Journal of Fractional Calculus and Applications,
Vol. 3(S). July, 11, 2012 (Proc. of the 4th. Symb. of Fractional Calculus and Applications)
No. 2, pp. 1– 36.
ISSN: 2090-5858.
http://www.fcaj.webs.com/

DYNAMIC INEQUALITIES ON TIME SCALES: A SURVEY

SAMIR H. SAKER

Abstract. In this paper, we present the recent results of dynamic inequalities on time scales. In particular, we will make a survey of the development of Gronnwall-Belamnn and Opial's type dynamic inequalities on time scales. Some applications of some results are included for illustrations.

1. Introduction

In 1919 Thomas Gronwall [19] proved that if β and u are real-valued continuous functions defined on J, where J is an interval in $\mathbb{R}$, $t_0 \in J$, and u is differentiable in the interior J^0 of J, then

$$u^{'}(t) \leq \beta(t)u(t), \text{ for } t \in J^0, \tag{1}$$

implies

$$u(t) \leq u(t_0) \exp\left(\int_{t_0}^{t} \beta(s)\right), \text{ for all } t \in J. \tag{2}$$

In 1943 Richard Bellman [8] considered the integral form of (1) and proved that if

$$u(t) \leq \alpha(t) + \int_{t_0}^{t} \beta(s)u(s)ds, \text{ for } t \in J, \tag{3}$$

then

$$u(t) \leq \alpha(t) + \int_{t_0}^{t} \alpha(s)\beta(s) \exp\left(\int_{s}^{t} \beta(\theta)d\theta\right) ds, \text{ for all } t \in J, \tag{4}$$

where J is an interval in $\mathbb{R}$, $t_0 \in J$, and α, β, $u \in C(J, \mathbb{R}^+)$. If in addition $\alpha(t)$ is nondecreasing, then (3) implies

$$u(t) \leq \alpha(t) \exp\left(\int_{t_0}^{t} \beta(s)ds\right), \text{ for all } t \in J. \tag{5}$$

Since the discovery of these inequalities much work has been done, and many papers which deal with new proofs, various generalizations and extensions have appeared in the literature, we refer to the results by Ou-Iang [39], Dafermos [17] and Pachpatte [40]. The inequalities of the form (4), which are called the Gronwall-Bellman type inequalities, are important tools to obtain various estimates in the theory of

2000 *Mathematics Subject Classification.* 26A15, 26D10, 26D15, 39A13, 34A40.
Key words and phrases. Hardy's inequality, Opial inequality, time scales.

differential equations. For example, Ou-Iang [39] in his study of the boundedness of certain second order differential equations established the following result which is generally known as Ou-Iang's inequality: If u and f are non-negative functions defined on $[0,\infty)$ such that

$$u^2(t) \leq k^2 + 2\int_0^t f(s)u(s)ds, \quad \text{for all} \quad t \in [0,\infty), \tag{6}$$

where $k \geq 0$ is a constant, then

$$u(t) \leq k + \int_0^t f(s)ds, \ \text{for all } t \in [0,\infty). \tag{7}$$

Dafermos [17] established a generalization of Ou-Iang's inequality in the process of investigating the connection between stability and the second law of thermodynamics. He proved that if $u \in L^\infty[0,r]$ and $f \in L^1[0,r]$ are non-negative functions satisfying

$$u^2(t) \leq M^2u^2(0) + 2\int_0^t [Nf(s)u(s) + Ku^2(s)]ds, \text{ for all } t \in [0,r], \tag{8}$$

where M, N, K are non-negative constants, then

$$u(t) \leq \left[Mu(0) + N\int_0^t f(s)ds\right]e^{Kt}.$$

Pachpatte [40] established the following further generalizations of the result of Dafermos [17] and proved that: If u, f, g are continuous non-negative functions on $[0,\infty)$ satisfying

$$u^2(t) \leq k^2 + 2\int_0^t [f(s)u(s) + g(s)u^2(s)]ds, \text{ for all } t \in [0,\infty), \tag{9}$$

where $k \geq 0$ is a constant, then

$$u(t) \leq \left(k + \int_0^t f(s)ds\right)\exp\left(\int_0^t g(s)ds\right), \text{ for all } t \in [0,\infty). \tag{10}$$

One of the main subjects of the qualitative analysis on time scales is to prove some new dynamic inequalities. These on the one hand generalize and on the other hand furnish a handy tool for the study of qualitative as well as quantitative properties of solutions of dynamic equations on time scales. The extension form of (1) on the time scale $\mathbb{T}$ has been studied in [10, Theorem 6.1]. In particular, it is proved that if u, a and $p \in C_{rd}$ and $p \in \mathcal{R}^+$, then

$$u^\Delta(t) \leq f(t) + p(t)u(t), \quad \text{for all } t \in [t_0,\infty)_{\mathbb{T}}, \tag{11}$$

implies

$$u(t) \leq u(t_0)e_p(t,t_0) + \int_{t_0}^t e_p(t,\sigma(s))f(s)\Delta s, \text{ for all } t \in [t_0,\infty)_{\mathbb{T}}, \tag{12}$$

where $\mathcal{R}^+ := \{a \in \mathcal{R} : 1 + \mu(t)a(t) > 0,\ t \in \mathbb{T}\}$ and $\mathcal{R}$ is the class of rd-continuous and regressive functions. The generalizations of (11) on time scales has been studied in [46] and some explicit upper bounds of the unknown function are obtained. Note that if we put $f(t) = 0$ in (11), then (11) and (12) can be considered as the time scale versions of (1) and (2). We mentioned here that the study of the general form of (11) on time scales is important in applications, especially in oscillation theory of dynamic equations on time scales. In particular, the application of the Riccati techniques on second and third order dynamic equations reduces these equations to a Riccati dynamic inequality of the form

$$w^{\Delta}(t) \leq f(t) + p(t)w(t) - q(t)w^{\lambda+1},$$

which is a generalization of (11). For contributions in this direction, we refer the reader to the book [50].

The Gronwall-Bellman dynamic inequality, which is the time scale version of (3) has been proved in [10, Theorem 6.4]. In particular it is proved that: If u, a and $p \in C_{rd}$ and $p \in \mathcal{R}^{+}$, then

$$u(t) \leq a(t) + \int_{t_0}^{t} p(s)u(s)\Delta s, \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}, \tag{13}$$

implies that

$$u(t) \leq a(t) + \int_{t_0}^{t} e_p(t, \sigma(s))a(s)p(s)\Delta s, \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}. \tag{14}$$

In 1960 Opial [37] proved that if y is an absolutely continuous function on $[a, b]$ with $y(a) = y(b) = 0$, then

$$\int_a^b |y(t)| \left|y^{'}(t)\right| dt \leq \frac{(b-a)}{4} \int_a^b \left|y^{'}(t)\right|^2 dt. \tag{15}$$

The inequalities of Opial types are the most important and fundamental integral inequalities in the analysis of qualitative properties of solutions of differential equations. In further simplifying the proof of the Opial inequality which had already been simplified by Olech [36], Beesack [9], Levinson [30], Mallows [34] and Pederson [38], it is proved that if y is real absolutely continuous on $(0, b)$ and with $y(0) = 0$, then

$$\int_0^b |y(t)| \left|y^{'}(t)\right| dt \leq \frac{b}{2} \int_0^b \left|y^{'}(t)\right|^2 dt. \tag{16}$$

Since the discovery of Opial's inequality much work has been done, and many papers which deal with new proofs, various generalizations, extensions and their discrete analogues have appeared in the literature. The discrete analogy of the (15) has been proved in [29] and the discrete analogy of (16) has been proved in [3, Theorem 5.2.2]. It worth to mentioned here that many results concerning differential inequalities carry over quite easily to corresponding results for difference inequalities, while other results seem to be completely different from their continuous counterparts. In [28] the authors proved the time scale version of Opial's inequality as follows: If $x : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $x(a) = 0$ (or $x(b) = 0$), then

$$\int_a^h |x(t) + x^{\sigma}(t)| \left|x^{\Delta}(t)\right| \Delta t \leq (b-a) \int_a^h \left|x^{\Delta}(t)\right|^2 \Delta t. \tag{17}$$

If r and q are positive rd-continuous functions on $[a, b]_{\mathbb{T}}$, $\int_a^b (\Delta t/r(t)) < \infty$, q nonincreasing and $x : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $x(a) = 0$ (or $x(b) = 0$) then

$$\int_a^b q^{\sigma}(t) \left|(x(t) + x^{\sigma}(t))\right| \left|x^{\Delta}(t)\right| \Delta t \leq \int_a^b \frac{\Delta t}{r(t)} \int_0^b r(t)q(t) \left|x^{\Delta}(t)\right|^2 \Delta t. \tag{18}$$

Our aim in this paper is to present the recent results of dynamic inequalities of Gronwall-Bellman and Opial types on time scales. The paper is organized as follows: In Section 2, we present the concepts related to the notion of time scales.

In Section 3, we present the results of Gronwall-Bellman's type inequalities with one independent variables and also with two independent variables. In Section 4, we state and prove some applications of Grownll-Belmann's type inequalities. In Section 5, we present the results of Opial's type inequalities. In Section 6, we state and prove some applications of Opial's type inequalities.

2. Concepts on Time Scales

In this section, for completeness, we recall the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The forward jump operator and the backward jump operator are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : \ s < t\},$$

where $\sup \emptyset = \inf \mathbb{T}$. A point $t \in \mathbb{T}$, is said to be left–dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, is right–dense if $\sigma(t) = t$, is left–scattered if $\rho(t) < t$ and right–scattered if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$.

The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. Fix $t \in \mathbb{T}$ and let $x : \mathbb{T} \to \mathbb{R}$. Define $x^{\Delta}(t)$ to be the number (if it exists) with the property that given any $\epsilon > 0$ there is a neighborhood U of t with

$$|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

In this case, we say $x^{\Delta}(t)$ is the (delta) derivative of x at t and that x is (delta) differentiable at t. We will frequently use the following results due to Hilger [22]. Throughout the paper will assume that $g : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}$.
(i) If g is differentiable at t, then g is continuous at t.
(ii) If g is continuous at t and t is right-scattered, then g is differentiable at t with $g^{\Delta}(t) = \frac{g(\sigma(t)) - g(t)}{\mu(t)}$.
(iii) If g is differentiable and t is right-dense, then

$$g^{\Delta}(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}.$$

(iv) If g is differentiable at t, then $g(\sigma(t)) = g(t) + \mu(t)g^{\Delta}(t)$.
Note that if $\mathbb{T} = \mathbb{R}$ then

$$\sigma(t) = t, \ \mu(t) = 0, \ f^{\Delta}(t) = f'(t), \ \int_a^b f(t)\Delta t = \int_a^b f(t)dt$$

if $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(t) = t + 1, \ \mu(t) = 1, \ f^{\Delta}(t) = \Delta f(t), \ \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t),$$

if $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\sigma(t) = t + h$, $\mu(t) = h$, and

$$y^{\Delta}(t) = \Delta_h y(t) := \frac{y(t+h) - y(t)}{h}, \quad \int_a^b f(t)\Delta t = \sum_{k=0}^{\frac{b-a-h}{h}} f(a+kh)h,$$

and if $\mathbb{T} = \{t : t = q^k,\ k \in \mathbb{N}_0,\ q > 1\}$, then $\sigma(t) = qt$, $\mu(t) = (q-1)t$,

$$x^{\Delta}(t) = \Delta_q x(t) = \frac{(x(q\,t) - x(t))}{(q-1)t}, \quad \int_{t_0}^{\infty} f(t)\Delta t = \sum_{k=n_0}^{\infty} f(q^k)\mu(q^k),$$

where $t_0 = q^{n_0}$, and if $\mathbb{T} = \mathbb{N}_0^2 := \{n^2 : n \in \mathbb{N}_0\}$, then $\sigma(t) = (\sqrt{t}+1)^2$,

$$\mu(t) = 1 + 2\sqrt{t},\ \Delta_N y(t) = \frac{y((\sqrt{t}+1)^2) - y(t)}{1 + 2\sqrt{t}}.$$

In this paper we will refer to the (delta) integral which we can define as follows. If $G^{\Delta}(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(s)\Delta s := G(t) - G(a)$. It can be shown (see [10]) that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^{\Delta}(t) = g(t)$, $t \in \mathbb{T}$. An infinite integral is defined as $\int_a^{\infty} f(t)\Delta t = \lim_{b\to\infty} \int_a^b f(t)\Delta t$. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$, here $g^{\sigma} = g \circ \sigma$) of two differentiable function f and g

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma}, \text{ and } \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}}. \tag{19}$$

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0$, $t \in \mathbb{T}$. The chain rule formula that we will use in this paper is

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 [hx^{\sigma} + (1-h)x]^{\gamma-1}\, dh x^{\Delta}(t), \tag{20}$$

which is a simple consequence of Keller's chain rule [10, Theorem 1.90]. Using the fact that $g^{\sigma}(t) = g(t) + \mu(t)g^{\Delta}(t)$, we obtain

$$(x^{\gamma}(t))^{\Delta} = \gamma \int_0^1 \left[x + h\mu(t)x^{\Delta}(t)\right]^{\gamma-1} dh x^{\Delta}(t). \tag{21}$$

The integration by parts formula is given by

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{22}$$

The Hölder inequality [10, Theorem 6.13] is given by

$$\int_a^b |u(t)v(t)|\,\Delta t \leq \left[\int_a^b |u(t)|^q\,\Delta t\right]^{\frac{1}{q}} \left[\int_a^b |v(t)|^p\,\Delta t\right]^{\frac{1}{p}}, \tag{23}$$

where $a,\ b \in \mathbb{T}$, $u,\ v \in C_{rd}(\mathbb{T},\ \mathbb{R})$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is regressive provided $1 + \mu(t)f(t) \neq 0$, $t \in \mathbb{T}$. The set of all regressive functions on a time scale $\mathbb{T}$ forms an Abelian group under the addition $\oplus$ defined by $p \oplus q := p + q + \mu pq$. The exponential function $e_p(t, s)$ on time scales is defined by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \text{ for } t \in \mathbb{T}, s \in \mathbb{T}^k,$$

where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

Alternatively, for $p \in \mathcal{R}$ one can define the exponential function $e_p(\cdot, t_0)$, to be the unique solution of the IVP $x^\Delta = p(t)x$, with $x(t_0) = 1$. If $p \in \mathcal{R}$, then $e_p(t, s)$ is real-valued and nonzero on $\mathbb{T}$. If $p \in \mathcal{R}^+$, then $e_p(t, t_0)$ is always positive, $e_p(t, t) = 1$ and $e_0(t, s) = 1$. Note that

$$\begin{cases} e_p(t,t_0) = \exp(\int_{t_0}^t p(s)ds), & \text{if } \mathbb{T} = \mathbb{R}, \\ e_p(t,t_0) =_{s=t_0}^{t-1} (1 + p(s)), & \text{if } \mathbb{T} = \mathbb{N}, \\ e_p(t,t_0) =_{s=t_0}^{t-1} (1 + (q-1)sp(s)), & \text{if } \mathbb{T} = q^{\mathbb{N}_0}. \end{cases}$$

3. Gronwall-Bellman Dynamic Inequalities

Since the Gronwall-Bellman's type inequalities provide explicit bounds to the unknown function $u(t)$ and a tool to the study of many qualitative as well as quantitative properties of solutions of dynamic equations, it has become one of the very few classic and most influential results in the theory and applications of dynamic inequalities. Because of its fundamental importance, over the years, many generalizations and analogous results of (14) have been established. In the following, we present some results for the general nonlinear dynamic inequality

$$u^\gamma(t) \leq a(t) + b(t)\int_{t_0}^t \left[f(s)u^\delta(s) + g(s)u^\alpha(s)\right]^\lambda \Delta s, \text{ for } t \in [t_0, \infty)_\mathbb{T}, \tag{24}$$

and the delay dynamic inequality

$$u^\gamma(t) \leq a(t) + b(t)\int_{t_0}^t \left[f(s)u^\delta(\tau(s)) + g(s)u^\alpha(\eta(s))\right]^\lambda \Delta s, \text{ for } t \in [t_0, \infty)_\mathbb{T}. \tag{25}$$

For (24) and (25), we assume the following hypotheses:

(H_1) $\begin{cases} u,\ a,\ b,\ f \text{ and } g \text{ are rd-continuous positive functions defined on } [t_0,\infty)_\mathbb{T}, \\ \alpha,\ \delta,\ \lambda \text{ and } \gamma \text{ are positive constants such that } \gamma \geq 1. \end{cases}$

(H_2) $a(t)$, $b(t)$ are nondecreasing functions, $\tau,\ \eta : \mathbb{T} \to \mathbb{T}$ such that $\tau(t) \leq t$, $\eta(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \eta(t) = \infty$.

The results has been proved in [47] by employed the Bernoulli inequality [35]

$$(1 + x)^\gamma \leq 1 + \gamma x, \quad \text{for } \ 0 < \gamma \leq 1 \text{ and } x > -1. \tag{26}$$

the Young inequality [35]

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \text{ where } a,\ b \geq 0,\ p > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \tag{27}$$

and the algebraic inequalities [35]

$$(a + b)^\lambda \leq 2^{\lambda-1}(a^\lambda + b^\lambda), \text{ for } a,\ b \geq 0, \text{ and } \lambda \geq 1, \tag{28}$$

$$(a + b)^\lambda \leq a^\lambda + b^\lambda, \text{ for } \ a,\ b \geq 0, \text{ and } 0 \leq \lambda \leq 1. \tag{29}$$

Before, we state the main results we present some basic Lemmas which played important roles in the proof of the main results.

Lemma 3.1 [12]. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Suppose that y, a, b, $p \in C_{rd}$ and b, $p \geq 0$. If*

$$y(t) \leq a(t) + b(t) \int_{t_0}^{t} p(s)y(s)\Delta s, \quad \text{for all } t \in [t_0, \infty)_{\mathbb{T}}, \tag{30}$$

then

$$y(t) < a(t) + b(t) \int_{t_0}^{t} a(s)p(s)e_{bp}(t, \sigma(s))\Delta s, \quad \text{for all } t \in [t_0, \infty)_{\mathbb{T}}. \tag{31}$$

Lemma 3.2. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Let $g_i : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ for $i = 1, 2, ..., n$ be functions with $g_i(t, x_1) \leq g_i(t, x_2)$ for all $t \in \mathbb{T}$ and $i = 1, 2, ..., n$, whenever $x_1 \leq x_2$. Let v, $w : \mathbb{T} \to \mathbb{R}$ be differentiable with*

$$v^{\Delta}(t) \leq \sum_{i=1}^{n} g_i(t, v(t)), \quad w^{\Delta}(t) \geq \sum_{i=1}^{n} g_i(t, w(t)), \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}. \tag{32}$$

Then $v(t_0) < w(t_0)$ implies $v(t) \leq w(t)$ for all $t \in [t_0, \infty)_{\mathbb{T}}$.

Lemma 3.3. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Suppose that $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing for $i = 1, 2, ..., n$ and $y : \mathbb{T} \to \mathbb{R}$ is such that $g_i(y)$ is rd-continuous. Let p_i be rd-continuous for $i = 1, 2, ..., n$ and $f : \mathbb{T} \to \mathbb{R}$ differentiable. Then*

$$y(t) \leq f(t) + \sum_{i=1}^{n} \int_{t_0}^{t} p_i(s)g_i(y(s))\Delta s, \quad \text{for all } t \geq t_0, \tag{33}$$

implies $y(t) \leq x(t)$ for all $t \geq t_0$, where x solves the initial value problem

$$x^{\Delta}(t) = f^{\Delta}(t) + \sum_{i=1}^{n} p_i(t)g_i(x(t)), \quad x(t_0) = x_0 > f(t_0) > 0. \tag{34}$$

Now, we consider the inequality (24) and present some explicit bounds of the unknown function $u(t)$ when $\lambda \geq 1$ and α, $\delta \leq \gamma$. For simplicity, we introduce the following notations:

$$\begin{aligned} F(t) \quad &: \quad = 2^{2(\lambda-1)} \int_{t_0}^{t} \left[f^{\lambda}(s) \left[a^{\frac{\delta}{\gamma}}(s) \right]^{\lambda} + g^{\lambda}(s) \left[a^{\frac{\alpha}{\gamma}}(s) \right]^{\lambda} \right] \Delta s, \\ F^{\Delta}(t) \quad &: \quad = 2^{2(\lambda-1)} \left[f^{\lambda}(t) \left[a^{\frac{\delta}{\gamma}}(t) \right]^{\lambda} + g^{\lambda}(t) \left[a^{\frac{\alpha}{\gamma}}(t) \right]^{\lambda} \right], \\ G(t) \quad &: \quad = 2^{2(\lambda-1)} \left(f^{\lambda}(s) \left[\frac{\delta}{\gamma} a^{\frac{\delta}{\gamma}-1}(s) \right]^{\lambda} + g^{\lambda}(s) \left[\frac{\alpha}{\gamma} a^{\frac{\alpha}{\gamma}-1}(s) \right]^{\lambda} \right). \end{aligned} \tag{35}$$

Theorem 3.1. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that (H_1) holds, $\lambda \geq 1$ and α, $\delta \leq \gamma$. Then*

$$u^{\gamma}(t) \leq a(t) + b(t) \int_{t_0}^{t} \left[f(s)u^{\delta}(s) + g(s)u^{\alpha}(s) \right]^{\lambda} \Delta s, \text{ for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{36}$$

implies that

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t)b(t)w(t), \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}, \tag{37}$$

where $w(t)$ solves the initial value problem

$$w^{\Delta}(t) = F^{\Delta}(t) + b^{\lambda}(t)G(t)w^{\lambda}(t), \quad w(t_0) = w_0 > 0. \tag{38}$$

Theorem 3.2. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that (H_1) holds,* $\lambda \geq 1$ and $\alpha, \delta \leq \gamma$. *Then (36) implies*

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + b^{\frac{1}{\gamma}}(t)w^{\frac{1}{\gamma}}(t), \text{ for all } t \in [t_0, \infty)_{\mathbb{T}}, \tag{39}$$

where $w(t)$ solves the initial value problem

$$\begin{cases} \quad w^{\Delta}(t) = F^{\Delta}(t) + G_1(t)w^{\lambda(\frac{\delta}{\gamma})}(t) + G_2(t)w^{\lambda(\frac{\alpha}{\gamma})}(t), \\ w(t_0) = w_0 > 0, \end{cases} \tag{40}$$

where $F(t)$ is defined as in (35) and

$$G_1(t) := 2^{2(\lambda-1)} \int_{t_0}^{t} f^{\lambda}(t) \left[b^{\frac{\delta}{\gamma}}(t)\right]^{\lambda}, \quad G_2 := 2^{2(\lambda-1)} g^{\lambda}(t) \left[b^{\frac{\alpha}{\gamma}}(t)\right]^{\lambda}. \tag{41}$$

As in Theorem 3.1 by employed the inequality (29) instead of the inequality (28), we obtained an explicit bound for $u(t)$ when $0 \leq \lambda \leq 1$. This will be presented below in Theorem 3.3. For simplicity, we introduce the following notations:

$$\begin{aligned} F_1(t) \quad &: \quad = \int_{t_0}^{t} \left[f^{\lambda}(s) \left[a^{\frac{\delta}{\gamma}}(s)\right]^{\lambda} + g^{\lambda}(s) \left[a^{\frac{\alpha}{\gamma}}(s)\right]^{\lambda} \right] \Delta s, \\ F_1^{\Delta}(t) \quad &: \quad = f^{\lambda}(t) \left[a^{\frac{\delta}{\gamma}}(t)\right]^{\lambda} + g^{\lambda}(t) \left[a^{\frac{\alpha}{\gamma}}(t)\right]^{\lambda}, \\ G_3(t) \quad &: \quad = \left(f^{\lambda}(t) \left[\frac{\delta}{\gamma} a^{\frac{\delta}{\gamma}-1}(t)\right]^{\lambda} + g^{\lambda}(t) \left[\frac{\alpha}{\gamma} a^{\frac{\alpha}{\gamma}-1}(t)\right]^{\lambda} \right). \end{aligned} \tag{42}$$

Theorem 3.3. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that (H_1) holds, $0 < \lambda \leq 1$, $\delta \leq \gamma$ and $\alpha \leq \gamma$. Then (36) implies that*

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t)b(t)s(t), \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{43}$$

where $s(t)$ solves the initial value problem

$$s^{\Delta}(t) = F_1^{\Delta}(t) + G_3(t)b^{\lambda}(t)s^{\lambda}(t), \quad s(t_0) = s_0 > 0. \tag{44}$$

The following theorems has been obtained by employed the Young inequality (27) to find a new explicit upper bound for $u(t)$ of (36) when $\lambda \geq 1$ and $0 \leq \lambda \leq 1$. First, we consider the case when $\lambda \geq 1$ and assume that $\lambda(\alpha/\gamma) < 1$ and $\lambda(\delta/\gamma) < 1$.

Theorem 3.4. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that (H_1) holds, $\lambda \geq 1$ and α, $\delta \leq \gamma$ such that $(\lambda\alpha/\gamma) < 1$ and $(\lambda\delta/\gamma) < 1$. Then (36) implies that*

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + b^{\frac{1}{\gamma}}(t)F_3^{\frac{1}{\gamma}}(t), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{45}$$

where

$$F_3(t) := F_0(t) + \beta \int_{t_0}^{t} F_0(s)e_{\beta}(t, \sigma(s))\Delta s, \ \beta = \lambda[\frac{\alpha}{\gamma} + \frac{\delta}{\gamma}],$$

$$\begin{aligned} F_0(t) \quad : \quad &= F(t) + \frac{(\gamma - \lambda\delta)}{\gamma} \int_{t_0}^{t} (G_1(s))^{\gamma/(\gamma-\lambda\delta)}\, \Delta s \\ &+ \frac{(\gamma - \lambda\alpha)}{\gamma} \int_{t_0}^{t} (G_2(s))^{\gamma/(\gamma-\lambda\alpha)}\, \Delta s, \end{aligned}$$

where F, G_1 and G_2 are defined as in (42) and (41).

Theorem 3.5. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that (H_1) holds, $0 < \lambda \leq 1$ and α, $\delta \leq \gamma$. Then (36) implies that*

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t) b(t) F_4(t), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{46}$$

where

$$\begin{aligned} F_4(t) \quad : \quad &= F_2(t) + \lambda \int_{t_0}^{t} F_2(s) e_{\lambda}(t, \sigma(s)) \Delta s, \\ F_2(t) \quad : \quad &= F_1(t) + (1-\lambda) \int_{t_0}^{t} (G_3(s))^{\frac{1}{1-\lambda}}\, \Delta s, \end{aligned}$$

where F_1 and G_3 are defined as in (42).

Next, in the following, we consider the delay dynamic inequality (25) and present some explicit bounds of the unknown function $u(t)$. First, we consider the case when $\lambda = 1$ and α, $\delta \leq \gamma$. For this case, we introduce the following notations:

$$\begin{aligned} A(t) \quad : \quad &= F^*(t) + \int_{t_0}^{t} F^*(s) G^*(s) e_G(t, \sigma(s)) \Delta s, \\ F^*(t) \quad : \quad &= \int_{t_0}^{t} [f(s) a^{\frac{\delta}{\gamma}}(s) + g(s) a^{\frac{\alpha}{\gamma}}(s)] \Delta s, \\ G^*(t) \quad : \quad &= b(t) \left[\frac{\delta}{\gamma} a^{\frac{\delta}{\gamma}-1}(t) f(t) + \frac{\alpha}{\gamma} a^{\frac{\alpha}{\gamma}-1}(t) g(t) \right]. \end{aligned}$$

Theorem 3.6. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that $(H_1) - (H_2)$ hold, $\lambda = 1$ and α, $\delta \leq \gamma$. Then (25) implies that*

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t) b(t) A(t), \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{47}$$

In the following, we consider (25) and present an upper bound for the function $u(t)$ in the case when $\lambda = 1$ and $\alpha = \delta \geq \gamma$. For simplicity, we introduce the following notations:

$$\begin{aligned} v(t) \quad : \quad &= 2^{\frac{\alpha}{\gamma}-1} \int_{t_0}^{t} a^{\frac{\alpha}{\gamma}}(s) \left[f(s) + g(s) \right] \Delta s, \\ R(t) \quad : \quad &= 2^{\frac{\alpha}{\gamma}-1} \int_{t_0}^{t} b^{\frac{\alpha}{\gamma}}(s) \left[g(s) + f(s) \right] \Delta s. \end{aligned}$$

Theorem 3.7. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that $(H_1) - (H_2)$ hold, $\lambda = 1$ and $\alpha = \delta \geq \gamma$. Then (25) implies that*

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t) b(t) V(t), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{48}$$

where $V(t)$ solves the initial value problem

$$V^{\Delta}(t) = v^{\Delta}(t) + R(t) V^{\frac{\alpha}{\gamma}}(t), \;\; V(t_0) = V_0 > 0. \tag{49}$$

Note that the results in Theorems 3.6, 3.7 can be extended to the cases when $\lambda \geq 1$ and $0 < \lambda \leq 1$. Also Theorem 3.7 can be extended as in Theorem 3.3 when $\alpha \neq \delta$. The details are left to the interested reader.

In the following, we present some new explicit upper bound for $u(t)$ of (25) when $\alpha,\, \delta \leq \gamma$. The results has been proved by employed the Young inequality (27). For simplicity, we introduce the following notations:

$$V_3(t) \;:\; = \int_{t_0}^{t} \left[f(s)(a^{\frac{\alpha}{\gamma}}(s) + \frac{b^{\frac{\alpha}{\gamma-\alpha}}(s)}{\frac{\gamma}{\gamma-\alpha}}) + g(s)(a^{\frac{\delta}{\gamma}}(s) + \frac{b^{\frac{\delta}{\gamma-\delta}}(s)}{\frac{\gamma}{\gamma-\delta}}) \right] \Delta s,$$

$$B_1(t) \;:\; = \int_{t_0}^{t} \left[\frac{\alpha}{\gamma} f(s) + \frac{\delta}{\gamma} g(s) \right] \Delta s.$$

Theorem 3.8. *Let $\mathbb{T}$ be an unbounded time scale with t_0 and $t \in \mathbb{T}$. Assume that $(H_1) - (H_2)$ hold, $\lambda = 1$ and $\alpha,\, \delta \leq \gamma$. Then (25) implies that*

$$u(t) \leq a^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} a^{\frac{1}{\gamma}-1}(t) b(t) V_1(t), \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{50}$$

where $V_1(t)$

$$V_1(t) = V_3(t) + \int_{t_0}^{t} V_3(s) B_1(s) e_{B_1}(t, \sigma(s)) \Delta s. \tag{51}$$

In [48] the author considered a dynamic inequality of the form

$$u^p(t) \leq a(t) + b(t) \int_{t_0}^{t} \left[f(s) u^q(s) - g(s) u^p(\sigma(s)) \right] \Delta s, \ t \in [t_0, \infty)_{\mathbb{T}}, \tag{52}$$

and proved that if a, f and g are positive rd-continuous functions defined on $[t_0,\infty)_{\mathbb{T}}$, $u(t) \geq 0$, for all $t \geq t_0$, where $t_0 \geq 0$ is a fixed number, p, q are positive constants such that $p > q \geq 1$, then (52) implies for $t \in [t_0, \infty)_{\mathbb{T}}$ that

$$u(t) \leq a^{\frac{1}{p}}(t) + \frac{q}{p} a^{\frac{1}{p}-1}(t) b(t) \left[\int_{t_0}^{t} e_{\left(a^{\frac{q}{p}} f\right)}(t, \sigma(s)) f(s) a^{\frac{q}{p}-1}(s) \Delta s \right]. \tag{53}$$

We note that the inequality (53) has been proved in the case when $p > q \geq 1$. Also in [48] the author considered the dynamic inequality

$$u^{\gamma}(t) \leq a(t) + b(t) \int_{t_0}^{t} \left[f(s) u^{\delta}(s) + g(s) u^{\alpha}(s) \right] \Delta s, \ \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$

when $\delta \leq \gamma$ and $\alpha \leq \gamma$, and established some explicit bounds for the function $u(t)$. The main results in [48] has been proved by employing the Bernoulli inequality.

In the following, we consider the dynamic inequalities in two independent variables. First, we present some basic definitions about calculus in two variables on time scales (for more details, we refer to [26]).

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales with at least two points and consider the time scale intervals $\Omega_1 = [t_0, \infty) \cap \mathbb{T}_1$ and $\Omega_2 = [s_0, \infty) \cap \mathbb{T}_2$ for $t_0 \in \mathbb{T}_1$ and $s_0 \in \mathbb{T}_2$. Let σ_1, ρ_1, Δ_1 and σ_2, ρ_2, Δ_2 denote the forward jump operators, backward jump operators and the delta differentiation operator respectively on $\mathbb{T}_1$ and $\mathbb{T}_2$. We say that a real valued function f on $\mathbb{T}_1 \times \mathbb{T}_2$ at $(t, s) \in \Omega \equiv \Omega_1 \times \Omega_2$ has a Δ_1 partial derivative $f^{\Delta_1}(t,s)$ with respect to t if for each $\epsilon > 0$ there exists a neighborhood U_t of t such that

$$|f(\sigma_1(t),s) - f(\eta,s) - f^{\Delta_1}(t,s)[\sigma_1(t)-\eta]| \leq \epsilon|\sigma(t)-\eta|, \quad \text{for all } \eta \in U_t.$$

In this case, we say $f^{\Delta_1}(t,s)$ is the (partial delta) derivative of $f(t,s)$ at t. We say that a real valued function f on $\mathbb{T}_1 \times \mathbb{T}_2$ at $(t,s) \in \Omega_1 \times \Omega_2$ has a Δ_2 partial derivative $f^{\Delta_2}(t,s)$ with respect to s if for each $\epsilon > 0$ there exists a neighborhood U_s of s such that

$$|f(t,\sigma_2(s)) - f(t,\xi) - f^{\Delta_2}(t,s)[\sigma_2(t)-\xi]| \leq \epsilon|\sigma(t)-\xi|, \quad \text{for all } \xi \in U_s.$$

In this case, we say $f^{\Delta_2}(t,s)$ is the (partial delta) derivative of $f(t,s)$ at s. The function f is called rd-continuous in t if for every $\alpha_2 \in \mathbb{T}_2$ the function $f(t,\alpha_2)$ is rd-continuous on $\mathbb{T}_1$. The function f is called rd-continuous in s if for every $\alpha_1 \in \mathbb{T}_1$ the function $f(\alpha_1,s)$ is rd-continuous on $\mathbb{T}_2$. Now, we are ready to present some results for dynamic inequalities in two independent variables on times scales. In [18] the authors proved that if a, f and u are positive rd-continuous functions and a is nonnegative and nondecreasing in each of its variables, then

$$u(x,y) \leq a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)u(s,t)\Delta t\Delta s, \tag{54}$$

for all $(x,y) \in [x_0,\infty)_{\mathbb{T}} \times [y_0,\infty)_{\mathbb{T}}$, implies

$$u(x,y) \leq a(x,y)e_F(x,x_0), \text{ where } F = \int_{y_0}^{y} f(x,t)\Delta t. \tag{55}$$

In [33] the author proved that if a, b, g, h and u are positive continuous real functions defined on $\mathbb{T} \times \mathbb{T}$ and $\gamma > 1$ is a real constant, then

$$u^{\gamma}(x,y) \leq a(x,y) + b(x,y)\int_{x_0}^{x}\int_{y_0}^{y} \left[g(s,t)u^{\gamma}(s,t) + h(s,t)u(s,t)\right]\Delta t\Delta s, \tag{56}$$

implies

$$u(x,y) \leq \left[a(x,y) + b(x,y)m(x,y)e_G(t,t_0)\right]^{1/\gamma}, \tag{57}$$

for all $(x,y) \in [x_0,\infty)_{\mathbb{T}} \times [y_0,\infty)_{\mathbb{T}}$, where

$$\begin{aligned} m(x,y) &= \int_{x_0}^{x}\int_{y_0}^{y} \left[a(s,t)g(s,t) + \left(\frac{\gamma-1}{\gamma} + \frac{a(s,t)}{\gamma}\right)h(s,t)\right]\Delta t\Delta s, \\ G(s,t) &= \int_{y_0}^{y} \left[g(x,t) + \frac{h(x,t)}{\gamma}\right] b(x,t)\Delta t. \end{aligned}$$

In the following, we are concerned with bounds of the double integral nonlinear dynamic inequality in two independent variables

$$u^{\gamma}(t,s) \leq a(t,s) + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s} \left[f(\tau,\eta)u^{\delta}(\tau,\eta) + g(\tau,\eta)u^{\alpha}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau, \tag{58}$$

for all $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [s_0,\infty)_{\mathbb{T}}$. The results are adapted from [45] and has been proved by employing the Bernoulli inequality, Young inequality, and the algebraic inequalities. We will assume that the equations or the inequalities possess such nontrivial solutions. the For (58), we will assume the following hypotheses:

$$(H)\ \begin{cases} u,\ a,\ b,\ f \text{ and } g \text{ are rd-continuous positive functions on } \Omega_1 \times \Omega_2, \\ \alpha,\ \delta,\ \lambda \text{ and } \gamma \text{ are positive constants.} \end{cases}$$

Lemma 3.4. *Let $\mathbb{T}$ be an unbounded time scale with (t_0, s_0) and $(t,\ s) \in \mathbb{T}\times\mathbb{T}$. Let $g_i : \mathbb{R} \to \mathbb{R}$* for *$i = 1, 2, ..., n$ be functions with $g_i(x_1(t, s)) \le g_i(x_2(t, s))$ for $i = 1, 2, ..., n$, where $x_i(t, s) : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$ for $i = 1, 2$, whenever $x_1 \le x_2$. Let v, $w : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$ be differentiable with*

$$v^{\Delta_t\Delta_s}(t,s) \le \sum_{i=1}^{n} a_i(t,s)g_i(v(t,s)), \quad w^{\Delta_t\Delta_s}(t,s) \ge \sum_{i=1}^{n} a_i(t,s)g_i(w(t,s)), \tag{59}$$

for all $(t, s) \in \mathbb{T}\times\mathbb{T}$. Then $v(t_0, s_0) < w(t_0, s_0)$ implies $v(t, s) \le w(t, s)$ for all $(t, s) \in [t_0, \infty)_{\mathbb{T}} \times [s_0, \infty)_{\mathbb{T}}$.

Lemma 3.5. *Let $\mathbb{T}$ be an unbounded time scale with (t_0, s_0) and $(t, s) \in \mathbb{T}\times\mathbb{T}$. Suppose that $g_i : \mathbb{R} \to \mathbb{R}$ is nondecreasing for $i = 1, 2, ..., n$ and $y : \mathbb{T}\times\mathbb{T} \to \mathbb{R}^+$ is such that $g_i(y)$ is rd-continuous. Let p_i be rd-continuous and positive for $i = 1, 2, ..., n$ and $f : \mathbb{T}\times\mathbb{T} \to \mathbb{R}^+$ differentiable. Then*

$$y(t,s) \le f(t,s) + \sum_{i=1}^{n}\int_{t_0}^{t}\int_{s_0}^{s} p_i(\eta,\tau)g_i(y(\eta,s))\Delta\eta\Delta\tau, \text{ for all } (t,s)\in\mathbb{T}\times\mathbb{T},$$

implies $y(t, s) \le x(t, s)$ for all *$(t, s) \in [t_0, \infty)_{\mathbb{T}} \times [s_0, \infty)_{\mathbb{T}}$, where $x(t, s)$ solves the initial value problem*

$$\left.\begin{array}{l} x^{\Delta_t\Delta_s}(t,s) = f^{\Delta_t\Delta s}(t,s) + \sum_{i=1}^{n} p_i(t,s)g_i(x(t,s)), \\ x(t_0,s_0) > f(t_0,s_0) > 0. \end{array}\right\}$$

Now, we are ready to state and prove the main results in this paper. First, we consider the case when $\lambda \ge 1$ and α, $\delta \le \gamma$. For simplicity, we introduce the following notations:

$$\begin{aligned} F(t,s) \quad &: \ = 2^{2(\lambda-1)}\int_{t_0}^{t}\int_{s_0}^{s}\left[f^{\lambda}(\tau,\eta)\left[a^{\frac{\delta}{\gamma}}(\tau,\eta)\right]^{\lambda}\right]\Delta\eta\Delta\tau \\ &\quad + 2^{2(\lambda-1)}\int_{t_0}^{t}\int_{s_0}^{s}\left[g^{\lambda}(\tau,\eta)\left[a^{\frac{\alpha}{\gamma}}(\tau,\eta)\right]^{\lambda}\right]\Delta\eta\Delta\tau, \\ G(t,s) \quad &: \ = 2^{2(\lambda-1)}\left(f^{\lambda}(t,s)\left[\frac{\delta}{\gamma}a^{\frac{\delta}{\gamma}-1}(t,s)\right]^{\lambda} + g^{\lambda}(t,s)\left[\frac{\alpha}{\gamma}a^{\frac{\alpha}{\gamma}-1}(t,s)\right]^{\lambda}\right). \end{aligned} \tag{60}$$

Theorem 3.9. *Let $\mathbb{T}$ be an unbounded time scale with $(t_0, s_0) \in \mathbb{T}\times\mathbb{T}$. Assume that (H) holds, λ, $\gamma \ge 1$ and α, $\delta \le \gamma$. Then*

$$u^{\gamma}(t,s) \le a(t,s) + b(t,s)\int_{t_0}^{t}\int_{s_0}^{s}\left[f(\tau,\eta)u^{\delta}(\tau,\eta) + g(\tau,\eta)u^{\alpha}(\tau,\eta)\right]^{\lambda}\Delta\eta\Delta\tau, \tag{61}$$

for $(t, s) \in \Omega$, *implies that*

$$u(t,s) \le a^{\frac{1}{\gamma}}(t,s) + \frac{1}{\gamma}a^{\frac{1}{\gamma}-1}(t,s)b(t,s)w(t,s), \text{ for all } (t,s)\in\Omega, \tag{62}$$

where $w(t)$ solves the initial value problem

$$w^{\Delta_t\Delta s}(t,s) = F^{\Delta_t\Delta s}(t,s) + b^{\lambda}(t,s)G(t,s)w^{\lambda}(t,s), \quad w(t_0,s_0) > 0. \tag{63}$$

Theorem 3.10. *Let $\mathbb{T}$ be an unbounded time scale with $(t_0, s_0) \in \mathbb{T}\times\mathbb{T}$. Assume that (H) holds, λ, $\gamma \ge 1$* and *α, $\delta \le \gamma$. Then (61) implies*

$$u(t,s) \le a^{\frac{1}{\gamma}}(t,s) + b^{\frac{1}{\gamma}}(t,s)w^{\frac{1}{\gamma}}(t,s), \text{ for all } (t,s)\in\Omega, \tag{64}$$

where $w(t)$ solves the initial value problem

$$\begin{cases} w^{\Delta_t\Delta_s}(t,s) = F^{\Delta_t\Delta_s}(t,s) + G_1(t,s)w^{\lambda(\frac{\delta}{\gamma})}(t,s) + G_2(t,s)w^{\lambda(\frac{\alpha}{\gamma})}(t,s), \\ \quad w(t_0,s_0) > 0, \end{cases} \tag{65}$$

where $F(t)$ is defined as in (60) and

$$G_1(t,s) := 2^{2(\lambda-1)} f^{\lambda}(t,s) b^{\frac{\lambda\delta}{\gamma}}(t,s),\ G_2 := 2^{2(\lambda-1)} g^{\lambda}(t,s) b^{\frac{\lambda\alpha}{\gamma}}(t,s). \tag{66}$$

By employed the inequality (29) instead of the inequality (28), we obtained an explicit bound for $u(t)$ when $0 \leq \lambda \leq 1$. This will be presented below in Theorem 3.11. For simplicity, we introduce the following notations:

$$\begin{aligned} F_1(t,s) \quad : \quad &= \int_{t_0}^{t}\int_{s_0}^{s}\left[f^{\lambda}(\tau,\eta)a^{\lambda(\frac{\delta}{\gamma})}(\tau,\eta)\right]\Delta\eta\Delta\tau \\ &+ \int_{t_0}^{t}\int_{s_0}^{s}\left[g^{\lambda}(\tau,\eta)a^{\lambda(\frac{\alpha}{\gamma})}(\tau,\eta)\right]\Delta\eta\Delta\tau, \\ G_3(t,s) \quad : \quad &= \left(f^{\lambda}(t,s)\left[\frac{\delta}{\gamma}a^{\frac{\delta}{\gamma}-1}(t,s)\right]^{\lambda} + g^{\lambda}(t,s)\left[\frac{\alpha}{\gamma}a^{\frac{\alpha}{\gamma}-1}(t,s)\right]^{\lambda}\right). \end{aligned} \tag{67}$$

Theorem 3.11. *Let $\mathbb{T}$ be an unbounded time scale with $(t_0,s_0) \in \mathbb{T}\times\mathbb{T}$. Assume that (H) holds, $\gamma \geq 1$, $0 < \lambda \leq 1$, $\delta \leq \gamma$ and $\alpha \leq \gamma$. Then (61) implies that*

$$u(t,s) \leq a^{\frac{1}{\gamma}}(t,s) + \frac{1}{\gamma}a^{\frac{1}{\gamma}-1}(t,s)b(t,s)z(t,s), \text{ for } (t,s)\in\Omega. \tag{68}$$

where $z(t)$ solves the initial value problem

$$z^{\Delta_t\Delta_t}(t) = F_1^{\Delta_t\Delta_s}(t) + G_3(t,s)b^{\lambda}(t,s)z^{\lambda}(t,s),\ \ z(t_0,s_0) > 0. \tag{69}$$

We applied the Young inequality (27) and found a new explicit upper bound for $u(t)$ of (61) when $\lambda \geq 1$ and $0 \leq \lambda \leq 1$. First, we consider the case when $\lambda \geq 1$ and assume that $\lambda(\alpha/\gamma) < 1$ and $\lambda(\delta/\gamma) < 1$.

Theorem 3.12. *Let $\mathbb{T}$ be an unbounded time scale with $(t_0,s_0) \in \mathbb{T}\times\mathbb{T}$. Assume that (H) holds, γ, $\lambda \geq 1$ and α, $\delta \leq \gamma$ such that $(\lambda\alpha/\gamma) < 1$ and $(\lambda\delta/\gamma) < 1$. Then (61) implies that*

$$u(t,s) \leq a^{\frac{1}{\gamma}}(t,s) + b^{\frac{1}{\gamma}}(t,s)F_3^{\frac{1}{\gamma}}(t,s), \text{ for all } (t,s)\in\Omega, \tag{70}$$

where

$$F_3(t,s) := F_0(t,s) + e_{\beta(s-s_0)}(t,t_0), \quad \beta = \lambda[\frac{\alpha}{\gamma}+\frac{\delta}{\gamma}],$$

$$\begin{aligned} F_0(t,s) \quad : \quad &= F(t,s) + \frac{(\gamma-\lambda\delta)}{\gamma}\int_{t_0}^{t}(G_1(\tau,\eta))^{\gamma/(\gamma-\lambda\delta)}\Delta\eta\Delta\tau \\ &+ \frac{(\gamma-\lambda\alpha)}{\gamma}\int_{t_0}^{t}(G_2(\tau,\eta))^{\gamma/(\gamma-\lambda\alpha)}\Delta\eta\Delta\tau, \end{aligned}$$

and F, G_1 and G_2 are defined as in (60) and (41).

Theorem 3.13. *Let $\mathbb{T}$ be an unbounded time scale with $(t_0,s_0) \in \mathbb{T}\times\mathbb{T}$. Assume that (H) holds, $\gamma \geq 1$, $0 < \lambda \leq 1$ and α, $\delta \leq \gamma$. Then (61) implies that*

$$u(t,s) \leq a^{\frac{1}{\gamma}}(t,s) + \frac{1}{\gamma}a^{\frac{1}{\gamma}-1}(t,s)b(t,s)F_4(t,s), \text{ for } (t,s)\in\Omega, \tag{71}$$

where

$$\begin{aligned} F_4(t,s) &:= F_2(t,s) + e_{\lambda(s-s_0)}(t,t_0), \\ F_2(t,s) &:= F_1(t,s) + (1-\lambda)\int_{t_0}^{t}\int_{s_0}^{s}(G_3(\tau,\eta))^{\frac{1}{1-\lambda}}\,\Delta\eta\Delta\tau, \end{aligned}$$

and F_1 and G_3 are defined as in (42).

Next in the following, we consider the case when $\gamma \leq 1$ and present some new explicit bounds of the unknown function $u(t,s)$ of (61).

Theorem 3.14 *Let $\mathbb{T}$ be an unbounded time scale with $(t_0,s_0) \in \mathbb{T}\times\mathbb{T}$. Assume that (H) holds, $\lambda \leq 1$, $\gamma \leq 1$ and $\alpha\lambda$, $\delta\lambda \leq \gamma$. Then (61) implies that*

$$u^{\gamma}(t,s) \leq a(t,s) + b(t,s)\left[H(t,s) + e_{\beta_1(s-s_0)}(t,t_0)\right], \text{ for } (t,s)\in\Omega, \tag{72}$$

where $\beta_1 = \frac{\lambda\alpha}{\gamma} + \frac{\lambda\delta}{\gamma}$ and

$$\begin{aligned} H(t,s) = {} & 2^{\lambda\delta\left(\frac{1}{\gamma}-1\right)}\int_{t_0}^{t}\int_{t_0}^{s} f^{\lambda}(\tau,\eta)a^{\frac{\lambda\delta}{\gamma}}(\tau,\eta)\Delta\eta\Delta\tau \\ & + 2^{\lambda\alpha\left(\frac{1}{\gamma}-1\right)}\int_{t_0}^{t}\int_{t_0}^{s} g^{\lambda}(\tau,\eta)a^{\frac{\lambda\alpha}{\gamma}}(\tau,\eta)\Delta\eta\Delta\tau \\ & + 2^{\lambda\delta\left(\frac{1}{\gamma}-1\right)}\frac{(\gamma-\lambda\delta)}{\gamma}\int_{t_0}^{t}\int_{t_0}^{s}\left(f^{\lambda}(\tau,\eta)b^{\frac{\lambda\delta}{\gamma}}(\tau,\eta)\right)^{\gamma/(\gamma-\lambda\delta)}\Delta\eta\Delta\tau \\ & + 2^{\lambda\alpha\left(\frac{1}{\gamma}-1\right)}\frac{(\gamma-\lambda\alpha)}{\gamma}\int_{t_0}^{t}\int_{t_0}^{s}\left(g^{\lambda}(\tau,\eta)b^{\frac{\lambda\alpha}{\gamma}}(\tau,\eta)\right)^{\gamma/(\gamma-\lambda\alpha)}\Delta\eta\Delta\tau. \end{aligned}$$

4. Applications of Grownll-Belmann inequalities

In this section, we give some applications of Grownll-Belmann inequalities on time scales. First, we apply the inequalities on the second-order half-linear delay dynamic equation

$$(r(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} + p(t)x^{\gamma}(\tau(t)) = 0, \text{ for } t\in[t_0,\infty)_{\mathbb{T}}, \tag{73}$$

on an arbitrary time scale $\mathbb{T}$, to establish an explicit upper bound of the nonoscillatory solutions, where $\gamma \geq 1$ is a quotient of odd positive integers, p is a positive $rd-$continuous function on $\mathbb{T}$, $r(t)$ is a positive and (delta) differentiable function and the so-called delay function $\tau : \mathbb{T} \to \mathbb{T}$ satisfies $\tau(t) \leq t$ for $t \in \mathbb{T}$ and $\lim_{t\to\infty}\tau(t) = \infty$. By a solution of (73) we mean a nontrivial real–valued function $x \in C_r^1[T_x,\infty)$, $T_x \geq t_0$ which has the property that $r(t)\left(x^{\Delta}(t)\right)^{\gamma} \in C_r^1[T_x,\infty)$ and satisfies equation (73) on $[T_x,\infty)$, where C_r is the space of $rd-$continuous functions. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration.

Lemma 4.1 [50]**.** *Assume that*

$$r^{\Delta}(t) \geq 0, \quad \text{and} \quad \int_{t_0}^{\infty}\tau^{\gamma}(t)p(t)\Delta t = \infty, \tag{74}$$

and

$$\int_{t_0}^{\infty}\frac{\Delta t}{r^{\frac{1}{\gamma}}(t)} = \infty. \tag{75}$$

Assume that (73) has a positive solution x on $[t_0,\infty)_{\mathbb{T}}$. Then there exists a $T\in[t_0,\infty)_{\mathbb{T}}$, sufficiently large, so that

(i) $x^{\Delta}(t)>0,\quad x^{\Delta\Delta}(t)<0,\quad x(t)>tx^{\Delta}(t),\quad$ *for* $t\in[T,\infty)_{\mathbb{T}}$;

(ii) $\frac{x(t)}{t}$ *is strictly decreasing on* $[T,\infty)_{\mathbb{T}}$.

The following theorem gives an upper bound of nonoscillatory solutions of (73).

Theorem 4.1. *Assume that (74) and (75) hold and $x(t)$ is a nonoscillatory solution of (73). Then $x(t)$ satisfies $x(t)\le x(t_1)e_K(t,t_1)$, where*

$$K(t)=\left[\frac{A}{\delta(t)r(t)}+\int_{t_1}^{t}\left[\frac{r(s)((\delta^{\Delta}(s))^{\gamma+1}}{\delta^{\gamma}(s)(\gamma+1)^{\gamma+1}}-\delta(s)p(s)\left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma}\right]\Delta s\right]^{\frac{1}{\gamma}},\tag{76}$$

and $\delta(t)$ is any positive $\Delta-$differentiable function and A is a positive constant and $t_1\in[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that there is a $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusions of Lemma 4.1 on $[t_1,\infty)_{\mathbb{T}}$ with $x(\tau(t))>0$ on $[t_1,\infty)_{\mathbb{T}}$. Let $\delta(t)$ be a positive Δ differentiable function and consider the Riccati substitution

$$w(t)=\delta(t)r(t)\left(\frac{x^{\Delta}(t)}{x(t)}\right)^{\gamma}.$$

Then by Lemma 4.1, we see that the function $w(t)$ is positive on $[t_1,\infty)_{\mathbb{T}}$. By the product rule and then the quotient rule (suppressing arguments)

$$\begin{aligned}w^{\Delta} &= \delta^{\Delta}\left(\frac{r(x^{\Delta})^{\gamma}}{x^{\gamma}}\right)^{\sigma}+\delta\left(\frac{r(x^{\Delta})^{\gamma}}{x^{\gamma}}\right)^{\Delta}\\ &= \frac{\delta^{\Delta}}{\delta^{\sigma}}w^{\sigma}+\delta\frac{x^{\gamma}(r(x^{\Delta})^{\gamma})^{\Delta}-r(x^{\Delta})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}x^{\gamma\sigma}}\\ &= \frac{\delta^{\Delta}}{\delta^{\sigma}}w^{\sigma}-p\delta\left(\frac{x^{\tau}}{x^{\sigma}}\right)^{\gamma}-\delta\frac{r(x^{\Delta})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}.\end{aligned}$$

Using the fact that $\frac{x(t)}{t}$ and $r(t)(x^{\Delta}(t))^{\gamma}$ are decreasing (from Lemma 3.1) we get

$$\frac{x^{\tau}(t)}{x^{\sigma}(t)}\ge\frac{\tau(t)}{\sigma(t)},\quad\text{and}\quad r(t)(x^{\Delta}(t))^{\gamma}\ge r^{\sigma}(t)(x^{\Delta}(t))^{\gamma\sigma}.$$

From these last two inequalities we obtain

$$w^{\Delta}\le\frac{\delta^{\Delta}}{\delta^{\sigma}}w^{\sigma}-\delta p\left(\frac{\tau}{\sigma}\right)^{\gamma}-\delta\frac{r^{\sigma}(x^{\Delta\sigma})^{\gamma}(x^{\gamma})^{\Delta}}{x^{\gamma}(x^{\sigma})^{\gamma}}.\tag{77}$$

By the chain rule and the fact that $x^{\Delta}(t)>0$, we obtain

$$\begin{aligned}(x^{\gamma}(t))^{\Delta} &= \gamma\int_0^1\left[x(t)+h\mu(t)x^{\Delta}(t)\right]^{\gamma-1}dh\,x^{\Delta}(t)\\ &\ge \gamma\int_0^1(x^{\sigma}(t))^{\gamma-1}dh\,x^{\Delta}(t)\\ &= \gamma(x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t).\end{aligned}\tag{78}$$

Using (77) and (78), we have that

$$w^{\Delta}\le\frac{\delta^{\Delta}}{\delta^{\sigma}}w^{\sigma}-\delta p\left(\frac{\tau}{\sigma}\right)^{\gamma}-\gamma\delta\frac{r^{\sigma}(x^{\Delta\sigma})^{\gamma}x^{\Delta}}{x^{\gamma}x^{\sigma}}.$$

Since

$$x^{\Delta}(t) \geq \frac{(r^{\sigma}(t))^{\frac{1}{\gamma}}(x^{\Delta}(t))^{\sigma}}{r^{\frac{1}{\gamma}}(t)}, \quad \text{and} \quad x^{\sigma}(t) \geq x(t),$$

we get that

$$w^{\Delta} \leq \frac{\delta^{\Delta}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma \frac{\delta r^{\sigma(1+\frac{1}{\gamma})}}{r^{\frac{1}{\gamma}}} \left(\frac{x^{\Delta\sigma}}{x^{\sigma}}\right)^{\gamma+1}.$$

Using the definition of w we finally obtain

$$w^{\Delta} \leq \frac{(\delta^{\Delta})_{+}}{\delta^{\sigma}} w^{\sigma} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma} - \gamma \frac{\delta}{(\delta^{\sigma})^{\lambda} r^{\frac{1}{\gamma}}} (w^{\sigma})^{\lambda}, \tag{79}$$

where $\lambda := \frac{\gamma+1}{\gamma}$. Define positive A and B by

$$A^{\lambda} := \frac{\gamma\delta}{(\delta^{\sigma})^{\lambda} r^{\frac{1}{\gamma}}} (w^{\sigma})^{\lambda}, \quad B^{\lambda-1} := \frac{r^{\frac{1}{\gamma+1}}}{\lambda(\gamma\delta)^{\frac{1}{\lambda}}} (\delta^{\Delta})_{+}.$$

Then, using the inequality $\lambda A B^{\lambda-1} - A^{\lambda} \leq (\lambda-1)B^{\lambda}$, we get that

$$\frac{(\delta^{\Delta})_{+}}{\delta^{\sigma}} w^{\sigma} - \gamma \frac{\delta}{(\delta^{\sigma})^{\lambda} r^{\frac{1}{\gamma}}} (w^{\sigma})^{\lambda} \leq \frac{r((\delta^{\Delta})_{+})^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}}.$$

From this last inequality and (79), we get

$$w^{\Delta} \leq \frac{r(\delta^{\Delta})^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma}.$$

Integrating both sides from t_1 to t we get

$$w(t) \leq w(t_1) + \int_{t_1}^{t} \left[\frac{r(\delta^{\Delta})^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma}\right] \Delta s,$$

which leads to

$$x^{\Delta}(t) \leq \left[\frac{w(t_1)}{\delta(t) r(t)} + \int_{t_1}^{t} \left[\frac{r(\delta^{\Delta})^{\gamma+1}}{\delta^{\gamma}(\gamma+1)^{\gamma+1}} - \delta p \left(\frac{\tau}{\sigma}\right)^{\gamma}\right] \Delta s\right]^{\frac{1}{\gamma}} x(t). \tag{80}$$

Applying the inequality (12), we get the desired inequality (76). The proof is complete.

When $\delta(t) = 1$, then $K(t)$ reduces to

$$K_1(t) = \left[\frac{A}{r(t)} - \int_{t_1}^{t} p(s) \left(\frac{\tau(s)}{\sigma(s)}\right)^{\gamma} \Delta s\right]^{\frac{1}{\gamma}}, \tag{81}$$

and then the upper bound of $x(t)$ of (73) is given by $x(t) \leq x(t_1) e_{K_1}(t, t_1)$.

Next, we consider the dynamic equation

$$(c(t)x^{\gamma}(t))^{\Delta} = a(t) + b(t)[f(t)x^{\frac{\delta}{\beta}}(t) + g(t)x^{\frac{\alpha}{\beta}}(t)]^{\beta}, \text{ for } t \in [t_0, \infty)_{\mathbb{T}}, \tag{82}$$

with $x(t_0) > 0$ and establish an upper bound for a positive solution $x(t)$. To prove the main results for equation (82), we introduce the following notations:

$$\begin{aligned} F_*(t) &:= \int_{t_0}^{t} \left[b(s) f^\beta(s) C^{\frac{\delta}{\gamma}}(s) + b(s) g^\beta(s) C^{\frac{\alpha}{\gamma}}(s) \right] \Delta s, \\ G_*(t) &:= \left(b(s) f^\beta(s) \frac{\delta}{\gamma} C^{\frac{\delta}{\gamma}-1}(s) + b(s) g^\beta(s) \frac{\alpha}{\gamma} C^{\frac{\alpha}{\gamma}-1}(s) \right), \qquad (83) \\ C(t) &= \frac{x(t_0)}{c(t)} + \frac{1}{c(t)} \int_{t_0}^{t} a(s)\Delta s, \quad B(t) = \frac{2^{\lambda-1}}{c(t)}. \end{aligned}$$

Theorem 4.2. *Assume that a, b, c, f and g are rd-continuous positive functions defined on $[t_0,\infty)_{\mathbb{T}}$, and γ, $\beta \geq 1$ and α, $\delta \leq \gamma$. Then*

$$x(t) \leq C^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} C^{\frac{1}{\gamma}-1}(t) B(t) W(t), \quad t \in [t_0, \infty)_{\mathbb{T}}, \qquad (84)$$

where $W_1(t)$ solves

$$W^\Delta(t) \leq F_*^\Delta(t) + B(t) G_*(t) W^\alpha(t), \quad W(t_0) = W_0 > 0. \qquad (85)$$

Proof. Since $\beta \geq 1$, we from (82) after application of (28), that

$$(c(t)x^\gamma(t))^\Delta \leq a(t) + 2^{\beta-1} b(t)[f^\beta(t) x^\delta(t) + g^\beta(t) x^\alpha(t)], \text{ for } t \in [t_0, \infty)_{\mathbb{T}}.$$

Integrating this inequality from t_0 to t, we have

$$x^\gamma(t) \leq C(t) + B(t) \int_{t_0}^{t} [b(s) f^\beta(s) x^\delta(s) + b(s) g^\beta(s) x^\alpha(s)] \Delta s,$$

Applying Theorem 3.1 with $\lambda = 1$, we get that

$$x(t) \leq C^{\frac{1}{\gamma}}(t) + \frac{1}{\gamma} C^{\frac{1}{\gamma}-1}(t) B(t) W(t), \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

where $W(t)$ solves the initial value problem (85). The proof is complete.

In the following, we present some applications of inequalities of two independent variables.

Example 4.1. Consider the partial dynamic equation on time scales

$$(u^\gamma(t,s))^{\Delta_t \Delta_s} = H(t, s, u(t,s) + h(t,s), \quad (t,s) \in \Omega \equiv [t_0, \infty)_{\mathbb{T}} \times [s_0, \infty)_{\mathbb{T}}, \qquad (86)$$

with initial boundary conditions

$$u(t, s_0) = a(t) > , \;\; u(t_0, s) = b(s) > , \; a(t_0) = b(s_0) = 0, \qquad (87)$$

where $\gamma \geq 1$ is a constant and H and h are rd-continuous functions on Ω, $a : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^+$ and $b : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^+$ are rd-continuous functions. Assume that

$$|H(t, s, u)| \leq f(t,s) \left| u(t,s) \right|^\delta + g(t,s) \left| u(t,s) \right|^\alpha, \qquad (88)$$

where $f(t,s)$ and $g(t,s)$ are nonnegative rd-continuous functions for $(t,s) \in \Omega$ and α, $\delta < \gamma$. If $u(t,s)$ is a solution of (86)-(87), then $u(t,s)$ satisfies

$$|u(t,s)| \leq a^{\frac{1}{\gamma}}(t,s) + A^{\frac{1}{\gamma}}(t,s), \text{ for all } (t,s) \in \Omega, \qquad (89)$$

where

$$a(t,s) = a^\gamma(t) + b^\gamma(s) + \int_{t_0}^{t} \int_{s_0}^{s} |h(\tau, \eta)| \Delta\eta \Delta\tau,$$

$$A(t,s) := H_0(t,s) + e_{\beta(s-s_0)}(t, t_0), \qquad \beta = [\frac{\alpha}{\gamma} + \frac{\delta}{\gamma}],$$

and

$$\begin{aligned} H_0(t,s) &= \int_{t_0}^{t}\int_{s_0}^{s}\left[f(\tau,\eta)a^{\frac{\delta}{\gamma}}(\tau,\eta)\right]\Delta\eta\Delta\tau \\ &+\int_{t_0}^{t}\int_{s_0}^{s}\left[g(\tau,\eta)a^{\frac{\alpha}{\gamma}}(\tau,\eta)\right]\Delta\eta\Delta\tau \\ &+\frac{(\gamma-\delta)}{\gamma}\int_{t_0}^{t}\int_{s_0}^{s}\left(f(\tau,s)\right)^{\gamma/(\gamma-\delta)}\Delta\eta\Delta\tau \\ &+\frac{(\gamma-\alpha)}{\gamma}\int_{t_0}^{t}\int_{s_0}^{s}\left(g(\tau,\eta)\right)^{\gamma/(\gamma-\alpha)}\Delta\eta\Delta\tau. \end{aligned}$$

In fact the solution of (86)-(87) satisfies

$$|u(t,s)|^{\gamma}=a^{\gamma}(t)+b^{\gamma}(t)+\int_{t_0}^{t}\int_{s_0}^{s}h(\tau,\eta)\Delta\eta\Delta\tau+\int_{t_0}^{t}\int_{s_0}^{s}H(\tau,\eta,u(\tau,\eta))\Delta\eta\Delta\tau,$$

for $(t,s)\in\Omega.$ Therefore

$$|u(t,s)|^{\gamma}\leq a(t,s)+\int_{t_0}^{t}\int_{s_0}^{s}|H(\tau,\eta,u(\tau,\eta))|\,\Delta\eta\Delta\tau,\ \text{ for } (t,s)\in\Omega. \tag{90}$$

It follows from (88) and (90) that

$$|u(t,s)|^{\gamma}\leq a(t,s)+\int_{t_0}^{t}\int_{s_0}^{s}f(\tau,s)\,|u(\tau,\eta)|^{\delta}+g(\tau,\eta)\,|u(\tau,\eta)|^{\alpha}\,\Delta\eta\Delta\tau, \tag{91}$$

for $(t,s)\in\Omega$. Applying Theorem 3.12 on (91) with $\lambda=1$ and $b(t,s)=1$, we obtain (89).

Example 4.2. Consider the equation

$$u^{\gamma}(t,s)=H(t,s,u(t,s)+h(t,s),\quad (t,s)\in\Omega\equiv[t_0,\infty)_{\mathbb{T}}\times[s_0,\infty)_{\mathbb{T}}, \tag{92}$$

where $\gamma\geq 1$ is a constant and H and h are rd-continuous on Ω, $a:[t_0,\infty)_{\mathbb{T}}\rightarrow\mathbb{R}^{+}$ and $b:[t_0,\infty)_{\mathbb{T}}\rightarrow\mathbb{R}^{+}$ are rd-continuous functions. Assume that

$$|H(t,s,u)|\leq f(t,s)\,|u(t,s)|^{\delta}+g(t,s)\,|u(t,s)|^{\alpha}\,, \tag{93}$$

where $f(t,s)$ and $g(t,s)$ are nonnegative rd-continuous functions for $(t,s)\in\Omega$, and α, $\delta<\gamma$. If $u(t,s)$ is a solution of (86)-(87), then $u(t,s)$ satisfies

$$|u(t,s)|\leq|h(t,s)|^{\frac{1}{\gamma}}+\frac{1}{\gamma}\,|h(t,s)|^{\frac{1}{\gamma}-1}\,B^{\frac{1}{\gamma}}(t,s),\ \text{for all } (t,s)\in\Omega, \tag{94}$$

where

$$B(t,s):=F^{*}(t,s)+(1-\lambda)\int_{t_0}^{t}\int_{s_0}^{t}\left(G^{*}(\tau,\eta)\right)^{\frac{1}{1-\lambda}}\Delta\eta\Delta\tau+e_{\lambda(s-s_0)}(t,t_0),$$

and

$$\begin{aligned} F^{*}(t,s) &:= \int_{t_0}^{t}\int_{s_0}^{s}\left[f^{\lambda}(\tau,\eta)a^{\lambda(\frac{\delta}{\gamma})}(\tau,\eta)+g^{\lambda}(\tau,\eta)a^{\lambda(\frac{\alpha}{\gamma})}(\tau,\eta)\right]\Delta\eta\Delta\tau, \\ G^{*}(t,s) &:= \left(f^{\lambda}(t,s)\left[\frac{\delta}{\gamma}a^{\frac{\delta}{\gamma}-1}(t,s)\right]^{\lambda}+g^{\lambda}(t,s)\left[\frac{\alpha}{\gamma}a^{\frac{\alpha}{\gamma}-1}(t,s)\right]^{\lambda}\right). \end{aligned}$$

In fact the solution of (92) satisfies

$$|u(t,s)|^{\gamma} \leq |h(t,s)| + \int_{t_0}^{t}\int_{s_0}^{s} |H(\tau,\eta,u(\tau,\eta))|\,\Delta\eta\Delta\tau, \text{ for } (t,s)\in\Omega. \tag{95}$$

It follows from (93) and (95) that

$$|u(t,s)|^{\gamma} \leq |h(t,s)| + \int_{t_0}^{t}\int_{s_0}^{s} \left[f(\tau,s)\,|u(\tau,\eta)|^{\delta} + g(\tau,\eta)\,|u(\tau,\eta)|^{\alpha}\right]^{\lambda} \Delta\eta\Delta\tau, \tag{96}$$

for $(t,s)\in\Omega$. Applying Theorem 3.13 on (96) with $b(t,s)=1$, we obtain (94).

Example 4.3. Assume that $\mathbb{T}=\mathbb{R}$ and consider the partial differential equation

$$\frac{\partial}{\partial s}((u^{\gamma-1}(t,s)\frac{\partial}{\partial t}u(t,s) + H(t,s,u(t,s)) = h(t,s), \ \ (t,s)\in\Omega^{*}, \tag{97}$$

where $\Omega^{*}=[0,\infty)\times[0,\infty)$, with initial boundary conditions

$$u(t,0)=a(t)>0, \ \ u(0,s)=b(s)>0, \ a(0)=b(0)=0. \tag{98}$$

Assume that $\gamma \geq 1$ is a constant and $H : [0,\infty)\times[0,\infty)\times\mathbb{R}\to\mathbb{R}$ and $h : [0,\infty)_{\mathbb{R}}\times[0,\infty)_{\mathbb{R}}\to\mathbb{R}$, $a:\mathbb{R}\to\mathbb{R}^{+}$ and $b:\mathbb{R}\to\mathbb{R}^{+}$ are continuous functions. Also, we assume that

$$|H(t,s,u)| \leq f(t,s)\,|u(t,s)|^{\delta} + g(t,s)\,|u(t,s)|^{\alpha}, \tag{99}$$

where $f(t,s)$ and $g(t,s)$ are nonnegative continuous functions for $(t,s)\in\Omega^{*}$ and α, $\delta<\gamma$. If $u(t,s)$ is a solution of (86)-(87), then $u(t,s)$ satisfies

$$|u(t,s)| \leq a^{\frac{1}{\gamma}}(t,s) + \gamma^{\frac{1}{\gamma}}B^{\frac{1}{\gamma}}(t,s), \text{ for all } (t,s)\in\Omega^{*}, \tag{100}$$

where

$$a(t,s) = a^{\gamma}(t) + b^{\gamma}(s) + \gamma\int_{t_0}^{t}\int_{s_0}^{s}|h(\tau,\eta)|\,d\eta d\tau,$$

$$B(t,s) := H_0(t,s) + e_{\beta(s-s_0)}(t,t_0), \quad \beta = [\frac{\alpha}{\gamma}+\frac{\delta}{\gamma}],$$

and

$$\begin{aligned} H_0(t,s) \ &= \ \int_{t_0}^{t}\int_{s_0}^{s}\left[f(\tau,\eta)a^{\frac{\delta}{\gamma}}(\tau,\eta)\right]\Delta\eta\Delta\tau + \int_{t_0}^{t}\int_{s_0}^{s}\left[g(\tau,\eta)a^{\frac{\alpha}{\gamma}}(\tau,\eta)\right]d\eta d\tau \\ &\quad + \frac{(\gamma-\delta)}{\gamma}\int_{t_0}^{t}\int_{s_0}^{s}(f(\tau,s))^{\gamma/(\gamma-\delta)}\,d\eta d\tau \\ &\quad + \frac{(\gamma-\alpha)}{\gamma}\int_{t_0}^{t}\int_{s_0}^{s}(g(\tau,\eta))^{\gamma/(\gamma-\alpha)}\,d\eta d\tau. \end{aligned}$$

In fact the solution formula of (97)-(98), after integration twice, is given by

$$\begin{aligned} &|u(t,s)|^{\gamma} - a^{\gamma}(t) - b^{\gamma}(t) + \gamma\int_{t_0}^{t}\int_{s_0}^{s}H(\tau,\eta,u(\tau,\eta))d\eta d\tau \\ = \ &\gamma\int_{t_0}^{t}\int_{s_0}^{s}h(\tau,\eta)d\eta d\tau, \ \text{ for } \ (t,s)\in\Omega^{*}. \end{aligned}$$

Therefore

$$|u(t,s)|^{\gamma} \leq a(t,s) + \gamma\int_{t_0}^{t}\int_{s_0}^{s}|H(\tau,\eta,u(\tau,\eta))|\,d\eta d\tau, \ \text{ for } (t,s)\in\Omega^{*}. \tag{101}$$

It follows from (99) and (101) that

$$|u(t,s)|^{\gamma} \leq a(t,s) + \gamma \int_{t_0}^{t} \int_{s_0}^{s} \left[f(\tau,s) |u(\tau,\eta)|^{\delta} + g(\tau,\eta) |u(\tau,\eta)|^{\alpha} \right] d\eta d\tau, \quad (102)$$

for $(t,s) \in \Omega^*$. Applying Theorem 3.12 on (102) with $\lambda = 1$ and $b(t,s) = \gamma$, we obtain (100).

5. Opial's Type Inequalities

In the following, we present the recent results of Opial type inequalities on time scales. The results are adapted from [41, 42, 43, 44].

Theorem 5.1. *Let $\mathbb{T}$ be a time scale with a, $X \in \mathbb{T}$. Assume that $s \in C_{rd}([a,X]_{\mathbb{T}}, \mathbb{R})$ and r be a positive rd-continuous function on $(a,X)_{\mathbb{T}}$ such that $\int_a^X r^{-1}(t)\Delta t < \infty$. If $y : [a,X]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(a) = 0$, then*

$$\int_a^X s(x) |y(x) + y^{\sigma}(x)| \left|y^{\Delta}(x)\right| \Delta x \leq K_1(a,X) \int_a^X r(x) \left|y^{\Delta}(x)\right|^2 \Delta x, \quad (103)$$

where

$$K_1(a,X) = \sqrt{2} \left(\int_a^X \frac{s^2(x)}{r(x)} \left(\int_a^x \frac{\Delta t}{r(t)} \right) \Delta x \right)^{\frac{1}{2}} + \sup_{a \leq x \leq X} \left(\mu(x) \frac{|s(x)|}{r(x)} \right). \quad (104)$$

In Theorem 5.1 , if $[a,X]$ is replaced by $[b,X]$ and putting $|y(x)| = \int_x^b \left|y^{\Delta}(t)\right| \Delta t$, then we have the following result.

Theorem 5.2. *Let $\mathbb{T}$ be a time scale with X, $b \in \mathbb{T}$. Assume that $s \in C_{rd}([a,X]_{\mathbb{T}}, \mathbb{R})$ and r be a positive rd-continuous function on $(a,X)_{\mathbb{T}}$ such that $\int_a^X r^{-1}(t)\Delta t < \infty$. If $y : [X,b]_{\mathbb{T}} \to \mathbb{R}$ is delta differentiable with $y(b) = 0$, then*

$$\int_X^b s(x) |y(x) + y^{\sigma}(x)| \left|y^{\Delta}(x)\right| \Delta x \leq K_2(X,b) \int_X^b r(x) \left|y^{\Delta}(x)\right|^2 \Delta x, \quad (105)$$

where

$$K_2(X,b) = \sqrt{2} \left(\int_X^b \frac{s^2(x)}{r(x)} \left(\int_x^b \frac{\Delta t}{r(t)} \right) \Delta x \right)^{\frac{1}{2}} + \sup_{X \leq x \leq b} \left(\mu(x) \frac{|s(x)|}{r(x)} \right). \quad (106)$$

Theorem 5.3. *Let $\mathbb{T}$ be a time scale with a, $X \in \mathbb{T}$ and p, q be positive real numbers such that $p \geq 1$, and let r, s be nonnegative rd-continuous functions on $(a,X)_{\mathbb{T}}$ such that $\int_a^X r^{\frac{-1}{p+q-1}}(t)\Delta t < \infty$. If $y : [a,X] \cap \mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(a) = 0$, then*

$$\int_a^X s(x) |y(x) + y^{\sigma}(x)|^p \left|y^{\Delta}(x)\right|^q \Delta x \leq K_1(a,X,p,q) \int_a^X r(x) \left|y^{\Delta}(x)\right|^{p+q} \Delta x, \quad (107)$$

where

$$\begin{aligned} K_1(a,X,p,q) &= 2^{2p-1}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \\ &\times \left(\int_a^X (s(x))^{\frac{p+q}{p}}(r(x))^{-\frac{q}{p}}\left(\int_a^x r^{\frac{-1}{p+q-1}}(t)\Delta t\right)^{(p+q-1)}\Delta x\right)^{\frac{p}{p+q}} \\ &+2^{p-1}\sup_{a\le x\le X}\left(\mu^p(x)\frac{s(x)}{r(x)}\right). \end{aligned}$$

Theorem 5.4. *Let $\mathbb{T}$ be a time scale with $X,\ b \in \mathbb{T}$ and $p,\ q$ be positive real numbers such that $p \geq 1$, and let r, s be nonnegative rd-continuous functions on $(X,b)_{\mathbb{T}}$ such that $\int_X^b r^{\frac{-1}{p+q-1}}(t)\Delta t < \infty$. If $y : [X,b]\cap\mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(b) = 0$, then we have*

$$\int_X^b s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x \le K_2(X,b,p,q)\int_X^b r(x)\left|y^\Delta(x)\right|^{p+q}\Delta x, \tag{108}$$

where

$$\begin{aligned} K_2(X,b,p,q) &= 2^{2p-1}\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \\ &\times \left(\int_X^b (s(x))^{\frac{p+q}{p}}(r(x))^{-\frac{q}{p}}\left(\int_x^b r^{\frac{-1}{p+q-1}}(t)\Delta t\right)^{(p+q-1)}\Delta x\right)^{\frac{p}{p+q}} \\ &+2^{p-1}\sup_{X\le x\le b}\left(\mu^p(x)\frac{s(x)}{r(x)}\right). \end{aligned} \tag{109}$$

Theorem 5.5. *Let $\mathbb{T}$ be a time scale with $a,\ X \in \mathbb{T}$ and $p,\ q$ be positive real numbers such that $p \le 1$, $p+q > 1$ and let $r,\ s$ be nonnegative rd-continuous functions on $(a,X)_{\mathbb{T}}$ such that $\int_a^X r^{\frac{-1}{p+q-1}}(t)\Delta t < \infty$. If $y : [a,X]\cap\mathbb{T} \to \mathbb{R}^+$ is delta differentiable with $y(a) = 0$, then*

$$\int_a^X s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x \le K_1(a,X,p,q)\int_a^X r(x)\left|y^\Delta(x)\right|^{p+q}\Delta x, \tag{110}$$

where

$$\begin{aligned} K_1(a,X,p,q) &= \sup_{a\le x\le X}\left(\mu^p(x)\frac{s(x)}{r(x)}\right)+2^p\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}} \\ &\times\left(\int_a^X \frac{(s(x))^{\frac{p+q}{p}}}{(r(x))^{\frac{q}{p}}}\left(\int_a^x r^{\frac{-1}{p+q-1}}(t)\Delta t\right)^{(p+q-1)}\Delta x\right)^{\frac{p}{p+q}} \end{aligned} \tag{111}$$

In Theorem 5.5 if we replaced $[a,X]$ by $[b,X]$ and putting $|y(x)| = \int_x^b \left|y^\Delta(t)\right|\Delta t$, we have the following result.

Theorem 5.6. *Let $\mathbb{T}$ be a time scale with $X,\ b \in \mathbb{T}$ and $p,\ q$ be positive real numbers such that $p \le 1$, $p+q > 1$ and let r, s be nonnegative rd-continuous* *functions on* $(X,b)_{\mathbb{T}}$ *such that* $\int_X^b r^{\frac{-1}{p+q-1}}(t)\Delta t<\infty$. *If* $y:[X,b]\cap\mathbb{T}\to\mathbb{R}^+$ *is delta differentiable with* $y(b)=0$, *then*

$$\int_X^b s(x)\left|y(x)+y^\sigma(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x\le K_2(X,b,p,q)\int_X^b r(x)\left|y^\Delta(x)\right|^{p+q}\Delta x, \tag{112}$$

where

$$\begin{aligned}K_2(X,b,p,q) &= \sup_{X\le x\le b}\left(\mu^p(x)\frac{s(x)}{r(x)}\right)+2^p\left(\frac{q}{p+q}\right)^{\frac{q}{p+q}}\\ &\times\left(\int_X^b\frac{(s(x))^{\frac{p+q}{p}}}{(r(x))^{\frac{q}{p}}}\left(\int_x^b(r(t))^{\frac{-1}{p+q-1}}\Delta t\right)^{(p+q-1)}\Delta x\right)^{\frac{p}{p+q}}\end{aligned} \tag{113}$$

Theorem 5.7. *Let* $\mathbb{T}$ *be a time scale with* $a,b\in\mathbb{T}$ *and* p,q *be positive real numbers such that* $p+q>1$, *and let* r, s *be nonnegative rd-continuous functions on* $(a,X)_{\mathbb{T}}$ *such that* $\int_a^X r^{\frac{-1}{p+q-1}}(t)\Delta t<\infty$. *If* $y:[a,X]\cap\mathbb{T}\to\mathbb{R}$ *is delta differentiable with* $y(a)=0$ *(and* y^Δ *does not change sign in* $(a,X)_{\mathbb{T}}$*), then*

$$\int_a^X s(x)\left|y(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x\le K_1(a,X,p,q)\int_a^X r(x)\left|y^\Delta(x)\right|^{p+q}\Delta x, \tag{114}$$

where

$$\begin{aligned}K_1(a,X,p,q) &= \left(\frac{q}{p+q}\right)^{\frac{q}{p+q}}\\ &\times\left(\int_a^X(s(x))^{\frac{p+q}{p}}(r(x))^{-\frac{q}{p}}\left(\int_a^x r^{\frac{-1}{p+q-1}}(t)\Delta t\right)^{(p+q-1)}\Delta x\right)^{\frac{p}{p+q}}.\end{aligned}$$

Theorem 5.8. *Let* $\mathbb{T}$ *be a time scale with* $a,b\in\mathbb{T}$ *and* p,q *be positive real numbers such that* $p+q>1$, *and let* r,s *be nonnegative rd-continuous functions on* $(b,X)_{\mathbb{T}}$ *such that* $\int_X^b r^{\frac{-1}{p+q-1}}(t)\Delta t<\infty$. *If* $y:[X,b]\cap\mathbb{T}\to\mathbb{R}$ *is delta differentiable with* $y(b)=0$, *(and* y^Δ *does not change sign in* $(X,b)_{\mathbb{T}}$*), then*

$$\int_X^b s(x)\left|y(x)\right|^p\left|y^\Delta(x)\right|^q\Delta x\le K_2(X,b,p,q)\int_X^b r(x)\left|y^\Delta(x)\right|^{p+q}\Delta x, \tag{115}$$

where

$$\begin{aligned}K_2(X,b,p,q) &= \left(\frac{q}{p+q}\right)^{\frac{q}{p+q}}\\ &\times\left(\int_X^b(s(x))^{\frac{p+q}{p}}(r(x))^{-\frac{q}{p}}\left(\int_x^b r^{\frac{-1}{p+q-1}}(t)\Delta t\right)^{(p+q-1)}\Delta x\right)^{\frac{p}{p+q}}.\end{aligned}$$

6. Applications of Opial's Type Inequalities

In this section, we present some applications of Opial's type inequalities on dynamic equations on time scales. First, we apply the inequalities on the dynamic equation

$$\left(r(t)y^\Delta(t)\right)^\Delta+q(t)y^\sigma(t)=0,\quad t\in[\alpha,\beta]_{\mathbb{T}}, \tag{116}$$

on an arbitrary time scale $\mathbb{T}$, where r, q are rd-continuous functions satisfying

$$\int_{\alpha}^{\beta} 1/r(t)\Delta t < \infty, \text{ and } \int_{\alpha}^{\beta} |q(t)|\, dt < \infty. \tag{117}$$

In particular, we will prove several results related to problems:

(i). obtain lower bounds for the spacing $\beta - \alpha$ where y is a solution of (116) satisfying $y(\alpha) = y^{\Delta}(\beta) = 0$, or $y^{\Delta}(\alpha) = y(\beta) = 0$,

(ii). obtain lower bounds for the spacing of generalized zeros of a solution of (116), and

(iii). obtain a lower bound for the smallest eigenvalue of the Sturm-Liouville eigenvalue problem

$$-y^{\Delta\Delta}(t) + q(t)y^{\sigma}(t) = \lambda y^{\sigma}(t), \;\; y(\alpha) = y(\beta) = 0.$$

By a solution of (116) on the interval $\mathbb{I}$, we mean a nontrivial real–valued function $y \in C_{rd}(\mathbb{I})$, which has the property that $r(t)y^{\Delta}(t) \in C^1_{rd}(\mathbb{I})$ and satisfies equation (116) on $\mathbb{I}$. We say that a solution y of (116) has a generalized zero at t if $y(t) = 0$, and has a generalized zero in $(t, \sigma(t))$ in case $y(t)y^{\sigma}(t) < 0$ and $\mu(t) > 0$. Equation (116) is disconjugate on the interval $[t_0, b]_{\mathbb{T}}$, if there is no nontrivial solution of (116) with two (or more) generalized zeros in $[t_0, b]_{\mathbb{T}}$. Equation (116) is said to be nonoscillatory on $[t_0, \infty)_{\mathbb{T}}$ if there exists $c \in [t_0, \infty)_{\mathbb{T}}$ such that this equation is disconjugate on $[c, d]_{\mathbb{T}}$ for every $d > c$. In the opposite case (116) is said to be oscillatory on $[t_0, \infty)_{\mathbb{T}}$. The oscillation of solutions of equation (116) may equivalently be defined as follows: A nontrivial solution $y(t)$ of (116) is called oscillatory if it has infinitely many (isolated) generalized zeros in $[t_0, \infty)_{\mathbb{T}}$; otherwise it is called nonoscillatory. So that the solution $y(t)$ of (116) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. This means that the property of oscillation or nonoscillation is the behavior in the neighborhood of the infinite points. We say that (116) is right disfocal (left disfocal) on $[\alpha, \beta]_{\mathbb{T}}$ if the solutions of (116) such that $y^{\Delta}(\alpha) = 0$ ($y^{\Delta}(\beta) = 0$) have no generalized zeros in $[\alpha, \beta]_{\mathbb{T}}$.

Theorem 6.1. *Suppose y is a nontrivial solution of (116) which satisfying $y(\alpha) = y^{\Delta}(\beta) = 0$, then*

$$\left[\sqrt{2}\left(\int_{\alpha}^{\beta} \frac{Q^2(t)}{r(t)}\left(\int_{\alpha}^{t} \frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}} + \sup_{\alpha \le t \le \beta}\left|\mu(t)\frac{Q(t)}{r(t)}\right|\right] \ge 1, \tag{118}$$

where $Q(t) = \int_t^{\beta} q(s)ds$. If $y^{\Delta}(\alpha) = y(\beta) = 0$, then

$$\left[\sqrt{2}\left(\int_{\alpha}^{\beta} \frac{Q^2(t)}{r(t)}\left(\int_{t}^{\beta} \frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}} + \sup_{\alpha \le t \le \beta}\left|\mu(t)\frac{Q(t)}{r(t)}\right|\right] \ge 1, \tag{119}$$

where $Q(t) = \int_{\alpha}^{t} q(s)ds$.

Proof. We prove (118). Multiplying (116) by y^{σ} and integrating by parts, we have

$$\begin{aligned}\int_{\alpha}^{\beta} y^{\sigma}(t)\left(r(t)y^{\Delta}(t)\right)^{\Delta}\Delta t &= y(t)r(t)y^{\Delta}(t)\big|_{\alpha}^{\beta} - \int_{\alpha}^{\beta} r(t)(y^{\Delta}(t))^2\Delta t \\ &= -\int_{\alpha}^{\beta} q(t)\left(y^{\sigma}(t)\right)^2\Delta t.\end{aligned}$$

Using the assumptions that $y(\alpha)=y^{\Delta}(\beta)=0$ and $Q(t)=\int_t^{\beta}q(s)\Delta s$, we get that

$$\int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^2\Delta t=\int_{\alpha}^{\beta} q(t)\left(y^{\sigma}(t)\right)^2\Delta t=-\int_{\alpha}^{\beta} Q^{\Delta}(t)\left(y^{\sigma}(t)\right)^2\Delta t.$$

Integrating by parts the right hand side and using the fact that $y(\alpha)=0=Q(\beta)$, we see that

$$\begin{aligned}\int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^2\Delta t &= \int_{\alpha}^{\beta} Q(t)\left(y(t)+y^{\sigma}(t)\right)y^{\Delta}(t)\Delta t\\ &\leq \int_{\alpha}^{\beta} |Q(t)|\,|y(t)+y^{\sigma}(t)|\,\left|y^{\Delta}(t)\right|\Delta t.\end{aligned}$$

This implies that

$$\begin{aligned}\int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^2\Delta t \leq& \left[\sqrt{2}\left(\int_{\alpha}^{\beta}\frac{Q^2(t)}{r(t)}\left(\int_{\alpha}^{t}\frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}}+\sup_{\alpha\leq t\leq\beta}\left|\mu(t)\frac{Q(t)}{r(t)}\right|\right]\\ &\times\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^2\Delta t.\end{aligned}$$

This implies that

$$\left[\sqrt{2}\left(\int_{\alpha}^{\beta}\frac{Q^2(t)}{r(t)}\left(\int_{\alpha}^{t}\frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}}+\sup_{\alpha\leq t\leq\beta}\left|\mu(t)\frac{Q(t)}{r(t)}\right|\right]\geq 1,$$

which is the desired inequality (118). The proof of (119) is similar to the proof of (118) by using the integration by parts and Theorem 5.2 instead of Theorem 5.1. The proof is complete.

As a special case of Theorem 6.1, when $r(t)=1$, we have the following results for the second order dynamic equation

$$y^{\Delta\Delta}(t)+q(t)y^{\sigma}(t)=0, \tag{120}$$

which are different from the results obtained by Karpuz, Kaymakçalan and Öcalan in [28].

Corollary 6.1. *Suppose y is a nontrivial solution of (120) which satisfying $y(\alpha)=y^{\Delta}(\beta)=0$, then*

$$\left[\sqrt{2}\left(\int_{\alpha}^{\beta}Q^2(t)\,(t-\alpha)\,\Delta t\right)^{\frac{1}{2}}+\sup_{\alpha\leq t\leq\beta}\left(\mu(t)\,|Q(t)|\right)\right]\geq 1,$$

where $Q(t)=\int_t^{\beta}q(s)ds$. If $y^{\Delta}(\alpha)=y(\beta)=0$, then

$$\left[\sqrt{2}\left(\int_{\alpha}^{\beta}Q^2(t)\,(\beta-t)\,\Delta t\right)^{\frac{1}{2}}+\sup_{\alpha\leq t\leq\beta}\left(\mu(t)\,|Q(t)|\right)\right]\geq 1,$$

where $Q(t)=\int_{\alpha}^{t}q(s)ds$.

Note that if $\mathbb{T}=\mathbb{R}$ then $\mu(t)=0$ and the equation (137) (when $r(t)=1$) becomes

$$y^{''}(t)+q(t)y(t)=0. \tag{121}$$

In this case the results in Corollary 3.1 reduce to the results that has been obtained by Brown and Hinton [16].

Corollary 6.2 ([16])**.** *If y is a solution of the equation (121) such that $y(\alpha) = y^{'}(\beta) = 0$, then*

$$2\int_{\alpha}^{\beta} Q^2(s)(s-\alpha)ds > 1, \tag{122}$$

where $Q(t) = \int_t^{\beta} q(s)ds$. If instead $y^{'}(\alpha) = y(\beta) = 0$, then

$$2\int_{\alpha}^{\beta} Q^2(s)(\beta - s)ds > 1, \tag{123}$$

where $Q(t) = \int_{\alpha}^{t} q(s)ds$.

Note that if $\mathbb{T} = \mathbb{N}$, then $\mu(t) = 1$ and the equation (137) (when $r(t) = 1$) becomes

$$\Delta^2 y(n) + q(n)y(n+1) = 0, \tag{124}$$

and the results in Corollary 6.1 reduce to the following results.

Corollary 6.3. *If y is a solution of the equation (124) such that $y(\alpha) = \Delta y(\beta) = 0$, then*

$$\sqrt{2}\left(\sum_{n=\alpha}^{\beta-1} (Q(n))^2 (n-\alpha)\right)^{\frac{1}{2}} + \sup_{\alpha\leq n\leq\beta} |Q(n)| > 1,$$

where $Q(n) = \sum_{s=n}^{\beta-1} q(s)$. If instead $\Delta y(\alpha) = y(\beta) = 0$, then

$$\sqrt{2}\left(\sum_{n=\alpha}^{\beta-1} (Q(n))^2 (\beta-n)\right)^{\frac{1}{2}} + \sup_{\alpha\leq n\leq\beta} |Q(n)| > 1,$$

where $Q(n) = \sum_{s=\alpha}^{n-1} q(s)$.

Theorem 6.2. *Suppose that y is a nontrivial solution of (116) which satisfying $y(\alpha) = y^{\Delta}(\beta) = 0$, then*

$$\sup_{\alpha\leq t\leq\beta}\left|\frac{Q(t)}{r(t)}\right| \left[\sqrt{2}\left(\int_{\alpha}^{\beta} \frac{1}{r(t)}\left(\int_{\alpha}^{t} \frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}} + \sup_{\alpha\leq t\leq\beta} \mu(t)\right] \geq 1, \tag{125}$$

where $Q(t) = \int_t^{\beta} q(s)ds$. If $y^{\Delta}(\alpha) = y(\beta) = 0$, then

$$\sup_{\alpha\leq t\leq\beta}\left|\frac{Q(t)}{r(t)}\right| \left[\sqrt{2}\left(\int_{\alpha}^{\beta} \frac{1}{r(t)}\left(\int_{t}^{\beta} \frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}} + \sup_{\alpha\leq t\leq\beta} \mu(t)\right] \geq 1, \tag{126}$$

where $Q(t) = \int_{\alpha}^{t} q(s)ds$.

Proof. We prove (125). Multiplying (116) by y^{σ} and integrating by parts and follows the proof of Theorem 6.1, we have

$$\int_{\alpha}^{\beta} r(t)\left(y^{\Delta}(t)\right)^2 \Delta t = \int_{\alpha}^{\beta} q(t)\left(y^{\sigma}(t)\right)^2 \Delta t = -\int_{\alpha}^{\beta} Q^{\Delta}(t)\left(y^{\sigma}(t)\right)^2 \Delta t.$$

Integrating by parts the right hand side and using the fact that $y(\alpha)=0=Q(\beta)$, we see that

$$\begin{aligned}\int_\alpha^\beta r(t)\left(y^\Delta(t)\right)^2\Delta t &\leq \int_\alpha^\beta |Q(t)|\,|y(t)+y^\sigma(t)|\left|y^\Delta(t)\right|\Delta t\\ &\leq \sup_{\alpha\leq t\leq\beta}\left|\frac{Q(t)}{r(t)}\right|\int_\alpha^\beta r(t)\,|y(t)+y^\sigma(t)|\left|y^\Delta(t)\right|\Delta t.\end{aligned}$$

Applying the Opial inequality in Section 5 and cancelling the term $\int_\alpha^\beta r(t)\left(y^\Delta(t)\right)^2\Delta t$, we get the desired inequality (125). The proof of (126) is similar to the proof of (125) by using the integration by parts and Corollary 6.2 instead of Corollary 6.1. The proof is complete.

As a special case of Theorem 6.2, when $r(t)=1$, we have the following result.

Corollary 6.4. *Suppose that y is a nontrivial solution of (120) which satisfying $y(\alpha)=y^\Delta(\beta)=0$, then*

$$\sup_{\alpha\leq t\leq\beta}|Q(t)|\left[(\beta-\alpha)+\sup_{\alpha\leq t\leq\beta}\mu(t)\right]\geq 1,$$

where $Q(t)=\int_t^\beta q(s)ds$. If $y^\Delta(\alpha)=y(\beta)=0$, then

$$\sup_{\alpha\leq t\leq\beta}|Q(t)|\left[(\beta-\alpha)+\sup_{\alpha\leq t\leq\beta}\mu(t)\right]\geq 1,$$

where $Q(t)=\int_\alpha^t q(s)ds$.

As special case of Corollary 6.4, when $\mathbb{T}=\mathbb{R}$, (note that in this case $\mu(t)=0$), we have the following result due to Harris and Kong [21] for the second order differential equation (121).

Corollary 6.5 [21]. *Suppose that y is a nontrivial solution of (121) which satisfying $y(\alpha)=y^{'}(\beta)=0$, then*

$$(\beta-\alpha)\max_{\alpha\leq t\leq\beta}\left|\left(\int_t^\beta q(s)ds\right)\right|\geq 1, \tag{127}$$

and

$$(\beta-\alpha)\max_{\alpha\leq t\leq\beta}\left|\int_\alpha^t q(s)ds\right|\geq 1, \tag{128}$$

when $y^{'}(\alpha)=y(\beta)=0$.

As special case of Corollary 6.4, when $\mathbb{T}=\mathbb{N}$, (note that in this case $\mu(t)=1$), we have the following result for the second order difference equation (124).

Corollary 6.6. *If y is a solution of the equation (124) such that $\Delta y(\alpha)=y(\beta)=0$, then*

$$\sup_{\alpha\leq n\leq\beta}|Q(n)|\,(\beta+1-\alpha)>1,$$

where $Q(n)=\sum_{s=n}^{\beta-1}q(s)$. If instead $y(a)=\Delta y(b)=0$, then

$$\sup_{\alpha\leq n\leq\beta}|Q(n)|\,(\beta+1-\alpha)>1>1,$$

where $Q(n)=\sum_{s=\alpha}^{n-1}q(s)$.

The above results yield sufficient conditions for disfocality of (137), i.e., sufficient conditions so that there does not exist a nontrivial solution y satisfying either $y(\alpha)=y^{\Delta}(\beta)=0$ or $y^{\Delta}(\alpha)=y(\beta)=0$.

Our concern in the following is to determine the lower bound for the distance between consecutive generalized zeros of the solutions of (116). Perhaps the best known existence results of this type for the dynamic equation (120) on a time scale $\mathbb{T}$ is due to Bohner et al. [13]. In particular they extended the classical Lyapunov inequality and proved that if $y(t)$ is a solution of (120) with $y(\alpha)=y(\beta)=0$ $(\alpha<\beta)$ then

$$\int_{\alpha}^{\beta}q(t)\Delta t>\frac{4}{f(d)},$$

where $q(t)$ is a positive $rd-$continuous function defined on $\mathbb{T}$, $f(d)=(d-\alpha)(d-\beta)$, and d is the closest element of $\mathbb{T}$ to the midpoint of the interval $[\alpha,\beta]$. As a particular case they derived that

$$\int_{\alpha}^{\beta}q(t)\Delta t>\frac{4}{\beta-\alpha}. \tag{129}$$

In the following, we assume that there exists a unique $h\in[\alpha,\beta]_{\mathbb{T}}$ such that

$$\int_{\alpha}^{h}\frac{\Delta t}{r(t)}=\int_{h}^{\beta}\frac{\Delta t}{r(t)},\text{ with }\int_{\alpha}^{\beta}\frac{\Delta t}{r(t)}<\infty. \tag{130}$$

Note again that when $r(t)=1$, we see that $(h-\alpha)=(\beta-h)$. So that the unique solution is given by $h=(\alpha+\beta)/2$.

Theorem 6.3. *Assume that (130) holds and* $Q^{\Delta}(t)=q(t)$. *Suppose that* y *is a nontrivial solution of (116) and* $y^{\Delta}(t)$ *does not change sign in* $(\alpha,\beta)_{\mathbb{T}}$. *If* $y(\alpha)=y(\beta)=0$, *then*

$$\left[\sqrt{2}\left(\int_{\alpha}^{\beta}\frac{Q^2(t)}{r(t)}\left(\int_{\alpha}^{h}\frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}}+\sup_{\alpha\le t\le\beta}\mu(t)\left|\frac{Q(t)}{r(t)}\right|\right]\ge 1. \tag{131}$$

Proof. As in the proof of Theorem 6.1 by multiplying (116) by $y^{\sigma}(t)$, integrating by parts and using $y(\alpha)=y(\beta)=0$, we have that

$$\int_{\alpha}^{\beta}r(t)\left|y^{\Delta}(t)\right|^2dt\le\int_{\alpha}^{\beta}|Q(t)|\,|y(t)+y^{\sigma}(t)|^{\gamma}\left|y^{\Delta}(t)\right|dt. \tag{132}$$

This implies that

$$\begin{aligned}\int_{\alpha}^{\beta}r(t)\left|y^{\Delta}(t)\right|^2dt &\le \left[\sqrt{2}\left(\int_{\alpha}^{\beta}\frac{Q^2(t)}{r(t)}\left(\int_{\alpha}^{h}\frac{\Delta u}{r(u)}\right)\Delta t\right)^{\frac{1}{2}}+\sup_{\alpha\le t\le\beta}\mu(t)\left|\frac{Q(t)}{r(t)}\right|\right]\\ &\quad\times\int_{\alpha}^{\beta}r(t)\left|y^{\Delta}(t)\right|^2dt.\end{aligned}$$

From the last inequality, after cancelling the term $\int_{\alpha}^{\beta}r(t)\left|y^{\Delta}(t)\right|^2\Delta t$, we get the desired inequality (131). This completes the proof.

As a special case of Theorem 6.3 when $r(t)=1$, (note that in this case $h=(\alpha+\beta)/2$)), we have the following result for the equation (120).

Theorem 6.4. *Assume that $Q^{\Delta}(t)=q(t)$. Suppose that y is a nontrivial solution of (120) and $y^{\Delta}(t)$ does not change sign in $(\alpha,\beta)_{\mathbb{T}}$. If $y(\alpha)=y(\beta)=0$, then*

$$\left[\sqrt{\beta-\alpha}\left(\int_{\alpha}^{\beta}Q^{2}(t)\Delta t\right)^{\frac{1}{2}}+\sup_{\alpha\leq t\leq\beta}\left(\mu(t)\left|Q(t)\right|\right)\right]\geq 1.$$

As special cases of Theorem 6.4, when $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{N}$, we have the following results for the second order differential equation (121) and second order difference equation (124).

Corollary 6.7. *Assume that $Q^{'}(t)=q(t)$. Suppose that y is a nontrivial solution of (17) and $y^{'}(t)$ does not change sign in (α,β). If $y(\alpha)=y(\beta)=0$, then*

$$\int_{\alpha}^{\beta}\left(\int_{\alpha}^{t}q(u)du\right)^{2}dt\geq\frac{1}{\beta-\alpha}. \tag{133}$$

Corollary 6.8. *Assume that $\Delta Q(n)=q(n)$. Suppose that y is a nontrivial solution of (124) and $\Delta y(n)$ does not change sign in (α,β). If $y(\alpha)=y(\beta)=0$, then*

$$\left[\sqrt{\beta-\alpha}\left(\sum_{n=\alpha}^{n-1}Q^{2}(n)\right)^{\frac{1}{2}}+\sup_{\alpha\leq n\leq\beta}\left|Q(n)\right|\right]\geq 1.$$

As an application, in the following we will show how Opial and Wirtinger inequalities may be used to find the lower bound for the smallest eigenvalue of a Sturm-Liouville eigenvalue problem on a time scale $\mathbb{T}$. For more details of Sturm-Liouville eigenvalue problems, we refer the reader to the paper [2]. Consider the Sturm-Liouville eigenvalue problem

$$-y^{\Delta\Delta}(t)+q(t)y^{\sigma}(t)=\lambda y^{\sigma}(t),\ \ y(0)=y(\beta)=0, \tag{134}$$

and assume that λ_0 is the smallest eigenvalue of (134). Our main aim in the following is to find the lower bound of λ_0. To find this lower bound, will apply some Opial's type inequalities and a Wirtinger inequality due to Hilscher [23] which is given by

$$\int_{\alpha}^{\beta}\frac{M(t)M^{\sigma}(t)}{|M^{\Delta}(t)|}\left(y^{\Delta}(t)\right)^{2}\Delta t\geq\frac{1}{\psi^{2}}\int_{\alpha}^{\beta}\left|M^{\Delta}(t)\right|(y^{\sigma}(t))^{2}\Delta t, \tag{135}$$

for a positive function $M\in C_{rd}^{1}(\mathbb{I})$ with either$M^{\Delta}(t)>0$ or $M^{\Delta}(t)<0$ on $\mathbb{I}$, $y\in C_{rd}^{1}(\mathbb{I})$ with $y(\alpha)=0=y(\beta)$, for $\mathbb{I}=[\alpha,\beta]_{\mathbb{T}}\subset\mathbb{T}$ and

$$\psi=\left(\sup_{t\in\mathbb{I}^{k}}\frac{M(t)}{M^{\sigma}(t)}\right)^{1/2}+\left[\left(\sup_{t\in\mathbb{I}^{k}}\frac{\mu(t)\left|M^{\Delta}(t)\right|}{M^{\sigma}(t)}\right)+\left(\sup_{t\in\mathbb{I}^{k}}\frac{M(t)}{M^{\sigma}(t)}\right)\right]^{1/2}.$$

We denote

$$A(Q)=\sqrt{(\beta-\alpha)}\left(\int_{\alpha}^{\beta}Q^{2}(t)\Delta t\right)^{\frac{1}{2}}+\sup_{\alpha\leq t\leq\beta}\mu(t)\left|Q(t)\right|.$$

Theorem 6.5. *Assume that λ_0 is the smallest eigenvalue of (134) and assume that $q(t)=Q^{\Delta}(t)+\gamma$, where $\gamma<\lambda_0$. Then*

$$|\lambda_{0}-\gamma|\geq\frac{1-A(Q)}{(1+\sqrt{2})^{2}\sigma^{2}(\beta)}. \tag{136}$$

Proof. Let $y(t)$ be the eigenfunction of (134) corresponding to λ_0. Multiplying (134) by $y^\sigma(t)$ and proceeding as in the proof of Theorem 6.1, we have

$$-\int_0^\beta y^{\Delta\Delta}y^\sigma(t)\Delta t+\int_0^\beta q(t)\left(y^\sigma(t)\right)^2\Delta t=\lambda_0\int_0^\beta \left(y^\sigma(t)\right)^2\Delta t.$$

This implies, after integrating by parts and using the fact that $y(0)=y(\beta)=0$, that

$$\begin{aligned}&(\lambda_0-\gamma)\int_0^\beta \left(y^\sigma(t)\right)^2\Delta t\\ =\ &\int_0^\beta \left(y^\Delta(t)\right)^2\Delta t+\int_0^\beta Q^\Delta(t)\left(y^\sigma(t)\right)^2\Delta t\\ =\ &\int_0^\beta \left(y^\Delta(t)\right)^2\Delta t-\int_0^\beta Q(t)\left[y(t)+y^\sigma(t)\right]y^\Delta(t)\Delta t\\ \geq\ &\int_0^\beta \left(y^\Delta(t)\right)^2\Delta t-\int_0^\beta \left|Q(t)\right|\left|y(t)+y^\sigma(t)\right|\left|y^\Delta(t)\right|\Delta t.\end{aligned}$$

Proceeding as in the proof of Theorem 6.1 by applying the inequality (**??**) with $r(t)=1$ and $s=Q$, on the term

$$\int_0^\beta \left|Q(t)\right|\left|y(t)+y^\sigma(t)\right|\left|y^\Delta(t)\right|\Delta t,$$

we obtain

$$\left|\lambda_0-\gamma\right|\int_0^\beta \left(y^\sigma(t)\right)^2\Delta t\geq\int_0^\beta \left(y^\Delta(t)\right)^2\Delta t-A(Q)\int_0^\beta \left|y^\Delta(t)\right|^2\Delta t.$$

Now, applying Wirtinger's inequality (135) by putting $M(t)=t$, we have that

$$\left|\lambda_0-\gamma\right|\psi_1^2\sigma^2(\beta)\int_0^\beta (y^\Delta(t))^2\Delta t\geq\int_0^\beta \left(y^\Delta(t)\right)^2\Delta t-A(Q)\int_0^\beta \left(y^\Delta(t)\right)^2\Delta t,$$

where

$$\psi_1=\left(\sup_{t\in\mathbb{I}^k}\frac{t}{\sigma(t)}\right)^{1/2}+\left[\left(\sup_{t\in\mathbb{I}^k}\frac{\mu(t)}{\sigma(t)}\right)+\left(\sup_{t\in\mathbb{I}^k}\frac{t}{\sigma(t)}\right)\right]^{1/2}\leq 1+\sqrt{2}.$$

This implies that

$$\left|\lambda_0-\gamma\right|(1+\sqrt{2})^2\sigma^2(\beta)\geq 1-A(Q).$$

From this we obtain the lower bound of λ_0 as given in (136). The proof is complete.

Also, in the following, we apply the dynamic inequalities of Opial's type presented in Section 5 to prove several results related to the problems $(i)-(ii)$ for the second-order half-linear dynamic equation

$$(r(t)(y^\Delta(t))^\gamma)^\Delta+q(t)\left(y^\sigma(t)\right)^\gamma=0,\text{ on }[a,b]_{\mathbb{T}},\tag{137}$$

on an arbitrary time scale $\mathbb{T}$, where $0<\gamma\leq 1$ is a quotient of odd positive integers, r and q are real $rd-$continuous functions defined on $\mathbb{T}$ with $r(t)>0$. The terminology half-linear arises because of the fact that the space of all solutions of (137) is homogeneous, but not generally additive. Thus, it has just "half" of the properties of a linear space. It is easily seen that if $y(t)$ is a solution of (137), then so also is $cy(t)$.

By a solution of (137) on an interval $\mathbb{I}$, we mean a nontrivial real–valued function $y \in C_{rd}(\mathbb{I})$, which has the property that $r(t)y^{\Delta}(t) \in C^1_{rd}(\mathbb{I})$ and satisfies equation (137) on $\mathbb{I}$. We say that a solution y of (137) has a generalized zero at t if $y(t) = 0$, and has a generalized zero in $(t, \sigma(t))$ in case $y(t)y^{\sigma}(t) < 0$ and $\mu(t) > 0$. Equation (137) is disconjugate on the interval $[t_0, b]_{\mathbb{T}}$, if there is no nontrivial solution of (137) with two (or more) generalized zeros in $[t_0, b]_{\mathbb{T}}$.

We say that (137) is right disfocal (left disfocal) on $[\alpha, \beta]_{\mathbb{T}}$ if the solutions of (137) such that $y^{\Delta}(\alpha) = 0$ ($y^{\Delta}(\beta) = 0$) have no generalized zeros in $[\alpha, \beta]_{\mathbb{T}}$.

Note that (137) in its general form involves some different types of differential and difference equations depending on the choice of the time scale $\mathbb{T}$. For example when $\mathbb{T} = \mathbb{R}$, (137) becomes a second-order half-linear differential equation and $\sigma(t) = t$. When $\mathbb{T} = \mathbb{Z}$, (137) becomes a second-order half-linear difference equation and $\sigma(t) = t + 1$. When $\mathbb{T} = h\mathbb{N}$, (137) becomes a generalized difference equation and $\sigma(t) = t + h$. When $\mathbb{T} = \{t : t = \varrho^k,\ k \in \mathbb{N}_0,\ \varrho > 1\}$, (137) becomes a quantum difference equation (see [27]) and $\sigma(t) = \varrho t$. Note also that the results in this paper can be applied on the time scales $\mathbb{T} = \mathbb{N}^2 = \{t^2 : t \in \mathbb{N}\}$, $\mathbb{T}_2=\{\sqrt{n} : n \in \mathbb{N}_0\}$, $\mathbb{T}_3=\{\sqrt[3]{n} : n \in \mathbb{N}_0\}$, and when $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}_0\}$ where $\{t_n\}$ is the set of harmonic numbers. In these cases we see that when $\mathbb{T} = \mathbb{N}_0^2 = \{t^2 : t \in \mathbb{N}_0\}$, we have $\sigma(t) = (\sqrt{t}+1)^2$ and when $\mathbb{T} = \mathbb{T}_n = \{t_n : n \in \mathbb{N}\}$ where $(t_n\}$ is the harmonic numbers that are defined by $t_0 = 0$ and $t_n = \sum_{k=1}^{n} \frac{1}{k}$, $n \in \mathbb{N}_0$, we have $\sigma(t_n) = t_{n+1}$. When $\mathbb{T}_2=\{\sqrt{n} : n \in \mathbb{N}_0\}$, we have $\sigma(t) = \sqrt{t^2+1}$, and when $\mathbb{T}_3=\{\sqrt[3]{n} : n \in \mathbb{N}_0\}$, we have $\sigma(t) = \sqrt[3]{t^3+1}$.

Saker [49] considered the second-order half-linear dynamic equation

$$\left(r(t)\varphi(x^{\Delta})\right)^{\Delta} + q(t)\varphi(x^{\sigma}(t)) = 0, \tag{138}$$

on an arbitrary time scale $\mathbb{T}$, where $\varphi(u) = |u|^{\gamma-1} u$, $\gamma \geq 1$ is a positive constant, r and q are real $rd-$continuous positive functions defined on $\mathbb{T}$ and proved that: if $x(t)$ is a positive solution of (137) which satisfies $x(a) = x(b) = 0$, $x(t) \neq 0$ for $t \in (a, b)$ and $x(t)$ has a maximum at a point $c \in (a, b)$, then

$$\left(\int_a^b \frac{\Delta t}{r^{\gamma}(t)}\right)^{\gamma} \int_a^b q(t)\Delta t \geq 2^{\gamma+1}. \tag{139}$$

Now, we are ready to state and prove the main results. To simplify the presentation of the results, we define

$$\begin{aligned} M(\beta) \ &: \ = \sup_{\alpha \leq t \leq \beta} \mu^{\gamma}(t)\frac{|Q(t)|}{r(t)}, \text{ where } Q(t) = \int_t^{\beta} q(s)\Delta s, \\ M(\alpha) \ &: \ = \sup_{\alpha \leq t \leq \beta} \mu^{\gamma}(t)\frac{|Q(t)|}{r(t)}, \text{ where } Q(t) = \int_{\alpha}^{t} q(s)\Delta s. \end{aligned}$$

Note that when $\mathbb{T} = \mathbb{R}$, we have $M(\alpha) = 0 = M(\beta)$, and when $\mathbb{T} = \mathbb{Z}$, we have

$$M(\beta) = \sup_{\alpha \leq t \leq \beta} \frac{\left|\sum_{s=t}^{\beta-1} q(s)\right|}{r(t)}, \text{ and } M(\alpha) = \sup_{\alpha \leq t \leq \beta} \frac{\left|\sum_{s=\alpha}^{t-1} q(s)\right|}{r(t)}. \tag{140}$$

Theorem 6.6. *Suppose that y is a nontrivial solution of (137) and y^{Δ} does not change sign in $(\alpha,\beta)_{\mathbb{T}}$. If $y(\alpha)=y^{\Delta}(\beta)=0$, then*

$$\frac{2}{(\gamma+1)^{\frac{1}{\gamma+1}}}\times\left(\int_{\alpha}^{\beta}\frac{|Q(x)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(x)}\left(\int_{\alpha}^{x}\frac{\Delta t}{r^{\frac{1}{\gamma}}(t)}\right)^{\gamma}\Delta x\right)^{\frac{\gamma}{\gamma+1}}+2^{1-\gamma}M(\beta)\geq 1, \quad (141)$$

where $Q(t)=\int_t^{\beta}q(s)\Delta s$. If $y^{\Delta}(\alpha)=y(\beta)=0$, then

$$\frac{2}{(\gamma+1)^{\frac{1}{\gamma+1}}}\left(\int_{\alpha}^{\beta}\frac{|Q(x)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(x)}\left(\int_{x}^{\beta}\frac{\Delta t}{r^{\frac{1}{\gamma}}(t)}\right)^{\gamma}\Delta x\right)^{\frac{\gamma}{\gamma+1}}+2^{1-\gamma}M(\alpha)\geq 1, \quad (142)$$

where $Q(t)=\int_{\alpha}^{t}q(s)\Delta s$.

Proof. We prove (141). Without loss of generality we may assume that $y(t)>0$ in $[\alpha,\beta]_{\mathbb{T}}$. Multiplying (137) by y^{σ} and integrating by parts, we have

$$\begin{aligned}\int_{\alpha}^{\beta}\left(r(t)\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}y^{\sigma}(t)\Delta t &= r(t)\left(y^{\Delta}(t)\right)^{\gamma}y(t)\Big|_{\alpha}^{\beta} \\ &\quad -\int_{\alpha}^{\beta}r(t)\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t \\ &= -\int_{\alpha}^{\beta}q(t)\left(y^{\sigma}(t)\right)^{\gamma+1}\Delta t.\end{aligned}$$

Using the assumptions that $y(\alpha)=y^{\Delta}(\beta)=0$ and $Q(t)=\int_t^{\beta}q(s)\Delta s$, we have

$$\int_{\alpha}^{\beta}r(t)\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t=\int_{\alpha}^{\beta}q(t)\left(y^{\sigma}(t)\right)^{\gamma+1}\Delta t=-\int_{\alpha}^{\beta}Q^{\Delta}(t)\left(y^{\sigma}(t)\right)^{\gamma+1}\Delta t. \quad (143)$$

Integrating by parts the right hand side (see 22), we see that

$$\int_{\alpha}^{\beta}r(t)\left(y^{\Delta}(t)\right)^{\gamma+1}\Delta t=-\,Q(t)(y(t))^{\gamma+1}\big|_{\alpha}^{\beta}+\int_{\alpha}^{\beta}Q(t)\left(y^{\gamma+1}(t)\right)^{\Delta}\Delta t.$$

Again using the facts that $y(\alpha)=0=Q(\beta)$, we obtain

$$\int_{\alpha}^{\beta}r(t)\left(y^{\Delta}(t)\right)^{\gamma+1}dt=\int_{\alpha}^{\beta}Q(t)\left(y^{\gamma+1}(t)\right)^{\Delta}dt. \quad (144)$$

Applying the chain rule formula, we see that

$$\begin{aligned}\left|\left(y^{\gamma+1}(t)\right)^{\Delta}\right| &\leq (\gamma+1)\int_0^1\left|hy^{\sigma}(t)+(1-h)y(t)\right|^{\gamma}dh\left|y^{\Delta}(t)\right| \\ &\leq (\gamma+1)\left|y^{\Delta}(t)\right|\int_0^1\left|hy^{\sigma}(t)\right|^{\gamma}dh \\ &\quad +(\gamma+1)\left|y^{\Delta}(t)\right|\int_0^1\left|(1-h)y(t)\right|^{\gamma}dh \\ &= \left|y^{\Delta}(t)\right|\left|y^{\sigma}(t)\right|^{\gamma}+\left|y^{\Delta}(t)\right|\left|y(t)\right|^{\gamma} \\ &\leq 2^{1-\gamma}\left|y^{\sigma}(t)+y(t)\right|^{\gamma}\left|y^{\Delta}(t)\right|. \quad (145)\end{aligned}$$

This and (144) imply that

$$\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t \leq 2^{1-\gamma}\int_{\alpha}^{\beta}|Q(t)|\,|y(t)+y^{\sigma}(t)|^{\gamma}\left|y^{\Delta}(t)\right|\Delta t.$$

Applying the inequality (103) with $s(t)=|Q(t)|$, $p=\gamma$ and $q=1$, we have

$$\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t \leq 2^{1-\gamma}K_1(\alpha,\beta,\gamma,1)\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t, \tag{146}$$

where

$$\begin{aligned} K_1(\alpha,\beta,\gamma,1) &= M(\beta)+2^{\gamma}\left(\frac{1}{\gamma+1}\right)^{\frac{1}{\gamma+1}} \\ &\quad\times\left(\int_{\alpha}^{\beta}|Q(x)|^{\frac{\gamma+1}{\gamma}}r^{-\frac{1}{\gamma}}(x)\left(\int_{\alpha}^{x}r^{\frac{-1}{\gamma}}(t)\Delta t\right)^{\gamma}\Delta x\right)^{\frac{\gamma}{\gamma+1}}. \end{aligned}$$

Then, we have from (146) after cancelling the term $\int_{\alpha}^{\beta} r(t)\left|y^{\Delta}(t)\right|^{\gamma+1}\Delta t$, that

$$2^{1-\gamma}M(\beta)+\frac{2}{(\gamma+1)^{\frac{1}{\gamma+1}}}\times\left(\int_{\alpha}^{\beta}\frac{|Q(x)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(x)}\left(\int_{\alpha}^{x}\frac{\Delta t}{r^{\frac{1}{\gamma}}(t)}\right)^{\gamma}\Delta x\right)^{\frac{\gamma}{\gamma+1}} \geq 1,$$

which is the desired inequality (141). The proof of (142) is similar to (141. The proof is complete.

As a special case of Theorem 6.6, when $r(t)=1$, we have the following result.

Corollary 6.1. *Suppose that y is a nontrivial solution of*

$$\left(\left(y^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+q(t)\left(y^{\sigma}(t)\right)^{\gamma}=0, \quad t\in[\alpha,\beta]_{\mathbb{T}}, \tag{147}$$

and y^{Δ} does not change sign in $(\alpha,\beta)_{\mathbb{T}}$. If $y(\alpha)=y^{\Delta}(\beta)=0$, then

$$\frac{2}{(\gamma+1)^{\frac{1}{\gamma+1}}}\times\left[\int_{\alpha}^{\beta}|Q(t)|^{\frac{1+\gamma}{\gamma}}(t-\alpha)^{\gamma}\Delta t\right]^{\frac{\gamma}{\gamma+1}}+2^{1-\gamma}\sup_{\alpha\leq t\leq\beta}\left(\mu^{\gamma}(t)\,|Q(t)|\right)\geq 1, \tag{148}$$

where $Q(t)=\int_{t}^{\beta}q(s)\Delta s$. If $y^{\Delta}(\alpha)=y(\beta)=0$, then

$$\frac{2}{(\gamma+1)^{\frac{1}{\gamma+1}}}\left[\int_{\alpha}^{\beta}|Q(t)|^{\frac{1+\gamma}{\gamma}}(\beta-t)^{\gamma}\Delta t\right]^{\frac{\gamma}{\gamma+1}}+2^{1-\gamma}\sup_{\alpha\leq t\leq\beta}\left(\mu^{\gamma}(t)\,|Q(t)|\right)\geq 1, \tag{149}$$

where $Q(t)=\int_{\alpha}^{t}q(s)\Delta s$.

Corollary 6.5. *Suppose that y is a nontrivial solution of (147) and y^{Δ} does not change sign in $(\alpha,\beta)_{\mathbb{T}}$, where $\gamma\leq 1$ is a quotient of odd positive integers. If $y(\alpha)=y^{\Delta}(\beta)=0$, then*

$$\frac{2(\beta-\alpha)^{\gamma}}{(\gamma+1)}\max_{\alpha\leq t\leq\beta}\left|\int_{t}^{\beta}q(s)\Delta s\right|+2^{1-\gamma}\sup_{\alpha\leq t\leq\beta}\left(\mu^{\gamma}(t)\left|\int_{t}^{\beta}q(s)\Delta s\right|\right)\geq 1, \tag{150}$$

and if $y^{\Delta}(\alpha)=y(\beta)=0$, then

$$\frac{2(\beta-\alpha)^{\gamma}}{(\gamma+1)}\max_{\alpha\leq t\leq\beta}\left|\int_{\alpha}^{t}q(s)\Delta s\right|+2^{1-\gamma}\sup_{\alpha\leq t\leq\beta}\left(\mu^{\gamma}(t)\left|\int_{\alpha}^{t}q(s)\Delta s\right|\right)\geq 1. \tag{151}$$

As a special when $\mathbb{T}=\mathbb{R}$, we have $M(\alpha)=M(\beta)=0$ and then the results in Corollary 6.5 reduce to the following results for the second order half-linear differential equation

$$\left((y^{'}(t))^{\gamma}\right)^{'}+q(t)(y(t))^{\gamma}=0,\ \ \alpha\leq t\leq\beta, \tag{152}$$

where $\gamma\leq 1$ is a quotient of odd positive integers.

Corollary 6.6. *Assume that $\gamma\leq 1$ is a quotient of odd positive integers. Suppose that y is a nontrivial solution of (152) and $y^{'}$ does not change sign in (α,β). If $y(\alpha)=y^{'}(\beta)=0$, then*

$$\frac{2}{(\gamma+1)}(\beta-\alpha)^{\gamma}\sup_{\alpha\leq t\leq\beta}\left|\int_t^{\beta}q(s)ds\right|\geq 1. \tag{153}$$

If instead $y^{'}(\alpha)=y(\beta)=0$, then

$$\frac{2}{(\gamma+1)}(\beta-\alpha)^{\gamma}\sup_{\alpha\leq t\leq\beta}\left|\int_{\alpha}^{t}q(s)ds\right|\geq 1. \tag{154}$$

As a special when $\mathbb{T}=\mathbb{Z}$, we see that $M(\alpha)$ and $M(\beta)$ are defined as in (140) and then the results in Corollary 6.5 reduce to the following results for the second order half-linear difference equation

$$\Delta\left((\Delta y(n))^{\gamma}\right)+q(n)(y(n+1))^{\gamma}=0,\ \ \alpha\leq n\leq\beta, \tag{155}$$

where $\gamma\leq 1$ is a quotient of odd positive integers.

Corollary 6.8. *Suppose that y is a nontrivial solution of (155) and $\Delta y(n)$ does not change sign in $(\alpha,\beta)_{\mathbb{T}}$, where $\gamma\leq 1$ is a quotient of odd positive integers. If $y(\alpha)=\Delta y(\beta)=0$, then*

$$\frac{2(\beta-\alpha)^{\gamma}}{(\gamma+1)}\max_{\alpha\leq n\leq\beta}\left|\sum_{s=n}^{\beta-1}q(s)\right|+2^{1-\gamma}\sup_{\alpha\leq n\leq\beta}\left(\left|\sum_{s=n}^{\beta-1}q(s)\right|\right)\geq 1,$$

and if $\Delta y(\alpha)=y(\beta)=0$, then

$$\frac{2(\beta-\alpha)^{\gamma}}{(\gamma+1)}\max_{\alpha\leq n\leq\beta}\left|\sum_{s=\alpha}^{n-1}q(s)\right|+2^{1-\gamma}\sup_{\alpha\leq n\leq\beta}\left(\left|\sum_{s=\alpha}^{n-1}q(s)\right|\right)\geq 1.$$

In the following, we determine the lower bound for the distance between consecutive zeros of solutions of (137). Note that the applications of the above results allows the use of arbitrary anti-derivative Q in the above arguments. In the following, we assume that $Q^{\Delta}(t)=q(t)$ and there exists $h\in(\alpha,\beta)$ which is the unique solution of the equation

$$K_1(\alpha,\beta)=K_1(\alpha,\beta,h)=K_1(\alpha,h,\beta)<\infty, \tag{156}$$

where

$$K_1(\alpha,\beta,h)=\frac{2^{\gamma}}{(\gamma+1)^{\frac{1}{\gamma+1}}}\times\left(\int_{\alpha}^{\beta}\frac{|Q(x)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(x)}\left(\int_{\alpha}^{h}\frac{\Delta t}{r^{\frac{1}{\gamma}}(t)}\right)^{\gamma}\Delta x\right)^{\frac{\gamma}{\gamma+1}},$$

and

$$K_1(\alpha,h,\beta)=\frac{2^\gamma}{(\gamma+1)^{\frac{1}{\gamma+1}}}\left(\int_\alpha^\beta\frac{|Q(x)|^{\frac{\gamma+1}{\gamma}}}{r^{\frac{1}{\gamma}}(x)}\left(\int_h^\beta\frac{\Delta t}{r^{\frac{1}{\gamma}}(t)}\right)^\gamma\Delta x\right)^{\frac{\gamma}{\gamma+1}}.$$

Theorem 6.7. *Assume that $Q^\Delta(t)=q(t)$. Suppose y is a nontrivial solution of (137) and $y^\Delta(t)$ does not change sign in (α,β). If $y(\alpha)=y(\beta)=0$, then*

$$K_1(\alpha,\beta)\geq 1, \tag{157}$$

where $K_1(\alpha,\beta)$ is defined as in (156).

Proof. Multiplying (137) by $y^\sigma(t)$, proceed as in Theorem 3.6 and using $y(\alpha)=y(\beta)=0$, to get

$$\int_\alpha^\beta r(t)\left(y^\Delta(t)\right)^{\gamma+1}\Delta t=\int_\alpha^\beta q(t)\left(y(t)\right)^{\gamma+1}\Delta t=\int_\alpha^\beta Q^\Delta(t)\left(y^\sigma(t)\right)^{\gamma+1}\Delta t.$$

Integrating by parts the right hand side, we see that

$$\int_\alpha^\beta r(t)\left(y^\Delta(t)\right)^{\gamma+1}\Delta t=Q(t)(y(t))^{\gamma+1}\Big|_\alpha^\beta+\int_\alpha^\beta(-Q(t))\left(y^{\gamma+1}(t)\right)^\Delta\Delta t.$$

Again using the facts that $y(\alpha)=0=y(\beta)$, we obtain

$$\int_\alpha^\beta r(t)\left|y^\Delta(t)\right|^{\gamma+1}\Delta t\leq\int_\alpha^\beta|Q(t)|\left|y(t)+y^\sigma(t)\right|^\gamma\left|y^\Delta(t)\right|\Delta t.$$

This implies that

$$\int_\alpha^\beta r(t)\left|y^\Delta(t)\right|^{\gamma+1}dt\leq 2^{1-\gamma}K_1(\alpha,\beta)\int_\alpha^\beta r(t)\left|y^\Delta(t)\right|^{\gamma+1}\Delta t,$$

From this inequality, after cancelling $\int_\alpha^\beta\left|y^\Delta(t)\right|^{\gamma+1}\Delta t$, we get the desired inequality (157). This completes the proof.

References

[1] R. P. Agarwal, M. Bohner, D. O'Regan and S. H. Saker, Some Wirtinger-type inequalities on time scales and their applications, Pacific J. Math. 252 (2011), 1-26.

[2] R. P. Agarwal, M. Bohner and P. J. Y. Wong, Sturm-Liouville eigenvalue problems on time scales, Appl. Math. Comp. 99 (1999), 153-166.

[3] R. P. Agarwal and P. Y. H. Pang, *Opial inequalities with Applications in Differential and Difference Equations*, Kluwer, Dordrechet (1995).

[4] C. D. Ahlbrandt and Ch. Morian, Partial differential equations on time scales, Journal of Comp. Appl. Math. 141 (2002), 35–55.

[5] D. A. Anderson, Young's integral inequality on time scales revisited, J. Ineq. Pure Appl. Math. 8 (2007), Issue 3, Art. 64, 5pp.

[6] D. A. Anderson, Nonlinear dynamic integral inequalities in two independent variables on time scale pairs, Advances Dyn. Syst. Appl. 3 (2008), 1-13.

[7] D. A. Anderson, Dynamic double integral inequalities in two independent variables on time scales, J. Math. Ineq. 2 (2008), 163-184.

[8] R. Bellman, The stability of solutions of linear differential equations, Duke Math. J. 10 (1943), 643-647.

[9] P. R. Bessack, On an integral inequality of Z.Opial, Trans. Amer. Math. Soc. 104 (1962), 470-475.

[10] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*: An Introduction with Applications, Birkhäuser, Boston, 2001.

[11] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[12] E. Akin-Bohner, M. Bohner, F. Akin, Pachpatte inequalities on time scales, JIPAM. J. Ineq. Pure Appl. Math. 6 (2005) 1–23.
[13] M. Bohner, S. Clark and J. Ridenhour, Lyapunov inequalities for time scales, J. Ineq. Appl. 7 (2002), 61-77.L.
[14] M. Bohner and G. Sh. Guseinov, Partial differentiation on time scales, Dyn. Syst. Appl. 13 (2004), 351–379.
[15] M. Bohner and G. Sh. Guseinov, Double integral calculus of variations on time scales, Comput. Math. Appl. 54 (2007), 45–57.
[16] R. C. Brown and D. B. Hinton, Opial's inequality and oscillation of 2nd order equations, Proc. Amer. Math. Soc. 125 (1997), 1123-1129.
[17] C. M. Dafermos, The second law of thermodynamics and stability, Arch. Rational Mech. Anal. 70 (1979), 167–179.
[18] R. A. C. Ferreira and D. F. M. Torres, Some linear and nonlinear integral inequalities on time scales in two independent variables, Nonlinear Dyn. Syst. Theor. 9 (2009), 161-169.
[19] T. H. Gronwall, Note on the derivative with respect to a parameter of the solutions of a system of differential equations, Ann. of Math. 20 (1919), 292-296.
[20] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities,* 2nd Ed. Cambridge Univ. Press 1952.
[21] B. J. Harris and Q. Kong, On the oscillation of differential equations with an oscillatory coefficient, Tran. Amer. Math. Soc. 347 (1995), 1831-1839.
[22] S. Hilger, Analysis on measure chains–a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.
[23] R. Hilscher, A time scale version of a Wirtinger type inequality and applications, J. Comp. Appl. Math. 141 (2002), 219-226.
[24] J. Hoffacker, Basic partial dynamic equations on time scales, J. Diff. Eqns. Appl. 8,(2002), 307–319.
[25] H. L. Hong, W. C. Lian and C. C. Yeh, The oscillation of half-linear differential equations with oscillatory coefficients, Matl. comp. Modelling 24 (1996), 77-86.
[26] B. Jackson, Partial dynamic equations on time scales, J. Comp. and Appl. Math. 186, (2006), 391–415.
[27] V. Kac and P. Cheung, *Quantum Calculus*, Springer, New York, 2001.
[28] B. Karpuz, B. Kaymakçalan and Ö. Öclan, A generalization of Opial's inequality and applications to second order dynamic equations, Diff. Eqns. Dyn. Sys. 18 (2010), 11-18.
[29] A. A. Lasota, A discrete boundary value problem, Ann Polon. Math. 20 (1968), 183-190.
[30] N. Levinson, On an inequality of Opial and Beesack, Proc. Amer. Math. Soc. 15 (1964), 565-566.
[31] W. N. Li, Some Pachpatte type inequalities on time scales, Comp. Math. Appl. 57 (2009), 275-282.
[32] W. N. Li, Some new dynamic inequalities on time scales, J. Math. Anal. Appl. 319 (2007), 802-814.
[33] W. N. Li, Nonlinear integral inequalities in two independent variables on time scales, Adv. Diff. Eqns. 2011, ID 283926 (2011), 1-11.
[34] C. L. Mallows, An even simpler proof of Opial inequality, Proc. Amer. Math. Soc. 16 (1965), 173.
[35] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publ. 1993.
[36] C. Olech, A simple proof of a certain result of Z.Opial, Ann. Polon. Math. 8 (1960), 61-63.
[37] Z. Opial, Sur uné inegalité, Ann. Polon. Math. 8 (1960), 92-32.
[38] R. N. Pederson, On an inequality of Opial, Beesack and Levinson, Proc. Amer. Math. Soc. 16 (1965), 174.
[39] L. Ou-Iang, The boundedness of solutions of linear differential equations $y^{''} + A(t)y = 0$, Shuxue Jinzhan 3 (1957) 409–415.
[40] B. G. Pachpatte, On some new inequalities related to certain inequalities in the theory of differential equations, J. Math. Anal. Appl. 189 (1995) 128–144.
[41] S. H. Saker, Opial's type inequalities on time scales and some applications, Annales. Polon. Math. 104 (2012), 243-260.
[42] S. H. Saker, Some new inequalities of Opial's type on time scales, Abstr. Appl. Anal. 2012 (in press).

[43] S. H. Saker, New inequalities of Opial's type on time scales and some of their applications, Discrt. Dyn. Nature & Soc. 2012 (2012), Doi: 10.1155/2012/ 362526, 23 pages.

[44] S. H. Saker, Some Opial-type inequalities on time scales, Abstr. Appl. Anal. 2011 (2011), ID 265316, 19 pages.

[45] S. H. Saker, Bounds of double integral dynamic inequalities in two independent variables on time scales, Discrete Dyn. Nature Soc. 2011 (2011), ID Article 10.1155/2011/7321164, 1-21.

[46] S. H. Saker, Some nonlinear dynamic inequalities on time scales and applications, Journal of Math. Ineq. 4 (2010), 561-579.

[47] S. H. Saker, Nonlinear dynamic inequalities of Gronwall-Bellman type on time scales, EJQTDE 86 (2011), 1-26.

[48] S. H. Saker, Some nonlinear dynamic inequalities on time scales, Math. Ineq. Appl. 14 (2011), 633-645.

[49] S. H. Saker, Lyapunov inequalities for half-linear dynamic equations on time scales and disconjugacy, Dyn. Contin. Discr.Impuls. Syst. Series B: Applications & Algorithms 18 (2001), 149-161.

[50] S. H. Saker, *Oscillation Theory of Dynamic Equations on Time Scales: Second and Third Orders*, Lambert Academic Publishing, Germany (2010).

[51] V. Spedding, Taming Nature's Numbers, New Scientist, July 19 (2003), 28–31.

[52] A. Tuna and S. Kutukcu, Some integrals inequalities on time scales, Appl. Math. Mech. Eng. Ed. 29 (2008), 23-29.

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt

E-mail address: shsaker@mans.edu.eg