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EXISTENCE OF CONTINUOUS SOLUTION FOR A QUADRATIC INTEGRAL EQUATION OF CONVOLUTION TYPE

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Abstract. We are concerned here with the existence of at least one continuous solution of the quadratic integral equation of convolution type

$$x(t) \ = \ a(t) \ + \int_0^t k_1(t-s)f_1(s,x(s))ds \int_0^t k_2(t-s)f_2(s,x(s))ds, \ t \in [0.T].$$

The maximal and minimal solutions are also proved.

1. Introduction and preliminaries

Quadratic integral equations (QIES) are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. The quadratic integral equations can be very often encountered in many applications (see[1]-[14]).
Let $I = [0,T]$, $C = C[0,T]$ be the space of continuous functions on I, and $L^1 = L^1[0,T]$ be the space of Lebesgue integrable functions on I.
The quadratic integral equation

$$x(t) = a(t) + \int_0^t f(s,x(s)) \ ds \int_0^t g(s,x(s)) \ ds \tag{1}$$

has been studied in [10]. The authors proved that it has at least one continuous solution, also they proved the existence of the maximal and minimal solutions.
The quadratic integral equation

$$x(t) = a(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s)) \ ds \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g(s,x(s)) \ ds, \ \alpha, \ \beta \in (0,1) \tag{2}$$

has been studied in [12]. The authors proved that it has at least one continuous solution, also they proved the existence of the maximal and minimal solutions.
We are concerned here with the existence of at least one continuous solution of the

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quadratic integral equation of convolution type

$$x(t) \ = \ a(t) \ + \int_0^t k_1(t-s)f_1(s,x(s))ds \int_0^t k_2(t-s)f_2(s,x(s))ds, \ t \in [0,T]. \quad (3)$$

The existence of the maximal and minimal solutions of the quadratic integral equation(3) will be proved.
The main result will be based on the following theorems.

Theorem 1. *Schauder fixed-point Theorem* [15]
Let S be a convex subset of a Banach space B, let the mapping $T: \ S \to S$ be compact, continuous. Then T has at least one fixed-point in S.

Theorem 2. *Arzela-Ascoli Theorem* [16]
Arzela-Ascoli Theorem Let E be a compact metric space and $C(E)$ the Banach space of real or complex valued continuous functions normed by

$$\| \ f \ \| \ = \ \max_{t \ \in \ E} \ | \ f(t) \ |.$$

If $A \ = \ \{f_n\}$ is a sequence in $C(E)$ such that f_n is uniformly bounded and equi-continuous. Then the closure of A is compact.

Theorem 3. *(Lebesgue Dominated Convergence Theorem)* [16] *Let $\{f_n\}$ be a sequence of functions converging to a limit f on A, and suppose that*

$$| \ f_n(t)| \ \leq \phi(t), \ t \in A, \ n = 1, 2, 3, ...$$

Where ϕ is an integrable function on A.Then f is integrable on A and

$$\lim_{n\to\infty} \int_A f_n \ (t) \ d\mu = \int_A f(t) \ d\mu.$$

2. Existence of solutions

Consider the quadratic integral equation (3) under the following assumptions

(i) $a : I = [0,T] \to R$ is continuous, $a \ = \ \sup \ |a(t)|, \ t \in [0,T]$.

(ii) $f_i : I \times R \to R$ are L^1-Carathèodary functions.
i.e f_i measurable in t for all $x \ \in R$ and continuous in x for almost all $t \in [0,T]$, and there exist two functions $m_i \in \ L^1[0,T]$ such that

$$| \ f_i(t,x) \ | \ \leq \ m_i(t)$$

$$\int_0^t m_i(s)ds \leq M, \ \forall \ t \in [0,T], \ i = 1, 2.$$

(iii) $k_i : \ [0,T] \ \to R$ are continuous, $|k_i(t)| \leq K, \ \forall \ t \in [0,T]$, i=1,2.

Now for the existence of at least one continuous solution of the quadratic integral equation (3) we have the following theorem.

Theorem 4. *If the assumptions (i)-(iii) are satisfied, then the quadratic integral equation (3) has at least one solution $x \ \in \ C[0,T]$.*

Proof. Let $C = C[0,T]$ and define the set S by

$$S = \{x \in \ C : |\ x(t)| \leq \ r\} \subset \ C[0,T],$$

where $r = a + K^2 M^2$.
It is clear that S is nonempty, bounded, convex, and closed.
Define the operator F associated with the quadratic integral equation (3) by

$$Fx(t) \ = \ a(t) \ + \int_0^t \ k_1 \ (t-s) \ f_1(s, x(s)) \ ds \int_0^t \ k_2 \ (t-s) f_2(s, x(s)) \ ds$$

to show that $F : S \ \to S$, let $x \ \in \ S$, then

$$\begin{aligned}
|Fx(t)\ | \quad &= \ |\ a(t) \ + \int_0^t k_1(t-s) \ f_1(s, x(s)) \ ds \int_0^t k_2(t-s) f_2(s, x(s)) \ ds \ | \\
&\leq \ |\ a(t)\ | + \int_0^t |k_1(t-s)| \ |f_1(s, x(s))| ds \int_0^t |k_2(t-s)| \ |f_2(s, x(s))| \ ds \\
&\leq \ |\ a(t)\ | + \int_0^t |k_1(t-s)| \ m_1(s) ds \int_0^t |k_2(t-s)| \ m_2(s) \ ds \\
&= \ |a(t)| \ + \int_0^t |k_1 \ (s) \ | \ m_1(t-s) ds \ \int_0^t |k_2 \ (s)| \ m_2(t-s) \ ds \\
&\leq \ |a(t)| \ + K^2 \int_0^t m_1(t-s) ds \ \int_0^t m_2(t-s) \ ds
\end{aligned}$$

set $t - s = \theta$, we get

$$\begin{aligned}
&\leq \ |a(t)| \ + K^2 \int_0^t m_1(\theta) d\theta \ \int_0^t m_2(\theta) \ d\theta \\
&\leq \ a \ + K^2 M^2
\end{aligned}$$

and $Fx(t) \in \ S$; which proves that $F : S \ \to S$.
This prove that the class of functions $\{F(x)\}$ is uniformly bounded.

Let $t_1, t_2 \in [0,T],\ t_1 < t_2$ and $|t_2 - t_1| \leq \delta$, then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= |a(t_2) - a(t_1) \\
&+ \int_0^{t_2} k_1(t_2-s)\ f_1(s,x(s))\ ds \int_0^{t_2} k_2(t_2-s) f_2(s,x(s))\ ds \\
&- \int_0^{t_1} k_1(t_1-s)\ f_1(s,x(s))\ ds \int_0^{t_1} k_2(t_1-s) f_2(s,x(s))\ ds| \\
&= |\ a(t_2) - a(t_1) \\
&+ \int_0^{t_2} k_1(t_2-s)\ f_1(s,x(s))\ ds \int_0^{t_2} k_2(t_2-s)\ f_2(s,x(s))\ ds \\
&- \int_0^{t_1} k_1(t_1-s)\ f_1(s,x(s))\ ds \int_0^{t_1} k_2(t_1-s)\ f_2(s,x(s))\ ds \\
&+ \int_0^{t_2} k_1(t_2-s)\ f_1(s,x(s))\ ds \int_0^{t_1} k_2(t_1-s)\ f_2(s,x(s))\ ds \\
&- \int_0^{t_2} k_1(t_2-s)\ f_1(s,x(s))\ ds \int_0^{t_1} k_2(t_1-s)\ f_2(s,x(s))\ ds| \\
&\leq |a(t_2) - a(t_1)| \\
&+ |\int_0^{t_1} k_2(t_1-s)\ f_2(s,x(s)) ds [\int_0^{t_2} k_1(t_2-s)\ f_1(s,x(s))\ ds \\
&- \int_0^{t_1} k_1(t_1-s)\ f_1(s,x(s))\ ds] \\
&+ \int_0^{t_2} k_1(t_2-s)\ f_1(s,x(s))\ ds [\int_0^{t_2} k_2(t_2-s)\ f_2(s,x(s))\ ds \\
&- \int_0^{t_1} k_2(t_1-s)\ f_2(s,x(s))\ ds]| \\
&\leq |a(t_2) - a(t_1)| \\
&+ |\int_0^{t_1} k_2(t_1-s)\ f_2(s,x(s))\ ds [\int_0^{t_1} k_1(t_2-s)\ f_1(s,x(s))\ ds \\
&+ \int_{t_1}^{t_2} k_1(t_2-s)\ f_1(s,x(s))\ ds - \int_0^{t_1} k_1(t_1-s)\ f_1(s,x(s))\ ds] \\
&+ \int_0^{t_2} k_1(t_2-s)\ f_1(s,x(s))\ ds [\int_0^{t_1} k_2(t_2-s)\ f_2(s,x(s))\ ds \\
&+ \int_{t_1}^{t_2} k_2(t_2-s)\ f_2(s,x(s))\ ds - \int_0^{t_1} k_2(t_1-s)\ f_2(s,x(s))\ ds]| \\
&\leq |a(t_2) - a(t_1)| \\
&+ \int_0^{t_1} |k_2(t_1-s)|\ |f_2(s,x(s))|\ ds \int_0^{t_1} |k_1(t_2-s) - k_1(t_1-s)|\ |f_1(s,x(s))|\ ds \\
&+ \int_0^{t_1} |k_2(t_1-s)|\ |f_2(s,x(s))|\ ds \int_{t_1}^{t_2} |k_1(t_2-s)|\ |f_1(s,x(s))|\ ds \\
&+ \int_0^{t_2} |k_1(t_2-s)|\ |f_1(s,x(s))|\ ds \int_0^{t_1} |k_2(t_2-s) - k_2(t_1-s)|\ |f_2(s,x(s))|\ ds \\
&+ \int_0^{t_2} |k_1(t_2-s)|\ |f_1(s,x(s))|\ ds \int_{t_1}^{t_2} |k_2(t_2-s)|\ |f_2(s,x(s))|\ ds
\end{aligned}$$

$$\begin{aligned}
&\leq \quad |a(t_2) - a(t_1)| \\
&+ \quad \int_0^{t_1} |k_2(t_1 - s)|\ m_2(s)ds\ \int_0^{t_1} |k_1(t_2 - s) - k_1\ (t_1 - s)|\ m_1(s)\ ds \\
&+ \quad \int_0^{t_1} |k_2(t_1 - s)|\ m_2(s)ds\ \int_{t_1}^{t_2} |k_1(t_2 - s)|\ m_1(s)ds \\
&+ \quad \int_0^{t_2} |k_1(t_2 - s)|\ m_1(s)ds\ \int_0^{t_1} |k_2(t_2 - s) - k_2\ (t_1 - s)|\ m_2(s)ds \\
&+ \quad \int_0^{t_2} |k_1(t_2 - s)|\ m_1(s)ds\ \int_{t_1}^{t_2} |k_2(t_2 - s)|\ m_2(s)ds.
\end{aligned}$$

This means that the class of functions $F\{x\}$ is equi-continuous on $[0, T]$. Using Arzela-Ascoli Theorem [16], we fined that F is compact.
Now we prove that $F : S \to S$ is continuous. Let $\{x_n\} \subset S$, and $x_n \to x$, then

$$Fx_n(t) \ = \ a(t) \ + \int_0^t \ k_1\ (t-s)\ f_1(s, x_n(s))\ ds \int_0^t \ k_2\ (t-s)\ f_2(s, x_n(s))\ ds$$

$$\lim_{n_k \to \infty} Fx_n(t) = \lim_{n \to \infty} a(t) + \lim_{n \to \infty} \{ \int_0^t k_1(t-s) f_1(s, x_n(s)) ds \int_0^t k_2(t-s) f_2(s, x_n(s)) ds \}$$

Now

$$f_i(s, x_{n_k}) \to f_i(s, x) \ \Rightarrow k_i(t-s) f_i(s, x_{n_k}) \to k_i(t-s) f_i(s, x),\ \ i = 1, 2.$$

Also

$$|k_i(t-s) f_i(s, x_{n_k})| \ \leq |k_i(t-s)| m_i(t) \in L^1[0, T],\ \ i = 1, 2.$$

Then by using Lebesgue dominated convergence Theorem [16], we have

$$Fx(t) = \lim_{n_k \to \infty} Fx_{n_k}(t) \ = \ a(t) + \int_0^t \ k_1\ (t-s) \ \lim_{n_k \to \infty} f_1(s, x_{n_k}(s))\ ds \int_0^t \ k_2\ (t-s) \ \lim_{n_k \to \infty} f_2(s, x_{n_k}(s))\ ds$$

$$Fx(t) \ = \ a(t) \ + \int_0^t \ k_1\ (t-s)\ f_1(s, x(s))\ ds \int_0^t \ k_2\ (t-s)\ f_2(s, x(s))\ ds.$$

Then $Fx_n(t) \to Fx(t)$. Which means that the operator F is continuous.
Since all conditions of Schauder fixed point Theorem [15] are satisfied, then the operator F has at least one fixed point $x \in C[0, T]$, which completes the proof. ■

Now let $k_1(t) = k_2(t) = 1$ in equation (3), then we have the following corollary;

Corollary 1. *Let the assumptions (i)-(ii) of Theorem 4 be satisfied, then the quadratic integral equation (1) has at least one continuous solution* $x \in C[0, T]$. ■

Remark 1. *Corollary 2.2 is the main result in* [10]. *This proves the generality of our result.*

3. Existence of the maximal and minimal solutions

Definition 1. .Let $q(t)$ be a solution of the quadratic integral equation (1). Then $q(t)$ is said to be a maximal solution of (1) if every solution $x(t)$ of (1) satisfies the inequality (see[17]).

$$x(t) < q(t), \ t \in [0, T]. \tag{4}$$

A minimal solution $s(t)$ can be defined by similar way by reversing the above inequality i.e

$$x(t) > s(t), \ t \in [0, T]. \tag{5}$$

Consider the following lemma

Lemma 1. *Let $f_1(t, x), \ f_2(t, x)$ are L^1-Caratheòdary and $x(t), \ y(t)$ are two continuous functions on $[0, T]$ satisfying*

$$x(t) \leq \ a(t) \ + \int_0^t \ k_1 \ (t-s) \ f_1(s, x(s))ds \int_0^t \ k_2 \ (t-s) \ f_2(s, x(s))ds, \ t \in [0, T]$$

$$y(t) \geq \ a(t) \ + \int_0^t \ k_1 \ (t-s) \ f_1(s, y(s)) \ ds \int_0^t \ k_2 \ (t-s) \ f_2(s, y(s)) \ ds, \ t \in [0, T]$$

and one of them is strict.
If $f_i, \ i = 1, 2$ are monotonic nondecreasing in x, then

$$x(t) < y(t), \ t > 0 \tag{6}$$

Proof. Let the conclusion (6) be false, then there exists t_1 such that

$$x(t_1) = y(t_1), \ t_1 > 0$$

and

$$x(t) < y(t), \ 0 < t < t_1.$$

From the monotonicity of $f_1, \ f_2$ in x, we get

$$\begin{aligned} x(t_1) \quad &\leq \quad a(t_1) \ + \int_0^{t_1} \ k_1 \ (t_1 - s) \ f_1(s, x(s)) \ ds \int_0^{t_1} \ k_2 \ (t_1 - s) \ f_2(s, x(s)) \ ds, \ t \in [0, T] \\ &< \quad a(t_1) \ + \int_0^{t_1} \ k_1 \ (t_1 - s) \ f_1(s, y(s)) \ ds \int_0^{t_1} \ k_2 \ (t_1 - s) \ f_2(s, y(s)) \ ds, \ t \in [0, T] \\ x(t_1) \quad &< \quad y(t_1) \end{aligned}$$

which contradicts the fact that $x(t_1) = y(t_1)$.
Then

$$x(t) < y(t). \ \blacksquare$$

Now, for the existence of the continuous maximal and minimal solutions of the quadratic integral equation (3) we have the following theorem.

Theorem 5. *Let the assumptions (i)-(iii) of Theorem 4 are satisfied. If $f_1(t, x), \ f_2(t, x)$ are monotonic nondecreasing in x for each $t \in [0, T]$, then the quadratic integral equation (3) has maximal and minimal solutions.*

Proof. Firstly we shall prove the existence of the maximal solution of (3) .
Let $\epsilon > 0$ be given, and consider the quadratic integral equation

$$x_\epsilon(t) \leq a(t) + \int_0^t k_1 (t-s) \, f_{1\epsilon}(s, x_\epsilon(s)) \, ds \int_0^t k_2 (t-s) \, f_{2\epsilon}(s, x_\epsilon(s)) \, ds, \; t \in [0, T] \tag{7}$$

where

$$f_{i\epsilon}(t, x_\epsilon(t)) = f_i(t, x_\epsilon(t)) + \epsilon, \; i = 1, 2.$$

Clearly the function $f_{i\epsilon}(t, x_\epsilon(t)), \; i = 1, 2$ are L^1- Carathèodary functions, therefore the equation (7) has a solution on $C[0, T]$.
Let $\epsilon_1, \; \epsilon_2$ be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$, then

$$\begin{aligned} x_{\epsilon_2}(t) &= a(t) + \int_0^t k_1 (t-s) \, f_{1\epsilon_2}(s, x_{\epsilon_2}(s)) \, ds \int_0^t k_2 (t-s) \, f_{2\epsilon_2}(s, x_{\epsilon_2}(s)) \, ds \\ &= a(t) + \int_0^t k_1(t-s) \, (f_1(s, x_{\epsilon_2}(s)) + \epsilon_2) \, ds \int_0^t k_2(t-s) \, (f_2(s, x_{\epsilon_2}(s)) + \epsilon_2) \, ds \end{aligned} \tag{8}$$

also

$$\begin{aligned} x_{\epsilon_1}(t) &= a(t) + \int_0^t k_1(t-s) \, f_{1\epsilon_1}(s, x_{\epsilon_1}(s)) \, ds \int_0^t k_2(t-s) \, f_{2\epsilon_1}(s, x_{\epsilon_1}(s)) \, ds \\ &= a(t) + \int_0^t k_1(t-s) \, (f_1(s, x_{\epsilon_1}(s)) + \epsilon_1) \, ds \int_0^t k_2(t-s) \, (f_2(s, x_{\epsilon_1}(s)) + \epsilon_1) \, ds \end{aligned}$$

$$x_{\epsilon_1}(t) > a(t) + \int_0^t k_1 (t-s) \, f_1(s, x_{\epsilon_1}(s)) \, ds \int_0^t k_2(t-s) \, f_2(s, x_{\epsilon_1}(s)) \, ds \tag{9}$$

Applying Lemma 1 on (8) and (9) we have

$$x_{\epsilon_2}(t) < x_{\epsilon_1}(t) \; for \; t \in [0, T].$$

As shown before, the family of functions $x_\epsilon(t)$ is equi-continuous and uniformly bounded. Hence, by Arzela-Ascoli Theorem (see[16]), there exists a decreasing sequence ϵ_n such that $\epsilon_n \to 0$ an $n \to \infty$, and $\lim_{n\to\infty} x_{\epsilon_n}(t)$ exists uniformly in $[0, T]$ and denote the limit by $q(t)$. From the continuity of the functions $f_{i\epsilon}(t, x_\epsilon(t)), \; i = 1, 2$ in the second argument, we get

$$f_{i\epsilon}(t, x_\epsilon(t)) \to f_i(t, x(t)) \; as \; n \to \infty, \; i = 1, 2$$

and

$$q(t) = \lim_{n\to\infty} x_{\epsilon_n}(t) = a(t) + \int_0^t k_1(t-s) f_1(s, q(s)) ds \int_0^t k_2(t-s) f_2(s, q(s)) ds$$

which implies that $q(t)$ is a solution of the quadratic integral equation (3).
Finally we shall show that $q(t)$ is the maximal solution of (3).
To do this let $x(t)$ be any solution of (3), then

$$x(t) = a(t) + \int_0^t k_1(t-s) f_1(s, x(s)) ds \int_0^t k_2(t-s) f_2(s, x(s)) ds \tag{10}$$

also

$$x_\epsilon(t) = a(t) + \int_0^t k_1(t-s) f_{1\epsilon}(s, x_\epsilon(s)) ds \int_0^t k_2(t-s) f_{2\epsilon}(s, x_\epsilon(s)) \, ds$$

$$x_\epsilon(t) = a(t) + \int_0^t k_1(t-s)(f_1(s, x_\epsilon(s)) + \epsilon)ds \int_0^t k_2(t-s)(f_2(s, x_\epsilon(s)) + \epsilon)\ ds$$

$$x_\epsilon(t) > a(t) + \int_0^t k_1(t-s) f_1(s, x_\epsilon(s))ds \int_0^t k_2(t-s) f_2(s, x_\epsilon(s))\ ds \tag{11}$$

Applying Lemma 1 on (10) and (11) we get

$$x(t) < x_\epsilon(t),\ for\ t \in [0, T].$$

From the uniqueness of the maximal solution (see[17]), it is clear that $x_\epsilon(t)$ tends to $q(t)$ uniformly in $[0, T]$ as $\epsilon \to \infty$.

By similar way we can prove the existence of the minimal solution. We set

$$f_{i_\epsilon}(t, x_\epsilon(t)) = f_i(t, x_\epsilon(t)) - \epsilon,\ i = 1, 2$$

and prove the existence of minimal solution. ■

Now let $k_1(t) = k_2(t) = 1$ in equation (3) then we have the following corollary;

Corollary 2. *Let the assumptions of Theorem 5 be satisfied, then the quadratic integral equation*

$$x(t) = a(t) + \int_0^t f(s, x(s))\ ds \int_0^t g(s, x(s))\ ds.$$

has a maximal and minimal solutions $x \in C[0, T]$, which is the same result obtained in (see[10]). ■

4. Quadratic integral equation of fractional orders

The quadratic integral equation of fractional orders (2) has been studied in [12]. The author proved the existence of at least one positive solution $x \in C[0, T]$ of (2) under the following assumptions;

(i) $a : I = [0, T] \to R_+$ is continuous function.

(ii) $f, g : I \times R_+ \to R_+$ such that f, g are measurable in t for all $x \in R_+$ and continuous in x for each fixed $t \in [0, T]$, and there exist two functions m_1 , $m_2 \in L^1(I)$ such that

$$| f(t, x) | \leq m_1(t)\ and\ | g(t, x) | \leq m_2(t)$$

Also they proved the existence of the maximal and minimal solutions when $f(t, x)$ and $g(t, x)$ are monotonic nondecreasing in x for each $t \in [0, T]$.

It must be noticed that the quadratic integral equation (2) is a spacial case of the quadratic integral equation (3), where

$$k_1(t) = \frac{(t)^{\alpha-1}}{\Gamma(\alpha)}\ and\ k_2(t) = \frac{(t)^{\beta-1}}{\Gamma(\beta)},\ \alpha,\ \beta > 0.$$

But these functions $k_1(t)$ and $k_2(t)$ does not satisfy our assumptions (iii) of Theorem 4 that is the two functions k_1 and k_2 are continuous.

This implies that the condition (iii) of continuity of the two functions k_1 and k_2 in Theorem 4 is sufficient condition. ■

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