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EXISTENCE OF AT LEAST ONE CONTINUOUS SOLUTION OF A COUPLED SYSTEM OF URYSOHN INTEGRAL EQUATIONS

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Abstract. In this work, we are concerning with a coupled system of nonlinear Urysohn functional integral equations. We study the existence of at least one continuous solution. The nonlinear Urysohn functional integral equation will be given as an special case. Some boundary value problems of coupled system of nonlinear Urysohn functional integro-differential equations will be studied as applications.

1. Introduction

It is known that integral equations have many useful applications in describing numerous events and problems of real world and the theory of integral equation is rapidly developing with the help of several tools of functional analysis, topology and fixed point theory [1]-[4],[6]-[8] and [10]-[13].

Let $\beta \in (0,1)$ and the function f be integrable on $[0,T]$. The Riemann-Liouville fractional order integral operator is given by the singular integral operator of convolution type [14]

$$I^{\beta} f(t) \ = \ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \ f(s) \ ds, \ t \in [0,T].$$

Let $\alpha \in (0,1)$ and $f \in AC[0,T]$. Then the Fractional order derivative is defined by the singular integro-differential operator [15]

$$D^{\alpha} \ f(t) \ = \ I^{1-\alpha} \ \frac{d}{dt} f(t)$$

Let $I = [0,1]$. Consider the coupled system of nonlinear Urysohn functional integral equations

$$x(t) \ = \ a_1(t) \ + \ \int_0^1 \ f_1(t,s,I^{\beta_1} y(s)) ds, \ t \in I \tag{1}$$

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$$y(t) \;=\; a_2(t) \;+\; \int_0^1 \; f_2(t,s,I^{\beta_2}x(s))ds,\;\; t\in I. \tag{2}$$

where I^{β_1} and I^{β_2} are integral operators of fractional orders β_1 and β_2.

The existence of at least one solution (x,y) of the coupled system (1)-(2) will be proved.
The special case, the nonlinear Urysohn functional integral equation

$$x(t) = a_1(t) + \int_0^1 \; f(t,s,I^{\beta}x(s))ds,\;\; t\in I \tag{3}$$

will be considered as an example.
Also the existence of the maximal and the minimal solution of (3) will be proved.

Finally, the coupled system of functional integro-differential equations

$$\frac{d}{dt}\,x(t) \;=\; a_1(t) \;+\; \int_0^1 \; f_1(t,s,D^{\alpha_1}y(s))ds,\;\; t\in I \tag{4}$$

$$\frac{d}{dt}\,y(t) \;=\; a_2(t) \;+\; \int_0^1 \; f_2(t,s,D^{\alpha_2}x(s))ds,\;\; t\in I \tag{5}$$

with the boundary conditions

$$x(0) \;=\; \gamma_1\; x(1), \;\;\text{ and } y(0) \;=\; \gamma_2\; y(1),\;\; \gamma_1,\;\; \gamma_2 \neq 1 \tag{6}$$

will be studied as an application.

2. Main results

Let $a_i \;:\; I=[0,1]\to R$ be continuous and $\sup_{t\in I}|a_i(t)| \;=\; a_i^*$.
Consider the following assumptions
(i) $f_i \;:\; I\times I\times R\to R$ are continuous in $t\in I$ for all $(s,x)\in I\times R$, measurable in $s\in I$ for all $(t,x)\in I\times R$ and continuous in $x\in R$ for all $(t,s)\in I\times I, i=1,2$.
(ii) There exist two integrable functions $m_i : I\times I\to R$ and two nonnegative constants $b_i,\; i=1,2$ such that

$$|f_i(t,s,x)| \leq |m_i(t,s)| + b_i|x|.$$

and

$$\sup_{t\in I}\int_0^1 m_i(t,s)ds \leq M_i.$$

Let X be the Banach space of all order pairs (x,y) with the norm

$$\|(x,y)\|_X = \|x\| + \|y\| = \sup_{t\in I}|x(t)| + \sup_{t\in I}|y(t)|.$$

Define the operator F by

$$F(x,y) = (T_1y, T_2x)$$

where

$$T_1y = a_1(t) + \int_0^1 \; f_1(t,s,D^{\alpha_1}y(s))ds,\;\; t\in I$$

and

$$T_2x = a_2(t) + \int_0^1 \; f_2(t,s,D^{\alpha_2}x(s))ds,\;\; t\in I$$

Define the set of functions

$$Q_r = \{u = (x, y) \in X \ : \ ||u|| \leq r, \ \|x\| \leq r_2, \|y\| \leq r_1, r_1 + r_2 = r\},$$

where

$$r_1 \ = \ \frac{(a_1^* \ + \ M_1)}{(1 - \frac{b_1}{\Gamma(\beta_1+2)})} \text{ and } \ r_2 \ = \ \frac{a_2^* \ + \ M_2}{(1 - \frac{b_2}{\Gamma(\beta_2+2)})}.$$

Definition 1. By a solution of the coupled system (1)-(2) we mean the ordered pair (x, y) such that $x, y \in C[0, T]$. This ordered pair satisfies the coupled system (1)-(2).

Now we prove some lemmas which will be used to prove the main Theorem.

Lemma 1. Let the assumptions (i)-(ii) be satisfied, then $F : Q_r \to Q_r$ and the set of functions FQ_r is uniformly bounded.

Proof. From our assumptions we have

$$\begin{aligned} |T_1 y(t)| &\leq |a_1(t)| + |\int_0^1 f_1(t, s, I^{\beta_1} y(s)) ds| \\ &\leq a_1^* + \int_0^1 m_1(t, s) ds \ + \ b_1 \|y\| \int_0^1 \int_0^s \frac{\theta^{\beta_1 - 1}}{\Gamma(\beta_1)} d\theta ds \\ &\leq a_1^* \ + \ M_1 + \frac{b_1 \ r_1}{\Gamma(\beta_1 + 2)} \leq r_1, \end{aligned}$$

then

$$\|T_1 y(t)\| \leq r_1.$$

Also

$$\begin{aligned} |T_2 x(t)| &\leq |a_2^*| + |\int_0^1 f_2(t, s, I^{\beta_2} x(s)) ds| \\ &\leq a_2^* + \int_0^1 m_2(t, s) ds \ + \ b_2 \ \|x\| \int_0^1 \int_0^s \frac{\theta^{\beta_2 - 1}}{\Gamma(\beta_2)} d\theta ds \\ &\leq a_1^* \ + \ M_2 + \frac{b_2 \ r_2}{\Gamma(\beta_2 + 2)} \leq r_2, \end{aligned}$$

then

$$\|T_2 x\| \leq r_2.$$

Now for $(x, y) \in Q_r$, we have

$$\begin{aligned} \|F(x, y)\| &= \|(T_1 y, T_2 x)\| = \|T_1 y\| + \|T_2 x\| \\ &\leq r_1 + r_2 = r. \end{aligned}$$

This proves that

$$F \ : \ Q_r \to Q_r,$$

and the set of function FQ_r is uniformly bounded.

Lemma 2. Let the assumptions (i)-(ii) be satisfied, then the set of functions FQ_r is equicontinuous.

Proof. Let $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| < \delta$, then

$$|T_1 y(t_2) - T_1 y(t_1)| \ = \ |a_1(t_2) - a_1(t_1) + \int_0^1 f_1(t_2, s, I^{\beta_1} \ y(s)) ds - \int_0^1 f_1(t_1, s, I^{\beta_1} y(s)) ds|$$

$$\leq |a_1(t_2) - a_1(t_1)| + \int_0^1 |f_1(t_2, s, I^{\beta_1} y(s)) - f_1(t_1, s, I^{\beta_1} y(s))| ds$$

Also

$$|T_2x(t_2) - T_2x(t_1)| = |a_2(t_2) - a_2(t_1) + \int_0^1 f_2(t_2, s, I^{\beta_2} x(s)) ds - \int_0^1 f_2(t_1, s, I^{\beta_2} y(s)) ds|$$

$$\leq |a_2(t_2) - a_2(t_1)| + \int_0^1 |f_2(t_2, s, I^{\beta_2} y(s)) - f_2(t_1, s, I^{\beta_2} y(s))| ds$$

Now

$$\begin{aligned} \|F(x(t_2), y(t_2)) - F(x(t_1), y(t_1))\| &= \|(T_1y(t_2), T_2x(t_2)) - (T_1y(t_1), T_2x(t_1))\| \\ &= \|(T_1y(t_2) - T_1y(t_1), T_2x(t_2) - T_2x(t1))\| \\ &= \|(T_1y(t_2) - T_1y(t_1)\| + \|T_2x(t_2) - T_2x(t1))\|. \end{aligned}$$

$$\leq |a_1(t_2) - a_1(t_1)| + \int_0^1 |f_1(t_2, s, I^{\beta_1} y(s)) - f_1(t_1, s, I^{\beta_1} y(s))| ds$$

$$+ |a_2(t_2) - a_2(t_1)| + \int_0^1 |f_2(t_2, s, I^{\beta_2} y(s)) - f_2(t_1, s, I^{\beta_2} y(s))| ds$$

Then the set of functions FQ_r is equicontinuous.

Lemma 3. Let the assumptions (i)-(ii) be satisfied, then the operator F is continuous in Q_r.

Proof. Let $(x_n, y_n) \in Q_r$ such that $(x_n, y_n) \to (x_0, y_0) \in Q_r$, then

$$F(x_n(t), y_n(t)) = (a_1(t) + \int_0^1 f_1(t, s, I^{\beta_1} y_n(s)) ds,\ a_2(t) + \int_0^1 f_2(t, s, I^{\beta_2} x_n(s) ds)$$

and

$$\lim_{n\to\infty} T_1y_n(t) = a_1(t) + \lim_{n\to\infty} \int_0^t f_1(t, s, I^{\beta_1} y_n(t)(s)) ds.$$

From the properties of the fractional calculus we have

$$I^{\beta_1} y_n(t) \to I^{\beta_1} y_o(t),$$

then from our assumptions (i)-(ii) we have

$$f_1(t, s, I^{\beta_1} y_n(t)\ y_n(s)) \to f_1(t, s, I^{\beta_1} y_n(t) y_o(s))$$

and

$$|f_1(t, s, I^{\beta_1} y_n(t) y_n(s))| \leq m_1(t, s) + \frac{b_1 r_1}{\Gamma(\beta_1 + 2)}.$$

Applying Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_0^1 f_1(t, s, I^{\beta_1} y_n(t)(s)) ds = \int_0^1 f_1(t, s, I^{\beta_1} y_n(t) y_0(s)) ds.$$

and

$$\lim_{n\to\infty} T_1y_n(t) = a_1(t) + \int_0^1 f_1(t, s, I^{\beta_1} y_o(s)) ds = T_1\ y_o(t).$$

By the same way we have

$$\lim_{n\to\infty} T_2x_n(t) = a_2(t) + \int_0^t f_2(t,s,I^{\beta_2}\ x_o(\varphi(s)))ds = T_2x_o(t).$$

Now we can deduced that

$$F(x_n(t), y_n(t)) \to F(x_0(t), y_0(t))$$

which implies that the operator F is continuous in Q_r.

Now for the existence of at least one solution of the coupled system of integral equations (1)-(2) we have the following theorem.

Theorem 1. Let the assumptions (i)-(ii) be satisfied. If

$$\frac{b_i}{\Gamma(\beta_i + 2} < 1,\ i = 1, 2,$$

then the coupled system of the integral equations (1)-(2) has at least one solution.
Proof. From lemmas (1)-(3) we deduced that F satisfied the axioms of Schauder fixed point theorem, then the operator F has a fixed point $(x, y) \in X$, then the coupled system of integral equations (1)-(2) has at least one continuous solution.

3. Urysohn functional integral equation

Let

$$x = y,\ f_1 = f_2 = f,\ \beta_1 = \beta_2 = \beta \text{ and } a_1 = a_2 = a,$$

then the coupled system (1)-(2) will be reduced to the Urysohn functional integral equation

$$x(t)\ =\ a(t)\ +\ \int_0^1\ f(t,s,I^{\beta}x(s))ds,\ t \in I \tag{7}$$

and we have the following corollary.

Corollary 1. Let $x = y,\ f_1 = f_2 = f,\ \beta_1 = \beta_2\ = \beta$ and $a_1 = a_2 = a$ in Theorem 1. Let the assumptions of Theorem 1. be satisfied , then the integral equation (7) has at least one continuous solution $x \in C[0, T]$.

4. Coupled system of Hammerstein functional integral equations

Let

$$f_1(t,s,y)\ =\ k_1(t,s)\ g_1(s,y(s)),\ \text{and}\ f_2(t,s,x)\ =\ k_2(t,s)\ g_2(s,x(s)).$$

Then the coupled system (1)-(2) will be the coupled system of Hammerstein functional integral equations (5)-(6)

$$x(t)\ =\ a_1(t)\ +\ \int_0^1\ k_1(t,s)g_1(s,I^{\beta_1}y(s))ds,\ t \in I \tag{8}$$

$$y(t)\ =\ a_2(t)\ +\ \int_0^1\ k_2(t,s)g(s,I^{\beta_2}x(s))ds,\ t \in I. \tag{9}$$

Consider the following assumptions
(iii) $g_i : I \times R \to R$ are measurable in $s \in I$ for all $x \in R$ and continuous in $x \in R$ for all $s \in I$ and there exist two functions $m_i^* \in L_1[0,T]$ and two positive constants $b_i^* > 0, i = 1,2$ such that

$$\begin{aligned} |g_1(t,y)| &\leq m_1^*(t) + b_1^*|x| \\ |g_2(t,x)| &\leq m_2^*(t) + b_2^*|y| \end{aligned}$$

(iv) $k_i : I \times R \to R$ are continuous in $t \in I$ for every $s \in I$ and measurable in $s \in I$ for all $t \in I$ such that

$$\sup_{t \in I} \int_0^1 |k_i(t,s)|\ |m_i^*(s)| ds \leq K_i,\ t \in I.$$

Now we have the following corollary.

Corollary 2. Let the assumptions (iii)-(iv) be satisfied.

$$\frac{b_i^*}{\Gamma(\beta_i + 2)} < 1,\ i = 1,2,$$

then the coupled system of integral equations (8)-(9) has at least one continuous solution.

Let

$$x = y,\ g_1 = g_2 = g,\ \beta_1 = \beta_2 = \beta,\ a_1 = a_2 = a \text{ and } k_1 = k_2,$$

then the coupled system (8)-(9) reduced to the Hammerstein functional integral equation

$$x(t) = a_1(t) + \int_0^1 k(t,s)\ g_1(s, y(\varphi_1(s)))ds,\ t \in I \tag{10}$$

and we have the following corollary

Corollary 3. Let $x = y,\ g_1 = g_2 = g,\ \beta_1 = \beta_2 = \beta,\ a_1 = a_2 = a$ and $k_1 = k_2$. If the assumption of Corollary 2 are satisfied then the functional integral equation (10) has at least one continuous solution.

5. Maximal and minimal solutions

Definition 2. Let q be a solution of (3), then q is said to be a maximal solution of (3) if for every solution of (3) satisfies the inequality $x(t) < q(t)\ t \in I$.
A minimal solution s can be defined by similar way by reversing the above inequality i.e. $x(t) > s(t)\ t \in I$.

The following lemma will be used later.

Lemma 4. Let the assumptions of Corollary 1 be satisfied. Let $u,\ v$ be continuous functions on I satisfying

$$u(t) \leq a(t) + \int_0^1 f(t,s,I^\beta u(s))ds, \quad t \in I.$$

$$v(t) \geq a(t) + \int_0^1 f(t, s, I^\beta v(s))ds, \quad t \in I,$$

and one of them is strict. If $f(t, s, x)$ is monotonic nondecreasing in x, then

$$x(t) < y(t), \quad t \in I. \tag{11}$$

Proof. Let the conclusion (11) be false, then there exists t_1 such that

$$u(t_1) = v(t_1), \quad t_1 > 0$$

and

$$u(t) < v(t), \quad 0 < t < t_1.$$

From the monotonicity of f, we get

$$\begin{aligned} u(t_1) &\leq a(t_1) + \int_0^1 f(t_1, s, I^\beta u(s))ds \\ &< a(t_1) + \int_0^1 f(t_1, s, I^\beta v(s))ds \\ &< v(t_1), \end{aligned}$$

which contradicts the fact $u(t_1) = v(t_1)$, then $u(t) < v(t), \quad t \in I$.

For the existence of the maximal and minimal solutions we have the following theorem.

Theorem 2. Let the assumption of Theorem 1 be satisfied. If $f(t, s, x)$ is nondecreasing in x on I, then there exist maximal and minimal solutions of the integral equation (3).

Proof. Firstly we shall prove the existence of the maximal solution of (3).
Let $\epsilon > 0$ be given and consider the integral equation

$$u_\epsilon(t) = a(t) + \int_0^1 f_\epsilon(t, sI^\beta u_\epsilon(s))ds, \quad t \in I. \tag{12}$$

where

$$f_\epsilon(t, s, I^\beta u_\epsilon(s) = f(t, s, I^\beta u(s)) + \epsilon$$

Clearly the function $f_\epsilon(t, s, I^\beta u_\epsilon(s)$ satisfies the assumptions of Theorem 1. and therefore equation (13) has at least one continuous solution $u_\epsilon(t) \in C[0, T]$.
Let ϵ_1 and ϵ_2 such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$, then

$$\begin{aligned} u_{\epsilon_2}(t) &= a(t) + \int_0^1 f_{\epsilon_2}(t, s, I^\beta u_{\epsilon_2}(s)ds \\ &= a(t) + \int_0^1 (f(t, s, I^\beta u_{\epsilon_2}(s)) + \epsilon_2)ds \end{aligned} \tag{13}$$

and

$$\begin{aligned} u_{\epsilon_1}(t) &= a(t) + \int_0^1 f_{\epsilon_1}(t, s, I^\beta u_{\epsilon_1}(s) ds \\ &= a(t) + \int_0^1 (f(t, s, I^\beta u_{\epsilon_1}(s)) + \epsilon_1) ds \\ &> a(t) + \int_0^1 (f(t, s, I^\beta u_{\epsilon_1}(s)) + \epsilon_2) ds \end{aligned} \tag{14}$$

Applying Lemma 4. on (14) and (15), we have

$$u_{\epsilon_2}(t) < u_{\epsilon_1}(t), \quad t \in I.$$

As shown before the family of functions $x_\epsilon(t)$ is equi-continuous and uniformly bounded. Hence by Arzela- Ascoli Theorem, there exists a decreasing sequence ϵ_n such that $\epsilon \to 0$ as $n \to \infty$, and

$$\lim_{n\to\infty} u_{\epsilon_n}(t)$$

exist uniformly in I.
Denote this limit by q, then from the continuity of the function $f_\epsilon(t, s, I^\beta u_\epsilon(s)$ in the third argument, we get

$$q(t) = \lim_{n\to\infty} u_{\epsilon_n}(t) = a(t) + \int_0^1 f(t, s, I^\beta u_\epsilon(s) ds$$

which implies that q is a solution of (3).
Finally, we shall show that q is the maximal solution of (3). To do this, let u be any solution of (3). Then

$$\begin{aligned} u_\epsilon(t) &= a(t) + \int_0^1 f_\epsilon(t, s, I^\beta u_\epsilon(s) ds \\ &= a(t) + \int_0^1 (f(t, s, I^\beta u_\epsilon(s)) + \epsilon) ds \\ &> a(t) + \int_0^1 f(t, s, I^\beta u_\epsilon(s) ds \end{aligned}$$

Also applying Lemma 4. we have

$$u(t) = a(t) + \int_0^1 f(t, s, I^\beta u_\epsilon(s) ds \Rightarrow x(t) < u_\epsilon(t) \quad for \;\; t \in I,$$

from the uniqueness of the maximal solution [14], it is clear that $u_\epsilon(t)$ tends to $q(t)$ uniformly in $t \in I$ as $\epsilon \to 0$.
By similar way as done above we set

$$f_\epsilon(t, s, I^\beta u_\epsilon(s) = f(t, s, I^\beta u(s) - \epsilon.$$

and prove the existence of the minimal solution.

6. Boundary value problems

Let $\alpha \in (0,1]$. Consider the boundary value problem of the functional integro-differential equation

$$\frac{d}{dt}x(t) \ = \ a(t) \ + \ \int_0^1 \ f(t,s,D^{\alpha}x(s))ds, \ t \in [0,1] \tag{15}$$

with the boundary condition

$$x(0) \ = \ \gamma \, x(1), \ \gamma \neq 1. \tag{16}$$

Let the assumptions of Corollary 1 be satisfied.
Letting $\frac{d}{dt}x(t) \ = \ y(t)$, then there exists at least one solution of the boundary value problem (15)-(16) is given by

$$x(t) \ = \ \frac{\gamma}{1-\gamma} \int_0^1 \ y(s)ds \ + \ \int_0^t \ y(s)ds \tag{17}$$

where y is the solution of the nonlinear Urysohn functional integral equation

$$y(t) \ = \ a(t) \ + \ \int_0^1 \ f(t,s,I^{1-\alpha}y(s))ds, \ t \in [0,1].$$

Consider now the boundary value problem of the coupled system of functional integro-differential equations (4)-(6).
Let the assumptions of Theorem 1 be satisfied.
Letting

$$\frac{d}{dt}x(t) \ = \ u(t) \text{ and } \frac{d}{dt}y(t) \ = \ v(t),$$

then there exits at least one solution (x,y) of the boundary value problem of the coupled system of functional integro-differential equations (4)-(6) is given by

$$x(t) \ = \ \frac{\gamma_1}{1-\gamma_1} \int_0^1 \ u(s)ds \ + \ \int_0^t \ u(s)ds$$

and

$$y(t) \ = \ \frac{\gamma_2}{1-\gamma_2} \int_0^1 \ v(s)ds \ + \ \int_0^t \ v(s)ds$$

where $u, \ v$ are the solution of the coupled system of the integral equations

$$u(t) \ = \ a(t) \ + \ \int_0^1 \ f(t,s,I^{1-\alpha_1}v(s))ds, \ t \in [0,1].$$

and

$$v(t) \ = \ a(t) \ + \ \int_0^1 \ f(t,s,I^{1-\alpha_2}u(s))ds, \ t \in [0,1].$$

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