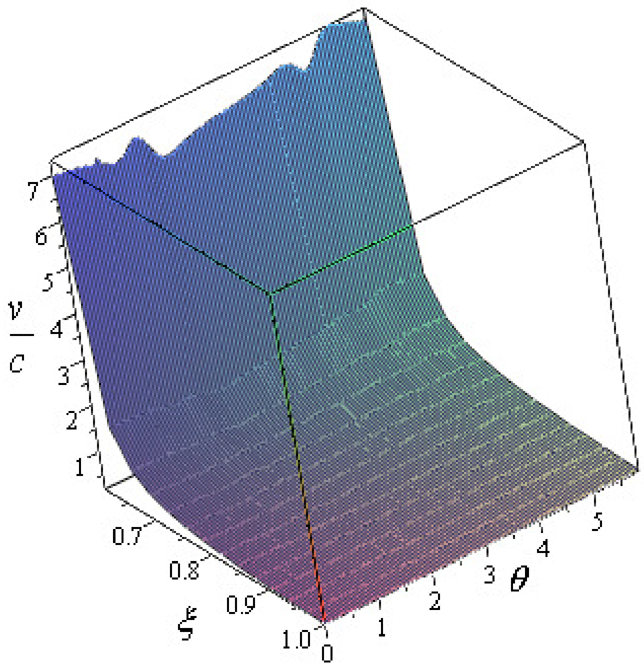

Fig. 15:- The displacement $\frac{v}{c}$

9. Conclusions

In this paper, we use the boundary integral method in numerical form to solve the first fundamental problem of elasticity for a long cylinder with cross section bounded by Cassini curve. The used computer program was tested to solve the first fundamental problem for a long cylinder with circular or elliptical cross sections and the results were compared with the numerical and analytical solutions in [3] and [4]. In the present case, no analytical solutions are available for comparison.

(1) In figures (3, 4, 5 and 6) the harmonic functions Φ, Φ^c, Ψ and Ψ^c were drawn on the boundary against the discretization angle θ_i, $i = 1,\ 2\ ,...M$, for three different values of the normal pressure. One observes that the harmonic functions Φ and Ψ have oppsite concavities, the same is true for Φ^c and Ψ^c.

(2) Figures (7 and 8) show the displacement vector $\frac{u}{c}$ and $\frac{v}{c}$ against the discretization angle θ_i, $i = 1,\ 2\ ,...M$, where the normal pressure takes different three values. One can observe that the displacement $\frac{u}{c}$ and $\frac{v}{c}$ looks like the harmonic functions Φ and Φ^c respectively.

(3) Figures (10, 11, 12 and 13) show the harmonic functions Φ, Φ^c, Ψ and Ψ^c inside the domain against the discretization ξ_i, $i = 1,\ 2\ ,...N$ and θ_i, $i = 1,\ 2\ ,...M$.

(4) Figures (14 and 15) show the displacement vector $\frac{u}{c}$ and $\frac{v}{c}$ inside the domain against the discretization ξ_i, $i = 1,\ 2\ ,...N$ and θ_i, $i = 1,\ 2\ ,...M$.

10. Appendix

10.1. **New Coordinates.** In this paper we introduce new coordinates, called Cassini coordinates, for which the position vector for any point is

$$\underline{r} = x\ \mathbf{i} + y\ \mathbf{j}.$$

From equation (1):

$$\underline{r} = \frac{c\cosh\xi\cos\theta}{\sinh^2\xi + \cos^2\theta}\mathbf{i} + \frac{\sinh\xi\sin\theta}{\sinh^2\xi + \cos^2\theta}\mathbf{j} \tag{44}$$

The unit vectors along the coordinates are

$$\mathbf{e}_\xi = -\frac{1}{\sinh^2\xi + \cos^2\theta}\frac{1}{\sqrt{\cosh^2\xi - \cos^2\theta}}\left\{\left(\cosh^2\xi + \sin^2\theta\right)\sinh\xi\cos\theta\ \mathbf{i} + \left(\sinh^2\xi - \cos^2\theta\right)\cosh\xi\sin\theta\ \mathbf{j}\right\}, \tag{45a}$$

$$\mathbf{e}_\theta = -\frac{1}{\sinh^2\xi + \cos^2\theta}\frac{1}{\sqrt{\cosh^2\xi - \cos^2\theta}}\left\{\left(\sinh^2\xi - \cos^2\theta\right)\cosh\xi\sin\theta\ \mathbf{i} - \left(\cosh^2\xi + \sin^2\theta\right)\sinh\xi\cos\theta\ \mathbf{j}\right\}, \tag{46}$$

and the scale factor is

$$h = \frac{c}{\sinh^2\xi + \cos^2\theta}\sqrt{\cosh^2\xi - \cos^2\theta}. \tag{47}$$

From equations (45a and 46) by scalar multiplication one gets:

$$\mathbf{e}_\xi\cdot\mathbf{e}_\theta = 0, \qquad \mathbf{e}_\xi\cdot\mathbf{e}_\xi = \mathbf{e}_\theta\cdot\mathbf{e}_\theta = 1.$$

10.1. **New Coordinates.**

$$\begin{aligned}
\dot{f}_i &= \frac{1}{\triangle\theta}\sum_{j=1}^{15}(-1)^{j+1}jC_j\left(f_{i+j}-f_{i-j}\right),\\
\ddot{f}_i &= -\frac{2}{(\triangle\theta)^2}\left(C_0 f_i+\sum_{j=1}^{15}(-1)^{j}C_j\left(f_{i+j}+f_{i-j}\right)\right),
\end{aligned}$$

with

$$\begin{aligned}
C_0 &= \frac{205234915681}{129859329600},\\
C_1 &= \frac{15}{16},\qquad C_2=\frac{105}{544},\qquad C_3=\frac{455}{7344},\\
C_4 &= \frac{455}{20672},\qquad C_5=\frac{1001}{129200},\qquad C_6=\frac{715}{279072},\\
C_7 &= \frac{195}{253232},\qquad C_8=\frac{195}{950912},\qquad C_9=\frac{455}{9627984},\\
C_{10} &= \frac{273}{29716000},\qquad C_{11}=\frac{21}{14382544},\qquad C_{12}=\frac{7}{38511936},\\
C_{13} &= \frac{1}{60264048},\qquad C_{14}=\frac{1}{1013434464},\qquad C_{15}=\frac{1}{34901442000}.
\end{aligned}$$

10.3. **Quantities appearing in boundary conditions.**

(1) For the first boundary condition

$$\begin{aligned}
A_i^{(1)} &= -\frac{2}{(\triangle\theta)^2}\frac{\Lambda_i}{\omega_i^3}C_0,\quad D_i^{(1)}=\frac{2}{(\triangle\theta)^2}\frac{\Sigma_i}{\omega_i^3}C_0,\\
G_i^{(1)} &= -\frac{2}{(\triangle\theta)^2}\frac{\dot{y}_i}{\omega_i^3}C_0,\quad L_i^{(1)}=\frac{2}{(\triangle\theta)^2}\frac{\dot{x}_i}{\omega_i^3}C_0,
\end{aligned}$$

$$\begin{aligned}
B_{i,k}^{(1)} &= (-1)^{k-i+1}\frac{C_{k-i}}{\triangle\theta}\left(\frac{W_i}{\omega_i^5}(k-i)+\frac{2}{\triangle\theta}\frac{\Lambda_i}{\omega_i^3}\right),\\
C_{i,k}^{(1)} &= (-1)^{i-k}\frac{C_{i-k}}{\triangle\theta}\left(\frac{W_i}{\omega_i^5}(i-k)-\frac{2}{\triangle\theta}\frac{\Lambda_i}{\omega_i^3}\right),
\end{aligned}$$

$$\begin{aligned}
E_{i,k}^{(1)} &= (-1)^{k-i+1}\frac{C_{k-i}}{\triangle\theta}\left(\frac{U_i}{\omega_i^5}(k-i)-\frac{2}{\triangle\theta}\frac{\Sigma_i}{\omega_i^3}\right),\\
F_{i,k}^{(1)} &= (-1)^{i-k}\frac{C_{i-k}}{\triangle\theta}\left(\frac{U_i}{\omega_i^5}(i-k)+\frac{2}{\triangle\theta}\frac{\Sigma_i}{\omega_i^3}\right),
\end{aligned}$$

$$\begin{aligned}
H_{i,k}^{(1)} &= (-1)^{k-i+1}\frac{C_{k-i}}{\triangle\theta}\left(\frac{\rho_i}{\omega_i^5}(k-i)+\frac{2}{\triangle\theta}\frac{\dot{y}_i}{\omega_i^3}\right),\\
J_{i,k}^{(1)} &= (-1)^{i-k}\frac{C_{i-k}}{\triangle\theta}\left(\frac{\rho_i}{\omega_i^5}(i-k)-\frac{2}{\triangle\theta}\frac{\dot{y}_i}{\omega_i^3}\right),
\end{aligned}$$

$$\begin{aligned} N_{i,k}^{(1)} &= (-1)^{k-i+1}\frac{C_{k-i}}{\triangle\theta}\left(\frac{\varrho_i}{\omega_i^5}(k-i)-\frac{2}{\triangle\theta}\frac{\dot{x}_i}{\omega_i^3}\right), \\ M_{i,k}^{(1)} &= (-1)^{i-k}\frac{C_{i-k}}{\triangle\theta}\left(\frac{\varrho_i}{\omega_i^5}(i-k)+\frac{2}{\triangle\theta}\frac{\dot{x}_i}{\omega_i^3}\right). \end{aligned}$$

(2) For the second boundary condition

$$\begin{aligned} A_i^{(2)} &= \frac{2}{(\triangle\theta)^2}\frac{\Sigma_i}{\omega_i^3}C_0, \quad D_i^{(2)} = \frac{2}{(\triangle\theta)^2}\frac{\Lambda_i}{\omega_i^3}C_0, \\ G_i^{(2)} &= \frac{2}{(\triangle\theta)^2}\frac{\dot{x}_i}{\omega_i^3}C_0, \quad L_i^{(2)} = \frac{2}{(\triangle\theta)^2}\frac{\dot{y}_i}{\omega_i^3}C_0, \end{aligned}$$

$$\begin{aligned} B_{i,k}^{(2)} &= (-1)^{k-i+1}\frac{C_{k-i}}{\triangle\theta}\left(\frac{V_i}{\omega_i^5}(k-i)-\frac{2}{\triangle\theta}\frac{\Sigma_i}{\omega_i^3}\right), \\ C_{i,k}^{(2)} &= (-1)^{i-k}\frac{C_{i-k}}{\triangle\theta}\left(\frac{V_i}{\omega_i^5}(i-k)+\frac{2}{\triangle\theta}\frac{\Sigma_i}{\omega_i^3}\right), \end{aligned}$$

$$\begin{aligned} E_{i,k}^{(2)} &= (-1)^{k-i}\frac{C_{k-i}}{\triangle\theta}\left(\frac{W_i}{\omega_i^5}(k-i)+\frac{2}{\triangle\theta}\frac{\Lambda_i}{\omega_i^3}\right), \\ F_{i,k}^{(2)} &= (-1)^{i-k+1}\frac{C_{i-k}}{\triangle\theta}\left(\frac{W_i}{\omega_i^5}(i-k)-\frac{2}{\triangle\theta}\frac{\Lambda_i}{\omega_i^3}\right), \end{aligned}$$

$$\begin{aligned} H_{i,k}^{(2)} &= (-1)^{k-i+1}\frac{C_{k-i}}{\triangle\theta}\left(\frac{\varrho_i}{\omega_i^5}(k-i)-\frac{2}{\triangle\theta}\frac{\dot{x}_i}{\omega_i^3}\right), \\ k_{i,k}^{(2)} &= (-1)^{i-k}\frac{C_{i-k}}{\triangle\theta}\left(\frac{\varrho_i}{\omega_i^5}(i-k)+\frac{2}{\triangle\theta}\frac{\dot{x}_i}{\omega_i^3}\right), \end{aligned}$$

$$\begin{aligned} N_{i,k}^{(2)} &= (-1)^{k-i}\frac{C_{k-i}}{\triangle\theta}\left(\frac{\rho_i}{\omega_i^5}(k-i)+\frac{2}{\triangle\theta}\frac{\dot{y}_i}{\omega_i^3}\right), \\ M_{i,k}^{(2)} &= (-1)^{i-k+1}\frac{C_{i-k}}{\triangle\theta}\left(\frac{\rho_i}{\omega_i^5}(i-k)-\frac{2}{\triangle\theta}\frac{\dot{y}_i}{\omega_i^3}\right). \end{aligned}$$

(3) For the seven additional conditions

$$\begin{aligned} K_k^{(1)} &= \frac{(3-4\nu)\,\Gamma_k}{x_k^2+y_k^2}\Delta\theta, \qquad K_k^{(2)} = \frac{(3-4\nu)\,\Pi_k}{x_k^2+y_k^2}\Delta\theta, \\ K_k^{(3)} &= \frac{\left(y_k^2-x_k^2\right)\dot{y}_k+2x_ky_k\dot{x}_k}{\left(x_k^2+y_k^2\right)^2}\Delta\theta, \qquad K_k^{(4)} = \frac{\left(y_k^2-x_k^2\right)\dot{x}_k-2x_ky_k\dot{y}_k}{\left(x_k^2+y_k^2\right)^2}\Delta\theta. \end{aligned}$$

with

$$\begin{aligned}
\mathcal{I}_k^{(1)} &= (-1)^k \frac{\Sigma_1}{\omega_1^2}\frac{k-1}{\triangle\theta}C_{k-1}, \quad \mathcal{I}_k^{(2)} = (-1)^{k-M+15}\frac{\Sigma_1}{\omega_1^2}\frac{M-k+1}{\triangle\theta}C_{M-k+1},\\
\mathcal{I}_k^{(3)} &= (-1)^k \frac{\Lambda_1}{\omega_1^2}\frac{k-1}{\triangle\theta}C_{k-1}, \quad \mathcal{I}_k^{(4)} = (-1)^{k-M+15}\frac{\Lambda_1}{\omega_1^2}\frac{M-k+1}{\triangle\theta}C_{M-k+1},\\
\mathcal{I}_k^{(5)} &= (-1)^k \frac{\dot{x}_1}{\omega_1^2}\frac{k-1}{\triangle\theta}C_{k-1}, \quad \mathcal{I}_k^{(6)} = (-1)^{k-M+15}\frac{\dot{x}_1}{\omega_1^2}\frac{M-k+1}{\triangle\theta}C_{M-k+1},\\
\mathcal{I}_k^{(7)} &= (-1)^k \frac{\dot{y}_1}{\omega_1^2}\frac{k-1}{\triangle\theta}C_{k-1}, \quad \mathcal{I}_k^{(8)} = (-1)^{k-M+15}\frac{\dot{y}_1}{\omega_1^2}\frac{M-k+1}{\triangle\theta}C_{M-k+1},
\end{aligned}$$

and

$$\begin{aligned}
\xi_i &= \dot{y_i}^2 - \dot{x_i}^2, \quad \zeta_i = 2\dot{x}_i\dot{y}_i, \quad \sigma_i = \dot{x}_i\ddot{y}_i - \ddot{x}_i\dot{y}_i, \quad \delta_i = \dot{x}_i\ddot{x}_i + \dot{y}_i\ddot{y}_i,\\
\omega_i^2 &= \dot{x_i}^2 + \dot{y_i}^2, \quad \varrho_i = \dot{x}_i\delta_i + \dot{y}_i\sigma_i, \quad \rho_i = \dot{x}_i\sigma_i - \dot{y}_i\delta_i,\\
\Pi_i &= x_i\dot{x}_i + y_i\dot{y}_i, \quad , \quad \Gamma_i = x_i\dot{y}_i - \dot{x}_i y_i,\\
\Lambda_i &= x_i\dot{y}_i + \dot{x}_i y_i, \qquad \Sigma_i = x_i\dot{x}_i - y_i\dot{y}_i,
\end{aligned}$$

$$\begin{aligned}
W_i &= 2\dot{x}_i\dot{y}_i\omega_i^2 - (\Lambda_i\delta_i - \Sigma_i\sigma_i), \qquad U_i = 2\dot{y_i}^2\omega_i^2 + \Lambda_i\sigma_i + \Sigma_i\delta_i,\\
V_i &= -2\dot{x_i}^2\omega_i^2 + \Lambda_i\sigma_i + \Sigma_i\delta_i.
\end{aligned}$$

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Amr Ramadan El Dhaba, Faculty of Science, Damanhour University, Damanhour, Egypt

E-mail address: amrramadaneg@yahoo.com