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SOME MATHEMATICS MOTIVATED BY FRACTALS (AN OVERVIEW)

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Abstract. Fractals are abundant in nature. To study them one should understand several mathematical concepts.
Here we concentrate on two of them. The first is Ulam-Hyers stability. The second is continuous nondifferentiable functions and their connection to fractional order differentiation.

1. Introduction

(I) Nature is an excellent source for mathematical topics that deserve to be studied.
Why fractals are almost everywhere (trees, lungs, kidneys, .nancial data, earthquakes, expansion of cities etc. . .)? [Farmer and Geanakoplos 2008].

(II) Also notice that ordinary differentiation is not suitable to study fractals since , although there is no formal de.nition of fractals, they are familiarly known to be continuous nowhere differentiable functions with Holder exponent less than one.

(III) (1) Schroeder equation [Wikipedia]:

$$x(f(t)) = cx(t),\ c \text{ constant},\ t > 0 \tag{1}$$

Set $f(t) = bt$, b is constant then one gets self similar equation
$x(bt) = ax(t) \Rightarrow\ x(t)$ is power law in time.

(2) Existence, uniqueness of its solution has been studied.

(3) Ulam-Hyers stability , I think is significant for approximation since it states that:
The system (1) is Ulam-Hyers stable if it has an exact solution and if

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$\forall\ \epsilon\ >\ 0$ there is $\delta > 0$ such that if $xa(t)$ is an approximation for the solution of (1) then there is an exact solution $x(t)$ of (1) which is close to xa i.e.

$$||xa(t)\ -\ xa(f(t))||\ <\ \delta\ ||x(t)\ -\ xa(t)||\ < \epsilon\ \forall\ t >\ 0 \tag{2}$$

(It can be shown that Ulam-Hyers stability ,in general, is independent of Lyapunov stability e.g. exponential map).

IV) Continuous but nowhere differentiable maps e.g. Weirstrass function:

$$W(t)\ =\ \sum_{j=0} c^{(-k)} \sin(t\ b^{-k})$$

Where $b > c > 1,\ b$ is a positive integer,
Takagi function:

$$T(x)\ =\ \sum_{n\geq\ 0} [dist(x\ b^n,\ Z)/(c^n)]$$

Where $b >= c,\ c > 1,\ b$ is natural number, $Dist(x,\ y)$ is the distance function
i.e.

$$dist(x)\ =\ x \text{ if } 0 \leq\ x\ <\ 1/2$$
$$dist(x)\ =\ 1\ -\ x \text{ if } 1/2 \leq x \leq 1$$

,Z is the set of integers.
and Riemann function

$$R(t)\ =\ \sin(k^2)/k^2$$

Proof of continuity, Holder continuity, nondifferentiability (almost everywhere) and functional equations of these functions have been given [Kanaapan 2009 and references therein]. Riemann function has intrinsic relation to some solutions of Schrodinger equation.
Using de Rham theorem [Kairies 1997] it can be shown that the functional equations satis.ed by Weierstrass and Takagi functions determine them uniquely. This not the case for Riemann function.
Existence of fractional order derivative for such maps have been given [Kolwanker and Gangal 1996, Falconer 2003, Spurrier 2004, Rocco and West 1999 and Liang et al 2008, Jumarie 2009].

$$D^{\alpha} f(t)\ =\ (1/\Gamma(1\ -\ \alpha))(d/dt) \int_0^t [f(s)\ -\ f(0)]/(t\ -\ s)^{\alpha} ds$$

(V) Schroeder equation and circle map
Recently [Cieplinski 2008] a relation has been derived between Schroeder equation and a kind of circle map. The standard circle map [Wikipedia, Ahmed et al 2006]

$x(t+1) = x(t) + w - k\ \sin(2\ \pi\ x(t)),\ k,\ w$ re positive constants.

are known to have a fractal structure in the relation between the parameter w and the rotation number of the map. The rotation number is the limit for large n for the difference

$$[f \circ f \circ f \circ \cdots \circ f(x)\ (n \text{ times}) - x]/n.$$

(VI) Fractal structure is intrinsic in 2-d dynamical systems (Julia set) [Devany 2003]. It is the boundary between two attractors of the dynamical system. The proof of the fractality of the Julia set is given.
But another interesting relation between continuous nowhere differentiable functions and chaos has been discovered by Yamaguti and Hata [Yamaguti and Hata 1983].
Fractals have many applications in medicine [Thamrin et al 2010] e.g. in the structures (trees and lungs etc... In fact the area of oxygen exchange is equal to the area of a Tennis court [Wikipedia lung].
Also it has been observed that some diseases change fractal properties of some organs e.g. heart rate variability.
Fractals are also intrinsically related to phase transition in statistical mechnics [Aharony and Stauffer 1991]. At phase transition the correlation length increases causing self similarity.
Fractals are also intrinsically related to number theory and numerical approximations.

(VII) Fractal space time is a theory [Nottale1993] that proposes that at small scales spacetime is fractal. It has many advantagese.g. Schrodinger equation is equivalent to Euler-Lagrange equation on a fractal. Possiblity of fractional order quantum mechanics and quantum .eld theory which are nonlocal, .nite and do not need Higg's particle.
The study of fractals and its related mathematics have included the work of Poincare, Kolmogorov, Arnold and Mandelbrot. I hope it will interest more fruitful studies soon.

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