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ADOMIAN'S DECOMPOSITION METHOD FOR GENERALIZED FIFTH ORDER TIME-FRACTIONAL KORTEWEG-DE VRIES EQUATIONS

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Abstract. Adomian's Decomposition Method (ADM) is applied to tackle generalized fifth order time- fractional Korteweg-de Vries (KdV) equations. The proposed technique is fully compatible with the complexity of these problems and obtained results are highly encouraging. Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the suggested algorithm.

1. Introduction

Nonlinear partial differential equations [1]- [25] are of extreme importance in applied and engineering sciences. The through study of literature reveals that most of the physical phenomena are nonlinear in nature and hence there is a dire need to find their appropriate solutions, see [1]- [25] and the references therein. Recently, scientists have observed that number of real time problems are modeled by fractional nonlinear differential equations [1],[6]-[9], [11]-[14], [16]-[18], [21]-[25] which are very hard to tackle. In the similar context, we apply Adomian's Decomposition Method (ADM) [10],[12],[20] and [22] to solve generalized fifth order time- fractional Korteweg-de Vries partial differential equations [17].

$$D_t^{\alpha} u + au^2 u_x + bu_x u_{xx} + cuu_{xxx} + du_{xxxxx} = 0, \tag{1}$$

(where $0 \leq \alpha \leq 1$).The fractional derivatives are considered in the Caputo sense. It is to be highlighted that such equations arise frequently in applied, physical and engineering sciences. The basic motivation of this paper is the extension of a very reliable and efficient technique which is called Adomian's Decomposition Method (ADM) to find approximate solutions of generalized fifth order time-fractional Korteweg-de-Vries partial differential equations. It is observed that the proposed algorithms is fully synchronized with the complexity of fractional differential equations,. Numerical results coupled with graphical representations explicitly

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reveal the complete reliability and efficiency of the proposed algorithm.

2. Definitions

Definition 2.1 [1]
A real function $f(x)$, $x > 0$, is said to be in the space $C_\mu, \mu \in R$ if there exists a real number$p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^∞ iff $f^m \in C_\mu, \mu \geq 1, m \in N$.
Definition 2.2 [1]
The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $fC_\mu, \mu - 1$, is defined as
$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\ dt.\ \alpha > 0, x > 0,$
$J^0 f(x) = f(x)$. Properties of the operator J can be found in [1, 6−8, 12, 13, 16, 19, 20], we mention only the following . For $f \in C_\mu,\ \mu \geq -1,\ \alpha\ ,\ \beta \geq 0$ and $\gamma > -1$:
1. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,
2. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$,
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

Definition 2.3 [1]
The fractional derivative of $f(x)$ in the Caputo sense is defined as
$D^\alpha f(x) = J^{m-\alpha} D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^m(t)\ dt.,$
for $m - 1 < \alpha \leq m, m \in Z, x > 0, f \in C_{-1}^m$.
also, we need here two of its basic properties
Lemma 2.1
if $m - 1 < \alpha \leq m, m \in N$ and $f \in C_\mu^m, \mu \geq -1$,
$D^\alpha J^\alpha f(x) = f(x)$,
$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} \frac{f^k(0^+) x^k}{k!}, x > 0.$

3. Analysis of Adoinian's Decomposition Method

The nonlinear differential equations (1) can be expressed in the operator form as

$$D_t^\alpha u + R(u) + N(u) = 0,\ n - 1 < \alpha \leq n, \tag{2}$$

subject to the initial conditions
$u(x, 0) = f(x),$
where D_t^α is the time-fractional differential operator, $N(u)$ is the nonlinear operator and $R(u)$ is some linear operator. Rearranging Eq. (2) and applying the operator J^α , inverse of the operator D^α , to both sides of Eq. (2) yields

$$u(x, t) = f(x) - J^\alpha [R(u) + N(u)], \tag{3}$$

we suppose the solution to Eq.(2) in the form of the decmposition series of the form

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \tag{4}$$

and the nonlinear term $N(u)$ is decomposed as

$$N(u) = \sum_{n=0}^{\infty} A_n, \tag{5}$$

where$\sum_{n=0}^{\infty} A_n$ are the so-called Adomians polynomials. Substituting the decomposition series Eq. (4) and Eq. (5) into the both sides of Eq. (3) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) - J^{\alpha}[R(u) + \sum_{i=n-1}^{\infty} A_n], \tag{6}$$

From Eq.(6), the iterates are determined by the following recursive way

$$\begin{aligned} u_0(x,t) &= f(x), \\ &\vdots \\ u_{k+1}(x,t) &= J^{\alpha}[R(u_k) - A_k] \qquad k \geq 0. \end{aligned} \tag{7}$$

The Adomians polynomial can be calculated for all the types of nonlinearities as described in [10, 12, 20, 22] and are given by

$$\begin{aligned} A_0 &= F(u_0), \\ A_1 &= u_1 F'(u_0), \\ A_2 &= u_2 F'(u_0) + \frac{u_1^2}{2!} F''(u_0), \\ A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{u_1^3}{3!} F'''(u_0), \\ &\ddots \end{aligned}$$

where $N(u) = F(u)$ is the nonlinear function in Eq. (6). Finally, we approximate the solution u(x,t) by the series $\phi_N(t) = \sum_{n=0}^{N-1} u_n(x,t)$, and $u(x,t) = \lim_{N\to\infty} \phi_N(t)$.

4. Solution Procedure

In this section, we apply Adomian's Decomposition Method (ADM) to solve time- fractional Korteweg-de Vries equations. Numerical results are very encouraging.

Case.1 For $a = 30, b = 30, c = 10$ and $d = 1$, the equation is known as Lax's fifth order KdV Equation. Consider the Lax's Equation.

$$D_t^{\alpha} u + 30u^2 u_x + 30u_x u_{xx} + 10uu_{xxx} + u_{xxxxx} = 0, \tag{8}$$

with initial conditions

$$u(x,0) = 2k^2(2 - 3\ tanh^2(kx)).$$

Applying the recurrence relation defined in Eq. (7), we get

$$\begin{aligned} u_0(x,t) &= 2k^2(2 - 3\ tanh^2(kx)), \\ u_1(x,t) &= \frac{k^7 96\ sinh(kx)(7\ cosh^4(kx) + 30\ cosh^2(kx) - 45)}{cosh^7(kx)} \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned}$$

$$u_2(x,t) = \frac{-1}{cosh^1 3(kx)}(768\, k^1 2(-238\, cosh^1 1(kx) - 4443\, cosh^9(kx) + 70800\, cosh^7(kx)$$

$$-299520\, ksinh(kx)\, cosh^4(kx) + 661500\, ksinh(kx) cosh^2(kx) - 396900\, k\, sin(kx) - 291600\, kcosh^4(kx) +$$

$$25920\, ksinh(kx)\, cosh^6(kx) + 1680\, k\, sinh(kx)\, cosh^8(kx) + 445500\, cosh^3(kx) - 222750\, cosh(kx))\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

The solution in the series form is given by

$$\begin{aligned} u(x,t) = &\quad 2k^2(2-3\, tanh^2(kx)) + \frac{k^7 96\, sinh(kx)(7\, cosh^4(kx) + 30\, cosh^2(kx) - 45)}{cosh^7(kx)}\frac{t^\alpha}{\Gamma(\alpha+1)} \\ + &\quad \frac{-1}{cosh^1 3(kx)}(768\, k^1 2(-238\, cosh^1 1(kx) - 4443\, cosh^9(kx) + 70800\, cosh^7(kx) \\ - &\quad 299520\, ksinh(kx)\, cosh^4(kx) + 661500\, ksinh(kx) cosh^2(kx) \\ - &\quad 396900\, k\, sin(kx) - 291600\, kcosh^4(kx) + cosh^6(kx) + 1680\, k\, sinh(kx)\, cosh^8(kx) \\ + &\quad 445500\, cosh^3(kx) - 222750\, cosh(kx))\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \ldots. \end{aligned} \tag{9}$$

For the special case $\alpha = 1$, we obtain from Eq.(9):

$$\begin{aligned} u(x,t) = &\quad 2k^2(2-3\, tanh^2(kx)) + \frac{k^7 96\, sinh(kx)(7\, cosh^4(kx) + 30\, cosh^2(kx) - 45)}{cosh^7(kx)}\frac{t}{1!} \\ + &\quad \frac{-1}{cosh^1 3(kx)}(768\, k^1 2(-238\, cosh^1 1(kx) - 4443\, cosh^9(kx) + 70800\, cosh^7(kx) \\ - &\quad 299520\, ksinh(kx)\, cosh^4(kx) + 661500\, ksinh(kx) cosh^2(kx) \\ - &\quad 396900\, k\, sin(kx) - 291600\, kcosh^4(kx) + cosh^6(kx) + 1680\, k\, sinh(kx)\, cosh^8(kx) \\ + &\quad 7445500\, cosh^3(kx) - 222750\, cosh(kx))\frac{t^2}{2!} + \ldots. \end{aligned} \tag{10}$$

The exact solution of the fifth order Laxs Equation [17] is given by

$$u(x,t) = 2k^2(2 - 3tanh^2(kx - 56k^4 t)). \tag{11}$$

The results for the first four iteration of the Adomians Decomposition Method for the Laxs Equation Eq.(8) for $\alpha = 0.50$ and 1 and the exact solution Eq. (11) and the graph for $|u_{exact} - u_{approx}|$,where u_{approx} is the approximate solution obtained from the first four components of Adomians Decomposition Method for $\alpha = 1$ are shown in Figure 1.
Table.1 shows the numerical results for the first four terms of the Adomians Decomposition method for the Eq. (8) for $\alpha = 0.50$ and $\alpha = 1$ in comparison with exact solution Eq. (11), also Error is calculated as $|u_{exact} - u_{approx}|$,where u_{approx} is the approximate solution obtained for the first four components of Adomians Decomposition Method for $\alpha = 1$. The result shows the accuracy of ADM up to four decimal places by considering the first four terms of the approximate solution Eq. (10). Higher accuracy can be achieved by considering more terms of the series solution.

Figure 3 gives a comparison of the approximate solution Eq. (10) for different values of α.

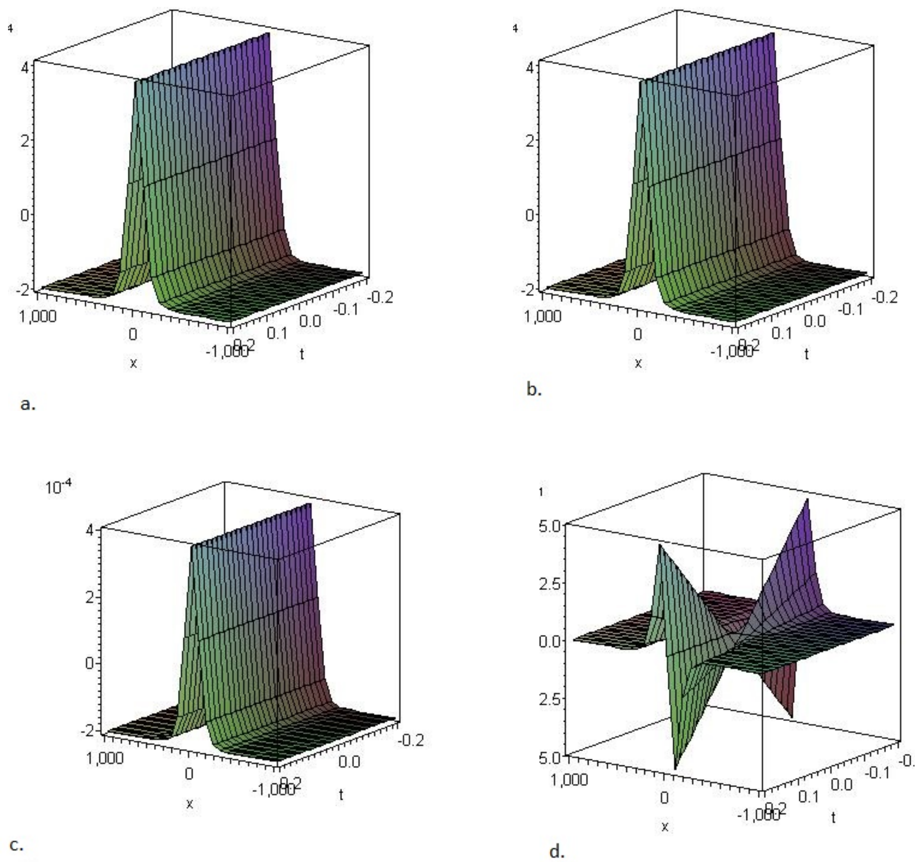


FIGURE 1. The surface shows solution $u(x,t)$ for $k = 0.01$ the Eq. (8) when (a) $\alpha = 0.5$, (b)$\alpha = 1$, (c) Exact solution Eq. (11), (d)$|u_{exact} - u_{approx}|$.

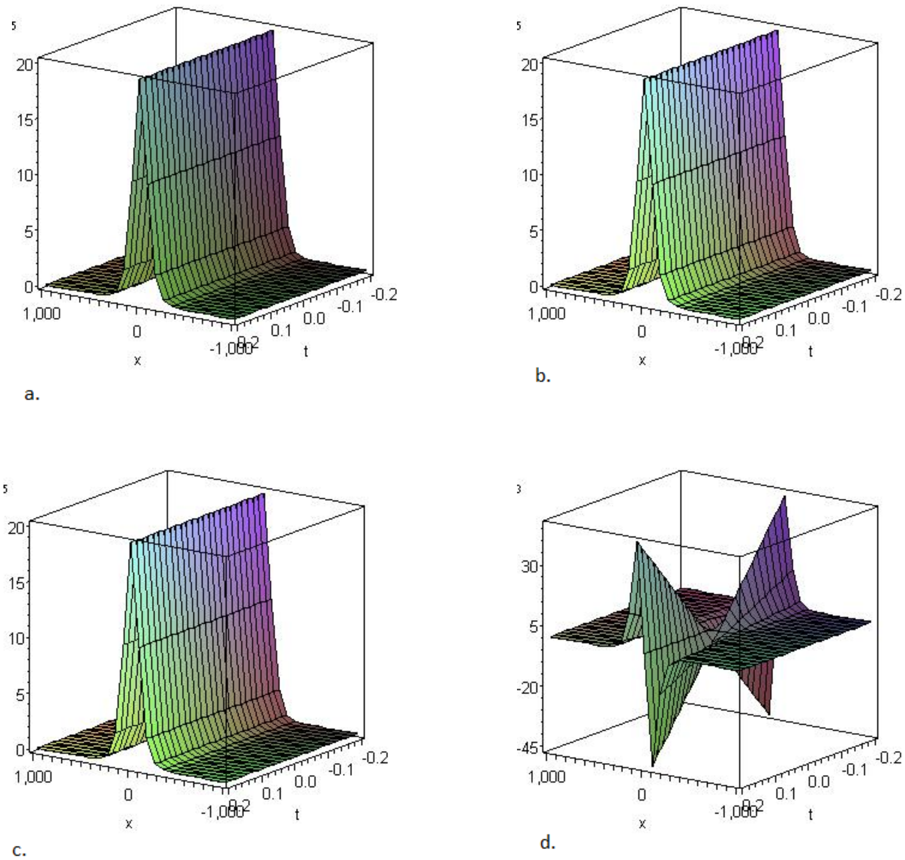
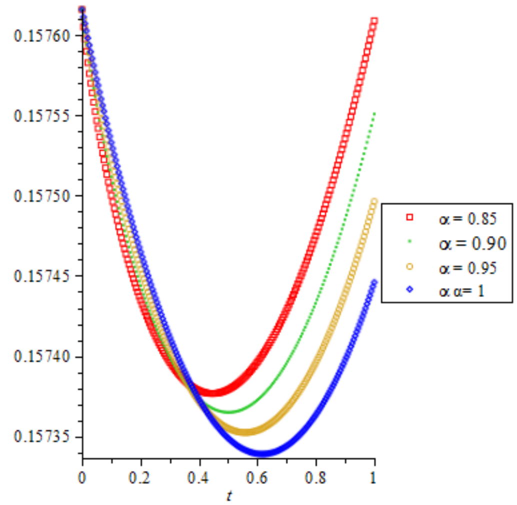
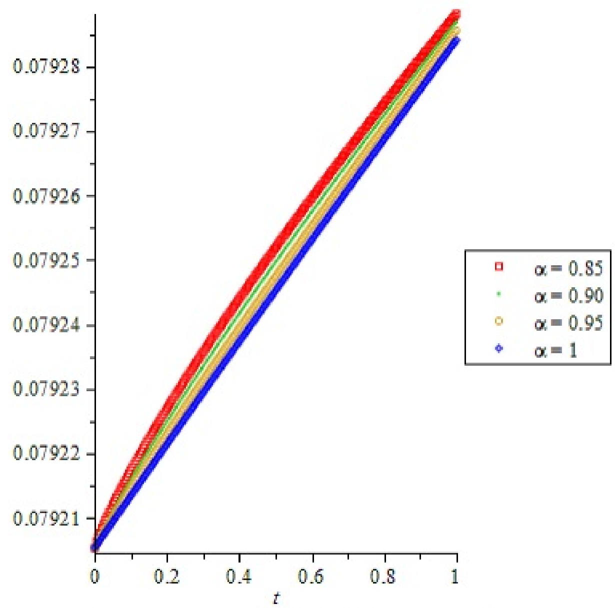


FIGURE 2. The surface shows solution $u(x,t)$ for $k = 0.01$ the Eq. (12) when (a) $\alpha = 0.5$, (b)$\alpha = 1$, (c) Exact solution Eq. (14), (d)$|u_{exact} - u_{approx}|$.



Comparison graph for approximate solution (4.3) for k=0.2 and x=1

FIGURE 3.



Comparison graph of (4.6) for different values of α

FIGURE 4.

Case.2 For $a = 45, b = 15, c = 15$ and $d = 1$, the equation is known as Sawada-Kotera fifth order KdV Equation. Consider the Sawada-Kotera Equation.

$$D_t^\alpha u + 45u^2 u_x + 15u_x u_{xx} + 15uu_{xxx} + u_{xxxxx} = 0, \tag{12}$$

with initial conditions

$$u(x,0) = 2k^2(sech^2(kx)),$$

applying the recurrence relation defined in Eq. (7), we get

$$u_0(x,t) = 2k^2(sech^2(kx)),$$

$$u_1(x,t) = \frac{k^7 84 sinh(kx)}{cosh^3(kx)} \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$\begin{aligned} u_2(x,t) = & \frac{1}{cosh^9(kx)}(512k^12(-720k cosh^2(kx)\ sinh(kx) - 345\ cosh^3(kx) + 66\ cosh^5(kx) + \\ + & 315\ cosh(kx) + 120\ k\ sinh(kx) cosh^4(kx) - 4\ cosh^7(kx) + 720\ k\ sinh(kx)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned} \tag{13}$$

$$\vdots$$

The solution in the series form is given by

$$\begin{aligned} u(x,t) = & \ 2k^2(sech^2(kx)) + \frac{k^7 84 sinh(kx)}{cosh^3(kx)} \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{cosh^9(kx)}(512k^12(-720k cosh^2(kx)\ sinh(kx) - \\ - & \ 345\ cosh^3(kx) + 66\ cosh^5(kx) + 315\ cosh(kx) + 120\ k\ sinh(kx) cosh^4(kx) - \\ - & \ 4\ cosh^7(kx) + 720\ k\ sinh(kx)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \dots . \end{aligned} \tag{14}$$

For the special case $\alpha = 1$, we obtain the from Eq. (13)

$$\begin{aligned} u(x,t) = 2k^2(sech^2(kx)) + \frac{k^7 84 sinh(kx)}{cosh^3(kx)} t + \frac{1}{cosh^9(kx)}(512k^12(-720k cosh^2(kx)\ sinh(kx) - 345 cosh^3(kx) + \\ + \quad & 66 cosh^5(kx) + \\ + \quad & cosh(kx) 315 + 120k sinh(kx)\ cosh^4(kx) - \\ - \quad & 4 cosh^7(kx) + 720k sinh(kx)) t^2 + \dots . \end{aligned} \tag{15}$$

The exact solution of the fifth order Sawada-Kotera Equation [17] is given by.

$$u(x,t) = 2k^2(sech^2(kx - 16k^4 t)). \tag{16}$$

The results for the first four iteration of the Adomians Decomposition Method for the fifth order Sawada-Kotera Equation Eq.(12) for $\alpha = 0.50$ and 1 and the exact solution Eq.(14) and the graph for $|u_{exact} - u_{approx}|$,where u_{approx} is the approximate solution obtained from the first four components of Adomians Decomposition Method for $\alpha = 1$ are shown in Figure 2.

Table.1 shows the numerical results for the first four terms of the Adomians Decomposition method for Eq.(12) for $\alpha = 0.50$ and 1 in comparison with exact solution Eq. (14), also Error is calculated as $|u_{exact} - u_{approx}|$,where u_{approx} is the approximate solution obtained for the first four components of Adomians Decomposition Method for $\alpha = 1$. The result shows the accuracy of ADM up to four decimal places by considering the first four terms of the approximate solution Eq. (13). Higher accuracy can be achieved by considering more terms of the series solution.The Figure.4 gives a comparison of the approximate solution Eq. (13) for different values of α.

TABLE 1.

t	x	$\alpha = 0.5$	$\alpha = 1$	u_{exact}	$\lvert u_{exact} - u_{approx} \rvert$
0.2	0.25	0.1595184136	0.1593408527	0.1594086287	0.6777600e-4
	0.50	0.1574539363	0.1574653814	0.1576310431	0.1656617e-3
	0.75	0.1542667411	0.1544596321	0.1547023186	0.2426865e-3
	1.00	0.1500522786	0.1503907619	0.1506794308	0.2886689e-3
0.4	0.25	0.1597779345	0.1593527078	0.1594162102	0.6350240e-4
	0.50	0.1575575066	0.1573731206	0.1576461292	0.2730086e-3
	0.75	0.1542073937	0.1542774632	0.1547246131	0.4471499e-3
	1.00	0.1498491048	0.1501464154	0.1507085040	0.5620886e-3
0.6	0.25	0.1600608355	0.1594365640	0.1594237429	0.1282110e-4
	0.50	0.1577047654	0.1573391275	0.1576611681	0.3220406e-3
	0.75	0.1542062702	0.1541334726	0.1547468627	0.6133901e-3
	1.00	0.1497108958	0.1499172761	0.1507375355	0.8202594e-3

TABLE 2.

t	x	$\alpha = 0.5$	$\alpha = 1$	u_{exact}	$\lvert u_{exact} - u_{approx} \rvert$
0.2	0.25	0.7982009936e-1	0.7980841449e-1	0.7983907898e-1	0.30664490e-4
	0.50	0.7924528293e-1	0.7922138993e-1	0.7928412302e-1	0.62733090e-4
	0.75	0.7828606788e-1	0.7825043558e-1	0.7834400446e-1	0.93568880e-4
	1.00	0.7696113036e-1	0.7691444541e-1	0.7703704455e-1	0.12259914e-3
0.4	0.25	0.7982779512e-1	0.7981632957e-1	0.7987366456e-1	0.57334990e-4
	0.50	0.7926136112e-1	0.7923731462e-1	0.7935889768e-1	0.12158306e-3
	0.75	0.7831021367e-1	0.7827405717e-1	0.7845749683e-1	0.18343966e-3
	1.00	0.7699287915e-1	0.7694530852e-1	0.7718706034e-1	0.24175182e-3
0.6	0.25	0.7983350342e-1	0.7982407766e-1	0.7990408237e-1	0.80004710e-4
	0.50	0.7927350409e-1	0.7925307647e-1	0.7942961173e-1	0.17653526e-3
	0.75	0.7832855415e-1	0.7829752336e-1	0.7856711378e-1	0.26959042e-3
	1.00	0.7701706486e-1	0.7697602669e-1	0.7733345622e-1	0.35742953e-3

5. Conclusion

Adomian's Decomposition Method (ADM) has been implemented to find appropriate solutions of generalized fifth order time-fractional Korteweg-de Vries equations. Numerical results coupled with graphical representations explicitly reveal the complete reliability and accuracy of the proposed algorithm.

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