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# INCLUSION RELATIONS FOR A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS BASED ON THE DZIOK-RAINA OPERATOR

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Abstract. The main objective of this paper is to obtain a set of inclusion relations for a subclass of normalized analytic functions using the Dziok-Raina operator. Special cases of these inclusion relations are also discussed.

## 1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}), \tag{1}$$

normalized by $f(0) = 0 = f'(0) - 1$ and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{U}$. The class of starlike functions $\mathcal{S}^*$ and convex functions $\mathcal{C}$ are well known subclasses of $\mathcal{S}$.

In 1993, Goodman [9, 10] introduced the concept of uniform convexity and uniform starlikeness of functions in $\mathcal{A}$. The classes conssisting of uniformly convex and uniformly starlike functions are denoted by $\mathcal{UCV}$ and $\mathcal{UST}$ respectively. Further, Rønning [17] introduced the class $\mathcal{S}_P$, the class of parabolic starlike functions.

Two interesting subclasses of $\mathcal{S}$, denoted by $k$-$\mathcal{UCV}$ and $k$-$\mathcal{ST}$ consisting, respectively, of functions which are $k$ - uniformly convex and $k$-uniformly starlike in $\mathbb{U}$, were studied by Kanas and Wisniowska [13, 14] whose analytic characterizations are as follows:

$$k-\mathcal{UCV} := \left\{ f \in \mathcal{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right|, 0 \le k < \infty \quad (z \in \mathbb{U})\right\} \text{ and}$$

$$k-\mathcal{ST} := \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right|, 0 \le k < \infty \quad (z \in \mathbb{U})\right\}.$$

We note that 1 - $\mathcal{UCV} = \mathcal{UCV}$ and 1 - $\mathcal{ST} = \mathcal{S}_P$.

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For functions $f$ of the form (1), if $f \in k-\mathcal{UCV}$, then the following holds true(cf. [13]):

$$|a_n| \leq \frac{(P_1)_{n-1}}{n!}, \quad n \in \mathbb{N}\backslash\{1\}, \tag{2}$$

where $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & k=0 \\ a(a+1)(a+2)\dots(a+k-1) & k=\mathbb{N} \end{cases}$$

and $P_1 = P_1(k)$ is the coefficient of $z$ in the function

$$p_k(z) = 1 + \sum_{n=2}^{\infty} P_n(k) z^n \tag{3}$$

which is the extremal function for the class $\mathcal{P}(p_k)$ related to the class $k$-$\mathcal{UCV}$ by the range of the expression $1 + \frac{zf''(z)}{f'(z)}$ $(z \in \mathbb{U})$.
Similarly, if $f \in \mathcal{A}$ of the form (1) belongs to the class $k$-$\mathcal{ST}$, then (cf. [14])

$$|a_n| \leq \frac{(P_1)_{n-1}}{(n-1)!}, \quad n \in \mathbb{N}\backslash\{1\} \tag{4}$$

where $P_1 = P_1(k)$ is as above, by (3).
A function $f \in \mathcal{A}$ is said to be in the class $\Re^{\tau}(A,B)$, $(\tau \in \mathbb{C}\backslash\{0\}$, $-1 \leq B < A \leq 1)$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau - B[f'(z)-1]}\right| < 1 \quad (z \in \mathbb{U}).$$

This class $\Re^{\tau}(A,B)$ was introduced earlier by Dixit and Pal [3].
The subclass $\mathcal{T} \subset \mathcal{A}$ consisting of univalent functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0), \tag{5}$$

was studied by Silverman [19].

Aqlan et al.[1] studied the class $\mathcal{U}(k,\alpha,\beta)$ of functions $f(z)$ with negative coefficients of the form (5)(see also [12]). In this paper, we consider the subclass $\mathcal{U}(k,\alpha,\beta)$ of $\mathcal{A}$ and obtain the corresponding inclusion relations.

For $0 \leq \alpha \leq 1$, $0 \leq \beta < 1$ and $k \geq 0$, let $\mathcal{U}(k,\alpha,\beta)$ be a subclass of $\mathcal{A}$ consisting of functions of the form (1) that satisfy the condition

$$\Re\left(\frac{zf'(z)+\alpha z^2 f''(z)}{(1-\alpha)f(z)+\alpha z f'(z)}\right) \geq k\left|\frac{zf'(z)+\alpha z^2 f''(z)}{(1-\alpha)f(z)+\alpha z f'(z)} - 1\right| + \beta.$$

In particular, the class $\mathcal{U}(1,\alpha,\beta)$ was studied by Suchithra et al. [25].
The Dziok-Raina operator $W_q^p[\alpha_1]$ was introduced by Dziok and Raina in [6], motivated by the Wright's generalized hypergeometric function as below.
For $\alpha_i \in \mathbb{C}(\frac{\alpha_i}{A_i} \neq 0,-1,-2,...,A_i > 0; i=1,2,...,p)$ and $\beta_i \in \mathbb{C}(\frac{\beta_i}{B_i} \neq 0,-1,-2,...,B_i > 0; i=1,2,...,q)$ such that $1+\sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \geq 0$, Wright's generalized hypergeometric function ${}_p\psi_q(z)$ ([24],[22]) is defined by

$${}_p\psi_q[z] = {}_p\,\psi_q[(\alpha_i,A_i)_{1,p},(\beta_i,B_i)_{1,q};z] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i+nA_i)}{\prod_{i=1}^{q}\Gamma(\beta_i+nB_i)} \frac{z^n}{n!} \tag{6}$$

which is analytic for bounded values of $|z|$.
In particular, if $A_i = 1$, $B_i = 1$, ${}_p\psi_q[z]$ reduces to the the generalized hypergeometric function ${}_pF_q[z]$ given by

$$ {}_PF_q[z] = {}_P F_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\beta_i)_n} \frac{z^n}{n!}, \quad p \leq 1 + q. \tag{7} $$

In view of (6), the Dziok-Raina operator [6] (see also [5], [7], [15], [16], [20]))

$$ W_q^p[\alpha_1] = W_q^p[(\alpha_i, A_i)_{1,p}; (\beta_i, B_i)_{1,q}] : \mathcal{S} \to \mathcal{S} $$

is defined by

$$ W_q^p[\alpha_1]f(z) = z \frac{\prod_{i=1}^{q} \Gamma(\beta_i)}{\prod_{i=1}^{p} \Gamma(\alpha_i)} \Big( {}_p\psi_q[(\alpha_i, A_i)_{1,p}, (\beta_i, B_i)_{1,q}; z] \Big) * f(z), $$

where * denotes convolution (Hadamard product) of two functions.
For $f(z)$ of the form (1), we have

$$ W_q^p[\alpha_1]f(z) = z + \sum_{n=2}^{\infty} a_n \Omega_n z^n, \; z \in \mathbb{U}, \tag{8} $$

where

$$ \Omega_n = \frac{\frac{\prod_{i=1}^{p} \Gamma(\alpha_i + (n-1)A_i)}{\Gamma(\alpha_i)}}{\frac{\prod_{i=1}^{q} \Gamma(\beta_i + (n-1)B_i)}{\Gamma(\alpha_i)}} \frac{1}{(n-1)!}, \; n \geq 2. \tag{9} $$

Taking $A_i = 1 (i = 1, 2, ..., p)$ and $B_i = 1 (i = 1, 2, ..., q)$ the linear operator $W_q^p[\alpha_1]$ given by(7) reduces to the Dziok -Srivastava operator $H_q^p[\alpha_1]$ [4], which inturn contains many other operators as special cases, such as the Hohlov operator [11], the Carlson - Shaffer operator [2], the Ruscheweyh derivative operator [18] denoted by $I$, $L$ and $D$ respectively as detailed below.

$$ I(\alpha_1, \alpha_2; \beta_1)f(z) = H_1^2(\alpha_1, \alpha_2; \beta_1)f(z) \tag{10} $$

$$ L(\alpha_1, ; \beta_1)f(z) = H_1^2(\alpha_1, 1; \beta_1)f(z) \tag{11} $$

$$ D^\lambda f(z) = H_1^2(\lambda + 1, 1; 1)f(z). \tag{12} $$

Motivated by the works of Srivastava et al.[23], Gangadharan et al. [8], Sivasubramanian et al. [21], in this paper by making use of the linear operator defined by (8), we establish a number of relations between the classes $k$-$\mathcal{UCV}$, $k$-$\mathcal{ST}$, $\Re^\tau(A, B)$ and $\mathcal{U}(k, \alpha, \beta)$.
In order to prove the main results, we need the following lemmas.
**Lemma 1**[1, 25] A function $f \in \mathcal{A}$ of the form (1) belongs to the class $\mathcal{U}(k, \alpha, \beta)$ if

$$ \sum_{n=2}^{\infty} [n(1 + k(1-\alpha) - \alpha\beta) + n(n-1)\alpha(1+k) - (k+\beta)(1-\alpha)]|a_n| \leq 1 - \beta. \tag{13} $$

**Lemma 2**[3] A function $f \in \Re^\tau(A, B)$ is of form (1), then

$$ |a_n| \leq (A - B)\frac{|\tau|}{n}, \quad n \in \mathbb{N}\backslash\{1\}. \tag{14} $$

The result is sharp.

## 2. Main Results

**Theorem 1** Suppose that $\alpha_i \in \mathbb{C}\backslash\{0\}(i = 1, ..., p)$, $\Re(\beta_i) > 0(i = 1, ..., q)$ and that $\Re(\sum_{i=1}^{q}(\beta_i)) > \sum_{i=1}^{p} |\alpha_i| + p - q$. If $f \in \Re^\tau(A, B)$ of the form (1), and let the inequality

$$\begin{aligned}
&(1 + k(1 - \alpha) - \alpha\beta)\Big[{}_p\psi_q((|\alpha_i|, A_i)_{1,p}, (\Re(\beta_i), B_i)_{1,q}; 1)\Big] \\
&\quad + \alpha(1 + k)\frac{|\alpha_1|...|\alpha_p|}{\Re(\beta_1)...\Re(\beta_q)}\Big[{}_p\psi_q((|\alpha_i| + 1, A_i)_{1,p}, (\Re(\beta_i) + 1, B_i)_{1,q}; 1)\Big] \\
&\quad - \frac{(k + \beta)(1 - \alpha)\Re(\beta_1)...\Re(\beta_q)}{|\alpha_1|...|\alpha_p|}\Big[{}_p\psi_q((|\alpha_i| - 1, A_i)_{1,p}, (\Re(\beta_i) - 1, B_i)_{1,q}; 1)\Big] \\
&\le (1 - \beta)\frac{1}{(A - B)|\tau|} + (1 + k(1 - \alpha) - \alpha\beta) \\
&\quad - (k + \beta)(1 - \alpha)\frac{\Re(\beta_1)...\Re(\beta_q)}{|\alpha_1|...|\alpha_p|}\Big[1 + \frac{(|\alpha_1| - 1)_{A_1}...(|\alpha_p| - 1)_{A_p}}{(\Re(\beta_1) - 1)_{B_1}...(\Re(\beta_q) - 1)_{B_q}}\Big]
\end{aligned} \tag{15}$$

hold. Then $W_q^p[|\alpha_1|](f(z)) \in \mathcal{U}(k, \alpha, \beta)$.

**Proof.** Let $f$ of the form (1) belong to the class $\Re^\tau(A, B)$ . In view of Lemma 1 and (8), it suffices to show that

$$\sum_{n=2}^{\infty}[n(1+k(1-\alpha)-\alpha\beta)+n(n-1)\alpha(1+k)-(k+\beta)(1-\alpha)]\,|\Omega_n a_n| \le 1-\beta, \tag{16}$$

where the coefficients $\Omega_n(n \in \mathbb{N}\backslash\{0\})$ are given by the equation(9). Using (14) and the relation $|(a)_{n-1}| \le (|a|)_{n-1}$, we deduce that

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n(1 + k(1 - \alpha) - \alpha\beta) + n(n - 1)\alpha(1 + k) - (k + \beta)(1 - \alpha)] \\
&\qquad \times \left|\frac{(\alpha_1)_{A_1(n-1)}...(\alpha_p)_{A_p(n-1)}}{(\beta_1)_{B_1(n-1)}...(\beta_q)_{B_q(n-1)}}\frac{1}{(n - 1)!}a_n\right| \\
&\le (A - B)|\tau|\Big[(1 + k(1 - \alpha) - \alpha\beta)\sum_{n=2}^{\infty}\frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}}\frac{1}{(n - 1)!} \\
&\quad + (1 + k)\alpha\sum_{n=2}^{\infty}\frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}}\frac{1}{(n - 2)!} \\
&\quad - (k + \beta)(1 - \alpha)\sum_{n=2}^{\infty}\frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}}\frac{1}{(n)!}\Big]
\end{aligned}$$

$$= (A-B)|\tau|\Big[(1+k(1-\alpha)-\alpha\beta)[{}_p\psi_q((|\alpha_i|,A_i)_{1,p},(\Re(\beta_i),B_i)_{1,q};1)-1]$$
$$+(1+k)\alpha\frac{(|\alpha_1|)...(|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)}[{}_p\psi_q((|\alpha_i|+1,A_i)_{1,p},(\Re(\beta_i)+1,B_i)_{1,q};1)]$$
$$-(k+\beta)(1-\alpha)\frac{\Re(\beta_1)...\Re(\beta_q)}{(|\alpha_1|)...(|\alpha_p|)}[{}_p\psi_q((|\alpha_i|-1,A_i)_{1,p},(\Re(\beta_i)-1,B_i)_{1,q};1)$$
$$-1-\frac{(|\alpha_1-1|)_{A_1}...(|\alpha_p|-1)_{A_p}}{\Re(\beta_1-1)_{B_1}...\Re(\beta_q-1)_{B_q}}]\Big]$$
$$\leq 1-\beta.$$

This completes the proof of Theorem 1 by virtue of (15).

With $A_i = 1$ , $B_i = 1$ we have,

**Corollary 1** Suppose that $\alpha_i \in \mathbb{C}\backslash\{0\}(i=1,...,p)$, $\Re(\beta_i) > 0(i=1,...,q)$ and that $\Re(\sum_{i=1}^{q}(\beta_i)) > \sum_{i=1}^{p}|\alpha_i| + p - q$. If $f \in \Re^{\tau}(A,B)$ of the form (1) and let the inequality

$$(1+k(1-\alpha)-\alpha\beta)\Big[{}_pF_q(|\alpha_1|,...,|\alpha_p|,\Re(\beta_1),...,\Re(\beta_q);1)\Big]$$
$$+\alpha(1+k)\frac{|\alpha_1|...|\alpha_p|}{\Re(\beta_1)...\Re(\beta_q)}\Big[{}_pF_q(|\alpha_1|+1,...,|\alpha_p|+1,\Re(\beta_1)+1,...,\Re(\beta_q)+1;1)\Big]$$
$$-\frac{(k+\beta)(1-\alpha)\Re(\beta_1)...\Re(\beta_q)}{|\alpha_1|...|\alpha_p|}\Big[{}_pF_q(|\alpha_1|-1,...,|\alpha_p|-1,\Re(\beta_1)-1,...,\Re(\beta_q)-1;1)\Big]$$
$$\leq (1-\beta)\frac{1}{(A-B)|\tau|}+(1+k(1-\alpha)-\alpha\beta)$$
$$-(k+\beta)(1-\alpha)\frac{\Re(\beta_1)...\Re(\beta_q)}{|\alpha_1|...|\alpha_p|}\Big[1+\frac{(|\alpha_1|-1)...(|\alpha_p|-1)}{(\Re(\beta_1)-1)...(\Re(\beta_q)-1}\Big]$$

hold. Then ${}_pF_q(f(z)) \in \mathcal{U}(k,\alpha,\beta)$.

**Theorem 2** Suppose that $\alpha_i \in \mathbb{C}\backslash\{0\}(i=1,...,p)$, $\Re(\beta_i) > 0(i=1,...,q)$ and that $\Re(\sum_{i=1}^{q}(\beta_i)) > \sum_{i=1}^{p}|\alpha_i| + p - q$. If $f \in \mathcal{S}$ of the form (1), and let the inequality

$$\alpha(1+k)\frac{|\alpha_1|...|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)}\frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{(\Re(\beta_1)+1)...(\Re(\beta_q)+1)}\frac{(|\alpha_1|+2)...(|\alpha_p|+2)}{(\Re(\beta_1)+2)...(\Re(\beta_q)+2)}$$
$$\times\Big[{}_p\psi_q((|\alpha_i|+3,A_i)_{1,p},(\Re(\beta_i)+3,B_i)_{1,q};1)\Big]$$
$$+[(1+k)(1+5\alpha)-\alpha(k+\beta)]\frac{|\alpha_1|...|\alpha_p|}{\Re(\beta_1)...\Re(\beta_q)}\frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{(\Re(\beta_1)+1)...(\Re(\beta_q)+1)}$$
$$\times\Big[{}_p\psi_q((|\alpha_i|+2,A_i)_{1,p},(\Re(\beta_i+2),B_i)_{1,q};1)\Big]$$
$$+[(1+k)(3+4\alpha)-(k+\beta)(1+2\alpha\beta)]\frac{|\alpha_1|...|\alpha_p|}{\Re(\beta_1)...\Re(\beta_q)}$$
$$\times\Big[{}_p\psi_q((|\alpha_i|+1,A_i)_{1,p},(\Re(\beta_i)+1,B_i)_{1,q};1)\Big]$$
$$+(1-\beta)\big[{}_p\psi_q((|\alpha_i|,A_i)_{1,p},(\Re(\beta_i),B_i)_{1,q};1)]\leq 2(1-\beta) \quad (17)$$

hold. Then $W_q^p[|\alpha_1|](f(z)) \in \mathcal{U}(k,\alpha,\beta)$.

**Proof.** Let $f \in \mathcal{S}$ be of the form (1). By virtue of the de Branges theorem it suffices to show that

$$\begin{aligned} S_1 &:= \sum_{n=2}^{\infty} n[n(1+k(1-\alpha)-\alpha\beta)+n(n-1)\alpha(1+k)-(k+\beta)(1-\alpha)] \\ &\qquad\qquad \times \left| \frac{(\alpha_1)_{A_1(n-1)}...(\alpha_p)_{A_p(n-1)}}{(\beta_1)_{B_1(n-1)}...(\beta_q)_{B_q(n-1)}} \frac{1}{(n-1)!} \right| \\ &\leq 1-\beta. \end{aligned} \tag{18}$$

Using the inequality $|(a)_{n-1}| \leq (|a|)_{n-1}$, we deduce that

$$\begin{aligned} S_1 \leq \sum_{n=2}^{\infty} & [n^3\alpha(1+k)+n^2[(1-\alpha)(1+k)-\alpha(k+\beta)]-n(k+\beta)(1-\alpha)] \\ &\qquad \times \frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}} \frac{1}{(n-1)!}. \end{aligned}$$

Writing $n^3 = (n-1)(n-2)(n-3)+6(n-1)(n-2)+7(n-1)+1$,
$n^2 = (n-1)(n-2)+3(n-1)+1$ and $n = (n-1)+1$,
the above inequality can be written as

$$\begin{aligned} S_1 &\leq \alpha(1+k) \sum_{n=2}^{\infty} \frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}} \frac{1}{(n-4)!} \\ &\quad + [(1+k)(1+5\alpha)-\alpha(k+\beta)] \sum_{n=2}^{\infty} \frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}} \frac{1}{(n-3)!} \\ &\quad + [(1+k)(3+4\alpha)-(k+\beta)(1+2\alpha)] \sum_{n=2}^{\infty} \frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}} \frac{1}{(n-2)!} \\ &\quad + (1-\beta) \sum_{n=2}^{\infty} \frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}} \frac{1}{(n-1)!} \\ &\quad \leq \alpha(1+k) \frac{(|\alpha_1|...|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)} \frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{\Re(\beta_1+1)...\Re(\beta_q+1)} \frac{(|\alpha_1|+2)...(|\alpha_p|+2)}{\Re(\beta_1+2)...\Re(\beta_q+2)} \\ &\qquad\qquad \times \sum_{n=4}^{\infty} \frac{(|\alpha_1|+3)_{A_1(n-4)}...(|\alpha_p|+3)_{A_p(n-4)}}{\Re(\beta_1+3)_{B_1(n-4)}...\Re(\beta_q+3)_{B_q(n-4)}} \frac{1}{(n-4)!} \\ &\quad + [(1+k)(1+5\alpha)-\alpha(k+\beta)] \frac{(|\alpha_1|...|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)} \frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{\Re(\beta_1+1)...\Re(\beta_q+1)} \\ &\qquad\qquad \times \sum_{n=3}^{\infty} \frac{(|\alpha_1|+2)_{A_1(n-3)}...(|\alpha_p|+2)_{A_p(n-3)}}{\Re(\beta_1+2)_{B_1(n-3)}...\Re(\beta_q+2)_{B_q(n-3)}} \frac{1}{(n-3)!} \\ &\quad + [(1+k)(3+4\alpha)-(k+\beta)(1+2\alpha)] \frac{(|\alpha_1|)...(|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)} \\ &\qquad\qquad \times \sum_{n=2}^{\infty} \frac{(|\alpha_1|+1)_{A_1(n-2)}...(|\alpha_p|+1)_{A_p(n-2)}}{\Re(\beta_1+1)_{B_1(n-2)}...\Re(\beta_q+1)_{B_q(n-2)}} \frac{1}{(n-2)!} \\ &\qquad\qquad\qquad + (1-\beta) \sum_{n=2}^{\infty} \frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}} \frac{1}{(n-1)!} \end{aligned}$$

$$
\begin{aligned}
&= \alpha(1+k)\frac{(|\alpha_1|...|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)}\frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{\Re(\beta_1+1)...\Re(\beta_q+1)}\frac{(|\alpha_1|+2)...(|\alpha_p|+2)}{\Re(\beta_1+2)...\Re(\beta_q+2)}\\
&\qquad\qquad \times\Big[{}_p\psi_q((|\alpha_i|+3,A_i)_{1,p},(\Re(\beta_i+3),B_i)_{1,q};1)\Big]\\
&+[(1+k)(1+5\alpha)-\alpha(k+\beta)]\frac{(|\alpha_1|)...(|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)}\frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{\Re(\beta_1+1)...\Re(\beta_q+1)}\\
&\qquad\qquad \times\Big[{}_p\psi_q((|\alpha_i|+2,A_i)_{1,p},(\Re(\beta_i+2),B_i)_{1,q};1)\Big]\\
&+[(1+k)(3+4\alpha)-(k+\beta)(1+2\alpha)]\frac{(|\alpha_1|)...(|\alpha_p|)}{\Re(\beta_1)...\Re(\beta_q)}\\
&\qquad\qquad \times\Big[{}_p\psi_q((|\alpha_i|+1,A_i)_{1,p},(\Re(\beta_i+1),B_i)_{1,q};1)\Big]\\
&+(1-\beta)\Big[{}_p\psi_q((|\alpha_i|,A_i)_{1,p},(\Re(\beta_i),B_i)_{1,q};1)-1\Big]\le(1-\beta),
\end{aligned}
$$

by using the inequality (17). Hence the theorem.

With $A_i = 1$ , $B_i = 1$ we have,
**Corollary 2** Suppose that $\alpha_i \in \mathbb{C}\backslash\{0\}(i=1,...,p)$, $\Re(\beta_i)>0(i=1,...,q)$ and that $\Re(\sum_{i=1}^{q}(\beta_i)) > \sum_{i=1}^{p}|\alpha_i|+p-q$. If $f\in\mathcal{S}$ of the form (1), and let the inequality

$$
\begin{aligned}
&\alpha(1+k)\frac{|\alpha_1|...|\alpha_p|}{\Re(\beta_1)...\Re(\beta_q)}\frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{(\Re(\beta_1)+1)...(\Re(\beta_q)+1)}\frac{(|\alpha_1|+2)...(|\alpha_p|+2)}{(\Re(\beta_1)+2)...(\Re(\beta_q)+2)}\\
&\quad\times\Big[{}_pF_q((|\alpha_1|+3,...,|\alpha_p|+3,\Re(\beta_1)+3,...,\Re(\beta_1)+3,;1)\Big]\\
&+[(1+k)(1+5\alpha)-\alpha(k+\beta)]\frac{|\alpha_1|...|\alpha_p|}{\Re(\beta_1)...\Re(\beta_q)}\frac{(|\alpha_1|+1)...(|\alpha_p|+1)}{(\Re(\beta_1)+1)...(\Re(\beta_q)+1)}\\
&\quad\times\Big[{}_pF_q((|\alpha_1|+2,...,|\alpha_p|+2,\Re(\beta_1)+2,...,\Re(\beta_q)+2;1)\Big]\\
&+[(1+k)(3+4\alpha)-(k+\beta)(1+2\alpha)]\frac{|\alpha_1|...|\alpha_p|}{\Re(\beta_1)...\Re(\beta_q)}\\
&\quad\times\Big[{}_pF_q(|\alpha_1|+1,...,|\alpha_p|+1,\Re(\beta_1)+1,...,\Re(\beta_q)+1;1)\Big]\\
&+(1-\beta)\Big[{}_pF_q((|\alpha_1|,...,|\alpha_p|,\Re(\beta_1),...,\Re(\beta_q);1)\Big]\le 2(1-\beta)
\end{aligned}
$$

hold. Then ${}_pF_q(f(z))\in\mathcal{U}(k,\alpha,\beta)$.

**Theorem 3** Suppose that $\alpha_i \in \mathbb{C}\backslash\{0\}(i=1,...,p)$, $\Re(\beta_i)>0(i=1,...,q)$ and that $\Re(\sum_{i=1}^{q}(\beta_i)) > \sum_{i=1}^{p}|\alpha_i|+P_1$, where $P_1 = P_1(k)$ is given by (3). If $f\in k\text{-}\mathcal{UCV}$ of the form (1), for some $k$ $(0\le k<\infty)$ and let the inequality

$$
\begin{aligned}
&(1+k(1-\alpha)-\alpha\beta)\big[{}_p\psi_q((|\alpha_i|,A_i)_{1,p},P_1;(\Re(\beta_i),B_i)_{1,q};1)\big]\\
&+\alpha(1+k)\frac{|\alpha_1|...|\alpha_p|(P_1(k))}{\Re(\beta_1)...\Re(\beta_q)}\big[{}_p\psi_q((|\alpha_i|+1,A_i)_{1,p},P_1;(\Re(\beta_i+1),B_i)_{1,q},2;1)\big]\\
&\qquad -(k+\beta)(1-\alpha)\big[{}_p\psi_q((|\alpha_i|,A_i)_{1,p},P_1;(\Re(\beta_i),B_i)_{1,q},2;1)\big]\\
&\le 2(1-\beta)
\end{aligned}
\tag{19}
$$

hold. Then $W_q^p[|\alpha_1|](f(z)) \in \mathcal{U}(k,\alpha,\beta)$.
**Proof.** Let $f$ of the form (1) belong $k$-$\mathcal{UCV}$. By virtue of (13) and (2), it is enough to prove that

$$\sum_{n=2}^{\infty}[n(1+k(1-\alpha)-\alpha\beta)+n(n-1)\alpha(1+k)-(k+\beta)(1-\alpha)] \times \left|\frac{(\alpha_1)_{A_1(n-1)}...(\alpha_p)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}}\frac{1}{(n-1)!}\frac{(P_1)_{n-1}}{n!}\right| \le 1-\beta.$$

Using $|(a)_{n-1}| \le (|a|)_{n-1}$, the proof is complete as in Theorem 1.

With $A_i = 1$ , $B_i = 1$ we have,
**Corollary 3.** Suppose that $\alpha_i \in \mathbb{C}\backslash\{0\}(i=1,...,p)$, $\Re(\beta_i) > 0(i=1,...,q)$ and that $\Re(\sum_{i=1}^{q}(\beta_i)) > \sum_{i=1}^{p}|\alpha_i|+P_1$, where $P_1 = P_1(k)$ is given by (3). If $f \in k$-$\mathcal{UCV}$ of the form (1), for some $k$ $(0 \le k < \infty)$ and let the inequality

$$\begin{aligned}&(1+k(1-\alpha)-\alpha\beta)[{}_pF_q((|\alpha_1|,...,|\alpha_p|,P_1;\Re(\beta_1),...,\Re(\beta_q);1)]\\&+\alpha(1+k)\frac{(|\alpha_1|...|\alpha_p|)(P_1)}{\Re(\beta_1)...\Re(\beta_q)}[{}_pF_q(|\alpha_1|+1,...,|\alpha_p|+1,P_1;\Re(\beta_1)+1,...,\Re(\beta_q)+1,2;1)]\\&\qquad-(k+\beta)(1-\alpha)[{}_pF_q(|\alpha_1|,...,|\alpha_p|,P_1;\Re(\beta_1),...,\Re(\beta_1),2;1)]\\&\le 2(1-\beta)\end{aligned}$$

hold. Then ${}_pF_q(f(z)) \in \mathcal{U}(k,\alpha,\beta)$.

**Theorem 4** Suppose that $\alpha_i \in \mathbb{C}\backslash\{0\}(i=1,...,p)$, $\Re(\beta_i) > 0(i=1,...,q)$ and that $\Re(\sum_{i=1}^{q}(\beta_i)) > \sum_{i=1}^{p}|\alpha_i|+P_1+1$, where
$P_1 = P_1(k)$ is given by (3). If $f \in k$-$\mathcal{ST}$ of the form (1) for some $k$ $(0 \le k < \infty)$ and let the inequality

$$\begin{aligned}&\frac{\alpha(1+k)}{2}\frac{(|\alpha_1|...|\alpha_p|)(P_1)}{\Re(\beta_1)...\Re(\beta_q)}\frac{(|\alpha_1|+1)...(|\alpha_p|+1)(P_1+1)}{(\Re(\beta_1)+1)...(\Re(\beta_q)+1)}\\&\qquad\times[{}_p\psi_q((|\alpha_i|+2,A_i)_{1,p},P_1+2;(\Re(\beta_i)+2,B_i)_{1,q}3;1)]\\&+[(1+k)(1+2\alpha)-\alpha(k+\beta)]\frac{(|\alpha_1|...|\alpha_p|)(P_1)}{\Re(\beta_1)...\Re(\beta_q)}\\&\qquad\times[{}_p\psi_q(|\alpha_i|+1,A_i)_{1,p},P_1+1;(\Re(\beta_i)+1,B_i)_{1,q},2;1]\\&\quad+(1-\beta)[{}_p\psi_q((|\alpha_i|,A_i)_{1,p},P_1;(\Re(\beta_i),B_i)_{1,q};1)] \le 2(1-\beta)\end{aligned}$$

hold. Then $W_q^p[|\alpha_1|](f(z)) \in \mathcal{U}(k,\alpha,\beta)$.
**Proof.** Let $f \in k-\mathcal{ST}$ be of the form (1). Applying the estimates for the coefficients given by (4), and using $|(a)_{n-1}| \le (|a|)_{n-1}$, we get

$$\sum_{n=2}^{\infty}[n(1+k(1-\alpha)-\alpha\beta)+n(n-1)\alpha(1+k)-(k+\beta)(1-\alpha)] \times \left|\frac{(\alpha_1)_{A_1(n-1)}...(\alpha_p)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}}\frac{1}{(n-1)!}\frac{(P_1)_{n-1}}{(n-1)!}\right|$$

$$\leq \sum_{n=2}^{\infty}[n(1+k(1-\alpha)-\alpha\beta)+n(n-1)\alpha(1+k)-(k+\beta)(1-\alpha)]$$
$$\times \frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}}\frac{1}{(n-1)!}\frac{(P_1)_{n-1}}{(n-1)!}$$
$$= \sum_{n=2}^{\infty}(\alpha(1+k)n^2+n[(1-\alpha)(1+k)-\alpha(k+\beta)]-(k+\beta)(1-\alpha))$$
$$\times \frac{(|\alpha_1|)_{A_1(n-1)}...(|\alpha_p|)_{A_p(n-1)}}{\Re(\beta_1)_{B_1(n-1)}...\Re(\beta_q)_{B_q(n-1)}}\frac{1}{(n-1)!}\frac{(P_1)_{n-1}}{(n-1)!}.$$

Rewriting $n^2 = (n-1)(n-2)+3(n-1)+1$, $n = (n-1)+1$, and proceeding as in Theorem 2, we get the required result.

With $A_i = 1$ , $B_i = 1$ we have,
**Corollary 4** Suppose that $\alpha_i \in \mathbb{C}\backslash\{0\}(i = 1, ..., p)$, $\Re(\beta_i) > 0(i = 1, ..., q)$ and that $\Re(\sum_{i=1}^{q}(\beta_i)) > \sum_{i=1}^{p}|\alpha_i| + P_1 + 1$, where $P_1 = P_1(k)$ is given by (3).
If $f \in k\text{-}\mathcal{ST}$ of the form (1) for some $k$ $(0 \leq k < \infty)$ and let the inequality

$$\frac{\alpha(1+k)}{2}\frac{(|\alpha_1|...|\alpha_p|)(P_1)}{\Re(\beta_1)...\Re(\beta_q)}\frac{(|\alpha_1|+1)...(|\alpha_p|+1)(P_1+1)}{(\Re(\beta_1)+1)...(\Re(\beta_q)+1)}$$
$$\times\Big[{}_pF_q((|\alpha_1|+2,...,|\alpha_p|+2,P_1+2;\Re(\beta_1)+2,...,\Re(\beta_1)+2,3;1)\Big]$$
$$+[(1+k)(1+2\alpha)-\alpha(k+\beta)]\frac{(|\alpha_1|...|\alpha_p|)(P_1)}{\Re(\beta_1)...\Re(\beta_q)}$$
$$\times\Big[{}_pF_q(|\alpha_1|+1,...,|\alpha_p|+1,P_1+1;\Re(\beta_1)+1,...,\Re(\beta_q)+1,2;1)\Big]$$
$$+(1-\beta)\Big[{}_pF_q(|\alpha_1|,...,|\alpha_p|,P_1;\Re(\beta_1),...,\Re(\beta_q);1)] \qquad \leq 2(1-\beta)$$

hold. Then ${}_pF_q(f(z)) \in \mathcal{U}(k,\alpha,\beta)$.

**Remark 1** By specialising the parameters $A_i$, $B_i$, $p$, $q$, $\alpha_i$ and $\beta_i$ , the main results derived can be easily restated interms of the operators defined by (10),(11) and (12).

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