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THE FRACTIONAL-ORDER LOGISTIC MODEL FOR THE INTERACTION OF DEMAND AND SUPPLY

H. A. A. EL-SAKA

Abstract. In this paper we shall consider the two-dimensional fractional-order logistic model for the interaction of demand and supply. The stability of equilibrium points are studied. Numerical solutions of this model are given. Numerical simulation have been used to verify the theoretical analysis.

1. Introduction

It is gradually recognized that ideas from the theory of complex dynamics are useful for economics and finance. In fact, in the early 1950s Richard Goodwin used nonlinear techniques in the study of dynamic economic processes. With the progress of the research on complex systems, some concepts and methods in nonlinear dynamics such as stability, bifurcation, catastrophe and chaos etc. have been applied to economic problems and some positive results have been achieved in the past decades [18]. Nonlinear dynamics has become an important approach to economic analysis.

A two-dimensional logistic model is used to describe the interactions and evolution of potential demand and supply [18].

The use of fractional-orders differential and integral operators in mathematical models has become increasingly widespread in recent years [17]. Several forms of fractional differential equations have been proposed in standard models.

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, economic, viscoelasticity, biology, physics and engineering. Recently, a large amount of literatures developed concerning the application of fractional differential equations in nonlinear dynamics [17].

In this paper we study the two-dimensional fractional-order logistic model for the interaction of demand and supply. The stability of equilibrium points are studied. Numerical solutions of this model are given.

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Now we give the definition of fractional-order integration and fractional-order differentiation:

Definition 1 The fractional integral of order $\beta \in R^+$ of the function $f(t)$, $t > 0$ is defined by

$$I^\beta f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s)\, ds \tag{1}$$

and the fractional derivative of order $\alpha \in (n-1, n)$ of $f(t)$, $t > 0$ is defined by

$$D_*^\alpha\ f(t) = I^{n-\alpha} D^n f(t),\ D_* = \frac{d}{dt}. \tag{2}$$

The following properties are some of the main ones of the fractional derivatives and integrals (see [6]-[8], [10], [16], [17]).

Let $\beta, \gamma \in R^+$ and $\alpha\ \in (0,1)$. Then

(i) $I_a^\beta : L^1 \to L^1$, and if $f(y) \in L^1$, then $I_a^\gamma\ I_a^\beta\ f(y)\ =\ I_a^{\gamma+\beta}\ f(y)$.

(ii) $\lim_{\beta \to n} I_a^\beta\ f(y) =\ I_a^n\ f(y)$ uniformly on $[a,b]$, $n\ =\ 1,\ 2,\ 3, \cdots$, where $I_a^1\ f(y) = \int_a^y f(s)\ ds$.

(iii) $\lim_{\beta \to 0} I_a^\beta\ f(y) = f(y)$ weakly.

(iv) If $f(y)$ is absolutely continuous on $[a,b]$, then $\lim_{\alpha \to 1} D_*^\alpha\ f(y) = \frac{df(y)}{dy}$.

(v) If $f(y) = k \neq 0$, k is a constant, then $D_*^\alpha\ k\ =\ 0$.

The following lemma can be easily proved (see [10]).

Lemma 1 Let $\beta \in (0,1)$ if $f \in C[0,T]$, then $I^\beta f(t)|_{t=0} = 0$.

2. Equilibrium points and their asymptotic stability

Let $\alpha \in (0,1]$ and consider the system ([1]-[3], [11], [13])

$$\begin{aligned} D_*^\alpha y_1(t) &= f_1(y_1, y_2) \\ D_*^\alpha y_2(t) &= f_2(y_1, y_2) \end{aligned} \tag{3}$$

with the initial values

$$y_1(0) = y_{o1} \text{ and } y_2(0) = y_{o2}. \tag{4}$$

To evaluate the equilibrium points, let

$$D_*^\alpha y_i(t) = 0 \Rightarrow f_i(y_1^{eq}, y_2^{eq}) = 0,\ i = 1, 2$$

from which we can get the equilibrium points $y_1^{eq},\ y_2^{eq}$.

To evaluate the asymptotic stability, let

$$y_i(t) = y_i^{eq} + \varepsilon_i(t),$$

so the the equilibrium point (y_1^{eq}, y_2^{eq}) is locally asymptotically stable if both the eigenvalues of the Jacobian matrix A

$$\begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix}$$

evaluated at the equilibrium point satisfies ([2], [3], [14])

$$(|\arg(\lambda_1)| > \alpha\pi/2, |\arg(\lambda_2)| > \alpha\pi/2). \tag{5}$$

The stability region of the fractional-order system with order α is illustrated in Fig. 1 (in which σ, ω refer to the real and imaginary parts of the eigenvalues, respectively, and $j = \sqrt{-1}$). From Fig. 1, it is easy to show that the stability region of the fractional-order case is greater than the stability region of the integer-order case.

3. Fractional-order Logistic model

The fractional-order Logistic model ([15], [18]) that determined the evolution of the potential demand and the potential supply are given by

$$D_*^{\alpha} y_1(t) = a y_1 \left(1 - \frac{c y_1}{2 y_2} - \frac{y_1}{2}\right), \tag{6}$$

$$D_*^{\alpha} y_2(t) = b y_2 \left(1 - \frac{y_2}{2 c y_1} - \frac{y_2}{2}\right), \tag{7}$$

where $a, b \geq 0$, $\alpha \in (0, 1]$, $c = M_d/M_s$, M_d is the sub-capacity for the potential demand and M_s is the sub-capacity for the potential supply.

To evaluate the equilibrium points, let

$$D_*^{\alpha} y_i(t) = 0, \quad i = 1, 2,$$

then $(y_1^{eq}, y_2^{eq}) = (0, 0)$, $(\frac{3}{2+c}, \frac{3c}{2c+1})$, are the equilibrium points.

For $(y_1^{eq}, y_2^{eq}) = (\frac{3}{2+c}, \frac{3c}{2c+1})$ we find that

$$A = \begin{bmatrix} -a & \frac{a(2c+1)^2}{2c(2+c)^2} \\ \frac{bc(2+c)^2}{2(2c+1)^2} & -b \end{bmatrix}$$

The characteristic polynomial of the equilibrium point $(y_1^{eq}, y_2^{eq}) = (\frac{3}{2+c}, \frac{3c}{2c+1})$ is given by:

$$\lambda^2 + (a + b)\lambda + \frac{3ab}{4} = 0, \tag{8}$$

and its eigenvalues are

$$\lambda_1 = \frac{-(a+b) + \sqrt{(a+b)^2 - 3ab}}{2},$$

$$\lambda_2 = \frac{-(a+b) - \sqrt{(a+b)^2 - 3ab}}{2}.$$

Hence the equilibrium point $(y_1^{eq}, y_2^{eq}) = (\frac{3}{2+c}, \frac{3c}{2c+1})$ is locally asymptotically stable [2].

4. Numerical methods and results

An Adams-type predictor-corrector method has been introduced and investigated further in ([4], [5], [9]). In this paper we use an Adams-type predictor-corrector method for the numerical solution of fractional integral equation.

The key to the derivation of the method is to replace the original problem (6)-(7) by an equivalent fractional integral equations

$$y_1(t) = y_1(0) + I^{\alpha}\left[a y_1 \left(1 - \frac{c y_1}{2 y_2} - \frac{y_1}{2}\right)\right], \tag{9}$$

$$y_2(t) = y_1(0) + I^{\alpha}\left[b y_2 \left(1 - \frac{y_2}{2 c y_1} - \frac{y_2}{2}\right)\right], \tag{10}$$

and then apply the **PECE** (Predict, Evaluate, Correct, Evaluate) method. The approximate solutions displayed in Figs. 2-7 for the step size 0.05 and different $0 < \alpha \leq 1$. In Figs. 2-4 we took $a = 2,\ b = 3,\ c = 1,\ y_1(0) = 0.5,\ y_2(0) = 0.3$ and found that the equilibrium point $(y_1^{eq}, y_2^{eq}) = (1, 1)$ is locally asymptotically stable. In Figs. 5-7 we took $a = 3,\ b = 4,\ c = 2,\ y_1(0) = 0.5, y_2(0) = 0.3$ and found that the equilibrium point $(y_1^{eq}, y_2^{eq}) = (0.75, 1.2)$ is locally asymptotically stable.

5. Conclusions

In this paper we studied the two-dimensional fractional-order logistic model for the interaction of demand and supply. The stability of equilibrium points are studied. Numerical solutions of this model are given.

The reason for considering a fractional order system instead of its integer order counterpart is that fractional order differential equations are generalizations of integer order differential equations.

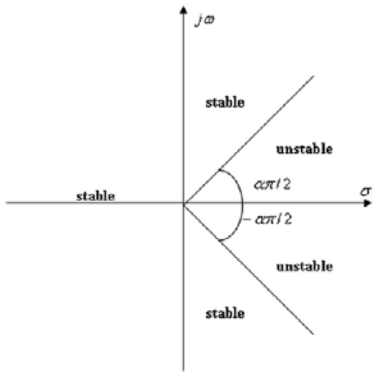

Fig. 1. Stability region of the fractional-order system.

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Fig. 2. alpha=0.8, a=2, b=3, c=1, y1(0)=0.5, y2(0)=0.3

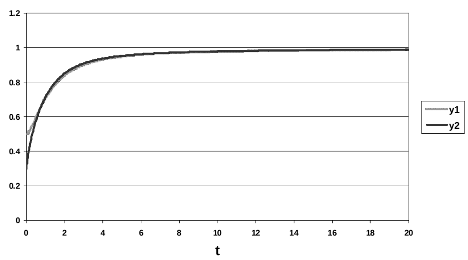


Fig. 3. alpha=0.9, a=2, b=3, c=1, y1(0)=0.5, y2(0)=0.3

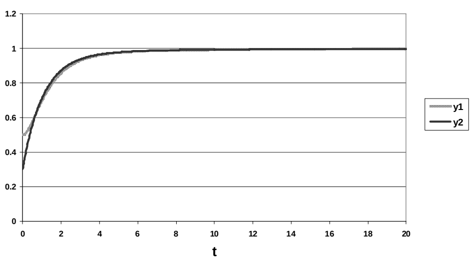


Fig. 4. alpha=1.0, a=2, b=3, c=1, y1(0)=0.5, y2(0)=0.3

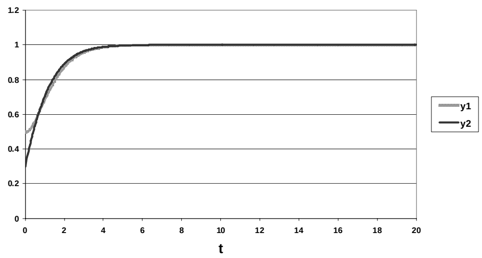


Fig. 5. alpha=0.8, a=3, b=4, c=2, y1(0)=0.5, y2(0)=0.3

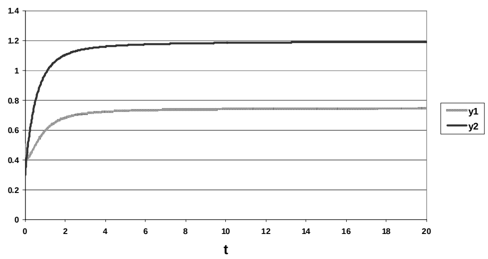


Fig. 6. alpha=0.9, a=3, b=4, c=2, y1(0)=0.5, y2(0)=0.3

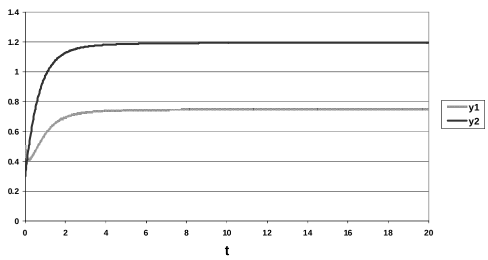
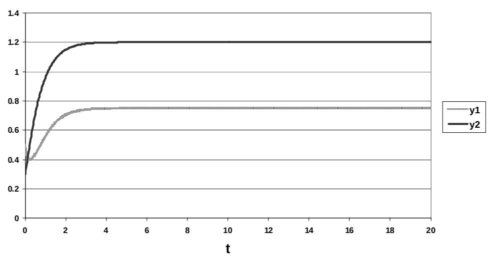


Fig. 7. alpha=1.0, a=3, b=4, c=2, y1(0)=0.5, y2(0)=0.3



References

[1] E. Ahmed, A. M. A. El-Sayed, E. M. El-Mesiry and H. A. A. El-Saka, Numerical solution for the fractional replicator equation, IJMPC, Vol. 16 (2005) No. 7, pp. 1-9.

[2] E. Ahmed, A. M. A. El-Sayed, H. A. A. El-Saka, On some Routh-Hurwitz conditions for fractional order differential equations and their applications in Lorenz, Rössler, Chua and Chen systems, Physics Letters A, Vol. 358 (2006) No.4.

[3] E. Ahmed, A. M. A. El-Sayed, H. A. A. El-Saka, Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models, J. Math. Anal. Appl., Vol. 325 (2007) pp. 542-553.

[4] Kai Diethelm, The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type (Lecture Notes in Mathematics), Springer-Verlag Berlin Heidelberg 2010.

[5] E. M. El-Mesiry, A. M. A. El-Sayed and H. A. A. El-Saka, Numerical methods for multi-term fractional (arbitrary) orders differential equations, Appl. Math. and Comput., Vol. 160 (2005) No. 3, pp. 683-699.

[6] A. M. A. El-Sayed, fractional differential-difference equations, Journal of Fractional Calculus, 10, 101-106, 1996.

[7] A. M. A. El-Sayed, Nonlinear functional differential equations of arbitrary orders, Nonlinear Analysis: Theory, Methods and Applications, Vol. 33 (1998) No. 2, pp. 181-186.

[8] A. M. A. El-Sayed and F. M. Gaafar, Fractional order differential equations with memory and fractional-order relayation-oscillation model, (PU.M.A) Pure Math. and Appl., Vol. 12 (2001).

[9] A. M. A. El-Sayed, E. M. El-Mesiry and H. A. A. El-Saka, Numerical solution for multi-term fractional (arbitrary) orders differential equations, Comput. and Appl. Math., Vol. 23 (2004) No. 1, pp. 33-54.

[10] A. M. A. El-Sayed, F. M. Gaafar and H. H. Hashem, On the mayimal and minimal solutions of arbitrary orders nonlinear functional integral and differential equations, Math. Sci. Res. J., Vol. 8 (2004) No. 11, pp. 336-348.

[11] A. M. A. El-Sayed, E. M. El-Mesiry and H. A. A. El-Saka, On the fractional-order logistic equation, AML, Vol. 20 (2007) No. 7, pp. 817-823.

[12] R. Gorenflo and F.Mainardi, Fractional Calculus: Integral and Differential Equations of Fractional Order, in A. Carpinteri and F. Mainardi (Eds), Fractals and Fractional Calculus in Continuum Mechanics, Springer, 223-276, 1997, Wien.

[13] H. A. El-Saka, E. Ahmed, M. I. Shehata and A. M. A. El-Sayed, On stability, persistence and Hopf Bifurcation in fractional order dynamical systems, Nonlinear Dyn, Vol. 56 (2009), pp. 121-126.

[14] D. Matignon, Stability results for fractional differential equations with applications to control processing, Computational Eng. in Sys. Appl. Vol. 2 (1996) Lille France 963.

[15] E. M. Elabbasy, H. Agiza and S. H. Saker, Global existence of positive periodic solutions of two-dimensional logistic model for the interaction demand and supply, Mansoura Journal of Mathematics, Vol. 32 (2005) No. 1.

[16] I. Podlubny and A. M. A. El-Sayed, On two definitions of fractional calculus, Solvak Academy of science-institute of eyperimental phys. UEF-03-96 ISBN 80-7099-252-2, 1996.

[17] I. Podlubny, Fractional differential equations, Academic Press, 1999.

[18] D. Zengru and M. Sanglier, A two-dimensional logistic model for the interaction of demand and supply and its bifurcations, Chaos, Solirons and Fractals, Vol. 7 (1996) No. 12, pp. 2259-2266.

Damietta Faculty of Science, Mansoura University, 34517, New Damietta, Egypt

E-mail address: halaelsaka@yahoo.com