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BESSEL FUNCTIONS ASSOCIATED WITH SAIGO-MAEDA FRACTIONAL DERIVATIVE OPERATORS

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Abstract. In this paper, we consider the operators of fractional derivative involving Appell's function $F_3(.)$ due to Saigo-Maeda to obtain the generalized fractional derivative formulas involving the Bessel function of the first kind. The results are expressed in terms of genealized Wright function and hypergeometric function ${}_pF_q$. Special cases of the results are also pointed out in the concluding section of this paper.

1. Introduction

The fractional derivative operators involving various special functions, have found significant importance and applications in various sub-field of applicable mathematical analysis. During last four decades, a number of workers have studied, in depth, the properties, applications and different extensions of various hypergeometric operators of fractional derivatives. A detailed account of such operators along with their properties and applications can be found in the research work by a number of authors (see, for example, [1, 2, 3, 4, 5, 6, 9, 10, 12, 13, 14, 15, 16, 17, 18, 21]).

A useful generalization of the hypergeometric fractional derivatives, including the Saigo operators [12, 13, 14], has been introduced by Marichev [10] (see details in Samko et al. [17, p. 194, Eq. (10.47) and whole Section 10.3]) and later extended and studied by Saigo and Maeda [15, p. 393, Eq. (4.12) and Eq. (4.13)] in term of any complex order with Appell function $F_3(.)$ in the kernel and Saigo-Maeda[15] introduced the fractional derivative operators, which is defined as follows:
Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$, $\mathbb{C}$ being the set of complex numbers and $x > 0$. Then the generalized fractional derivative operators (Saigo-Maeda operators)(see, [15]) are defined by the following equations:

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$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x) = \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f\right)(x) \tag{1}$$

$$= \left(\frac{d}{dx}\right)^k \left(I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}f\right)(x) \tag{2}$$

$(\Re(\gamma) > 0; k = [\Re(\gamma)] + 1)$;

$$\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x) = \left(I_{0-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f\right)(x) \tag{3}$$

$$= \left(-\frac{d}{dx}\right)^k \left(I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}f\right)(x) \tag{4}$$

$(\Re(\gamma) > 0; k = [\Re(\gamma)] + 1)$.

Following Saigo *et.al.*[15] (see also, [18]), the left-hand sided and right-hand sided generalized integration for a power function are given by:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\right)(x) = \Gamma\left[\begin{array}{c}\rho, \rho+\gamma-\alpha-\alpha', \rho+\beta'-\alpha' \\ \rho+\beta', \rho+\gamma-\alpha-\alpha', \rho+\gamma-\alpha'-\beta\end{array}\right] x^{\rho-\alpha-\alpha'+\gamma-1} \tag{5}$$

where $\Re(\gamma) > 0, \Re(\rho) > \max\{0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\}$ and

$$\left(I_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\right)(x) = \Gamma\left[\begin{array}{c}1-\rho-\gamma+\alpha+\alpha', 1-\rho+\alpha+\beta'-\gamma, 1-\rho-\beta \\ 1-\rho, 1-\rho+\alpha+\alpha'+\beta'-\gamma, 1-\rho+\alpha-\beta\end{array}\right] x^{\rho-\alpha-\alpha'+\gamma-1} \tag{6}$$

where $\Re(\gamma) > 0, \Re(\rho) < 1 + \min\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}$.

The symbol occurring in (5) and (6) is given by $\left[\begin{array}{c}a, b, c \\ d, e, f\end{array}\right] = \dfrac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)}$.

The Bessel function of the first kind $J_\nu(z)$ is defined for $z \in \mathbb{C} \setminus \{0\}$ and $\nu \in \mathbb{C}$ with $\Re(\nu) > -1$ by the following series (see, for example, [11, p. 217, Entry 10.2.2] and [22, p. 40, Eq. (8)]):

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}, \tag{7}$$

where $\mathbb{C}$ denotes the set of complex numbers and $\Gamma(z)$ is the familiar Gamma function (see [19, Section 1.1]).

An interesting further generalization of the generalized hypergeometric series ${}_pF_q$ (11) is due to Fox [8] and Wright [23, 24, 25] who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [20, p. 21])

$$ {}_p\Psi_q \left[\begin{matrix} (\alpha_1,\, A_1)\,,\, \ldots,\, (\alpha_p,\, A_p)\,; \\ (\beta_1,\, B_1)\,,\, \ldots,\, (\beta_q,\, B_q)\,; \end{matrix} \; z \right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma\,(\alpha_j + A_j\, k)}{\prod_{j=1}^{q} \Gamma\,(\beta_j + B_j\, k)} \frac{z^k}{k!}, \tag{8} $$

where the coefficients $A_1,\, \ldots,\, A_p$ and $B_1,\, \ldots,\, B_q$ are positive real numbers such that

$$ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0. \tag{9} $$

A special case of (8) is

$$ {}_p\Psi_q \left[\begin{matrix} (\alpha_1,\, 1)\,,\, \ldots,\, (\alpha_p,\, 1)\,; \\ (\beta_1,\, 1)\,,\, \ldots,\, (\beta_q,\, 1)\,; \end{matrix} \; z \right] = \frac{\prod_{j=1}^{p} \Gamma\,(\alpha_j)}{\prod_{j=1}^{q} \Gamma\,(\beta_j)}\, {}_pF_q \left[\begin{matrix} \alpha_1,\, \ldots,\, \alpha_p\,; \\ \beta_1,\, \ldots,\, \beta_q\,; \end{matrix} \; z \right], \tag{10} $$

where ${}_pF_q$ is the *generalized hypergeometric series* defined by (see [19, Section 1.5])

$$ \begin{aligned} {}_pF_q \left[\begin{matrix} \alpha_1,\, \ldots,\, \alpha_p\,; \\ \beta_1,\, \ldots,\, \beta_q\,; \end{matrix} \; z \right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \\ &= {}_pF_q(\alpha_1,\, \ldots,\, \alpha_p;\, \beta_1,\, \ldots,\, \beta_q;\, z), \end{aligned} \tag{11} $$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [19, p. 2 and pp. 4–6]):

$$ \begin{aligned} (\lambda)_n :&= \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N} := \{1,\, 2,\, 3,\, \ldots\}) \end{cases} \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{aligned} \tag{12} $$

and $\mathbb{Z}_0^-$ denotes the set of nonpositive integers.

In this paper, we apply the derivative operators (1) and (3) to the Bessel function of the first kind $J_v(x)$ and express the image in terms of generalized Wright and hypergeometric functions.

2. Main Results

In this section, we establish image formulas for the Bessel function of the first kind involving Saigo-Meada fractional derivative operators (1) and (3), in term of generalized Wright function. These formulas are given by the following theorems:

Theorem 1. *Let* $\alpha, \alpha', \beta, \beta', \gamma, v, \rho \in \mathbb{C}$ *and* $x > 0$ *be such that* $\Re(\gamma) < 0, \Re(v) > -1$,

$$ \Re(\rho + v) > max[0, \Re(\gamma - \alpha - \alpha' - \beta), \Re(\beta - \alpha)]. \tag{13} $$

Then the following formula holds:

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_v(t)\right)(x) = \frac{x^{\rho+v+\alpha+\alpha'-\gamma-1}}{2^v} {}_3\psi_4 \left[\begin{array}{c} (\rho+v,2),(\rho+v+\alpha+\alpha'+\beta'-\gamma,2), \\ (\rho+v-\beta,2),(\rho+v-\gamma+\alpha+\alpha',2), \end{array} \begin{array}{c} (\rho+v+\alpha-\beta,2) \\ (\rho+v+\alpha+\beta'-\gamma,2),(v+1,1) \end{array} \mid \frac{-x^2}{4} \right]. \quad (14)$$

Proof : Using (1) and (7), after a little simplification, we get

$$\sum_{k=0}^{\infty} \frac{(-1)^k (1/2)^{v+2k}}{\Gamma(v+k+1)k!} \left(I_{0+}^{-\alpha',-\alpha,-\beta'-\beta,-\gamma} t^{\rho+v+2k-1}\right)(x). \quad (15)$$

Following the conditions(13), for any $k = 0,1,2,...,\Re(\rho+v+2k) \geq \Re(\rho+v) > \max[0,\Re(\gamma-\alpha-\alpha'-\beta'),\Re(\beta-\alpha)]$

Applying the known result (5) with ρ replaced by $\rho+v+2k$, we obtain

$$\left(D_{0+}^{(\alpha,\alpha',\beta,\beta',\gamma)} t^{\rho-1} J_v(t)\right)(x) = \frac{x^{\rho+v+\alpha+\alpha'-\gamma-1}}{2^v} \sum_{k=0}^{\infty} \frac{\Gamma(\rho+v+2k)}{\Gamma(\rho+v-\beta+2k)}$$
$$\frac{\Gamma(\rho+v+\alpha+\alpha'+\beta'-\gamma+2k)\Gamma(\rho+v-\beta+\alpha+2k)}{\Gamma(\rho+v-\gamma+\alpha+\alpha'+2k)\Gamma(\rho+v-\gamma+\alpha+\beta'+2k)\Gamma(v+k+1)} \frac{(-x^2)^k}{4^k k!}. \quad (16)$$

Interpreting the right-hand side of the above equation, in view of the definition (7), we arrive at the required result (14).

Theorem 2. *Let $\alpha,\alpha',\beta,\beta',\gamma,v,\rho \in \mathbb{C}$ and $x > 0$ be such that $\Re(\gamma) < 0, \Re(v) > -1$,*

$$\Re(\rho - v) < 1 + min[\Re(\beta'),\Re(\gamma-\alpha-\alpha'),\Re(\gamma-\alpha'-\beta)]. \quad (17)$$

Then the following formula holds:

$$\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_v(1/t)\right)(x) = \frac{x^{\rho-v+\alpha+\alpha'-\gamma-1}}{2^v} {}_3\psi_4 \left[\begin{array}{c} (1-\rho+v+\gamma-\alpha-\alpha',2), \\ (1-\rho+v,2),(1-\rho+v-\alpha-\alpha'-\beta+\gamma,2), \end{array} \begin{array}{c} (1-\rho-\alpha'-\beta+\gamma+v,2),(1-\rho+v+\beta',2) \\ (1-\rho+v-\alpha'+\beta',2),(v+1,1) \end{array} \middle| \frac{-1}{4x^2} \right]. \quad (18)$$

Proof. The proof of the fractional integral formula (18) would run parallel to the (14) asserted by Theorem 1. We, therefore, choose to skip the details involved.

Now, we consider other variations of the above theorems, that is, we establish image formulas for the Bessel function $J_v(x)$ under the operators (1) and (3), in terms of the generalized hypergeometric function ${}_pF_q$. To do this, we recall the well-known Legendre duplication formula for the Gamma function Γ:

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \qquad (z\in\mathbb{C}), \tag{19}$$

which is equivalently written in terms of the Pochhammer symbol (12) as follows (see, for example, [19, p. 6]):

$$(\lambda)_{2n}=2^{2n}\left(\frac{1}{2}\lambda\right)_n\left(\frac{1}{2}\lambda+\frac{1}{2}\right)_n \qquad (n\in\mathbb{N}_0). \tag{20}$$

First we consider the formulae (14) of Theorem 1.

Corollary 1. *Let the conditions of Theorem 1 be satisfied, and let* $\rho+v, \rho+v+\alpha+\alpha'+\beta'-\gamma, \rho+v+\alpha-\beta\neq 0,-1,....$ *Then the following formula holds:*

$$\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}J_v(t)\right)(x)=\frac{x^{\rho+v+\alpha+\alpha'-\gamma-1}}{2^v}\frac{\Gamma(\rho+v)}{\Gamma(\rho+v-\beta)}$$
$$\frac{\Gamma(\rho+v+\alpha+\alpha'+\beta'-\gamma)\Gamma(\rho+v-\beta+\alpha)}{\Gamma(\rho+v-\gamma+\alpha+\alpha')\Gamma(\rho+v-\gamma+\alpha+\beta')(v+1)}$$
$${}_6F_7\left[\begin{matrix}\frac{\rho+v}{2},\frac{\rho+v+1}{2},\frac{\rho+v+\alpha+\alpha'+\beta'-\gamma}{2},\frac{\rho+v+\alpha+\alpha'+\beta'-\gamma+1}{2},\\ v+1,\frac{\rho+v-\beta}{2},\frac{\rho+v-\beta+1}{2},\frac{\rho+v-\gamma+\alpha+\alpha'}{2},\frac{\rho+v-\gamma+\alpha+\alpha'+1}{2},\end{matrix}\right.$$
$$\left.\begin{matrix}\frac{\rho+v-\beta+\alpha}{2},\frac{\rho+v-\beta+\alpha+1}{2}\\ \frac{\rho+v-\gamma+\alpha+\beta'}{2},\frac{\rho+v-\gamma+\alpha+\beta'+1}{2}\end{matrix}\;\middle|\;\frac{-x^2}{4}\right]. \tag{21}$$

Proof. To prove the above result, we make use of the well-known identity

$$\Gamma(z+k)=(z)_k\Gamma(z) \qquad (z\in\mathbb{C},k\in\mathbb{N}) \tag{22}$$

and the formula (22), in Equation (16), then we have

$$\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}J_v(t)\right)(x)=\frac{x^{\rho+v+\alpha+\alpha'-\gamma-1}}{2^v}\sum_{k=0}^{\infty}\frac{1}{\Gamma(v+1)(v+1)_k k!}$$
$$\frac{\Gamma(\rho+v)\Gamma(\rho+v+\alpha+\alpha'+\beta'-\gamma)\Gamma(\rho+v-\beta+\alpha)}{\Gamma(\rho+v-\beta)\Gamma(\rho+v-\gamma+\alpha+\alpha')\Gamma(\rho+v-\gamma+\alpha+\beta')}$$
$$\frac{(\rho+v)_{2k}(\rho+v+\alpha+\alpha'+\beta'-\gamma)_{2k}(\rho+v-\beta+\alpha)_{2k}}{(\rho+v-\beta)_{2k}(\rho+v-\gamma+\alpha+\alpha')_{2k}(\rho+v-\gamma+\alpha+\beta')_{2k}}\left(\frac{-x^2}{4}\right)^k$$
$$=\frac{x^{\rho+v+\alpha+\alpha'-\gamma-1}}{2^v}\frac{\Gamma(\rho+v)\Gamma(\rho+v+\alpha+\alpha'+\beta'-\gamma)\Gamma(\rho+v-\beta+\alpha)}{\Gamma(\rho+v-\beta)\Gamma(\rho+v-\gamma+\alpha+\alpha')\Gamma(\rho+v-\gamma+\alpha+\beta')}$$
$$\frac{1}{\Gamma(v+1)}\sum_{k=0}^{\infty}\frac{\left(\frac{\rho+v}{2}\right)_k\left(\frac{\rho+v+1}{2}\right)_k\left(\frac{\rho+v+\alpha+\alpha'+\beta'-\gamma}{2}\right)_k\left(\frac{\rho+v+\alpha+\alpha'+\beta'-\gamma+1}{2}\right)_k}{\left(\frac{\rho+v-\beta}{2}\right)_k\left(\frac{\rho+v-\beta+1}{2}\right)_k\left(\frac{\rho+v-\gamma+\alpha+\alpha'}{2}\right)_k\left(\frac{\rho+v-\gamma+\alpha+\alpha'+1}{2}\right)_k}$$
$$\frac{\left(\frac{\rho+v-\beta+\alpha}{2}\right)_k\left(\frac{\rho+v-\beta+\alpha+1}{2}\right)_k}{\left(\frac{\rho+v-\gamma+\alpha+\beta'}{2}\right)_k\left(\frac{\rho+v-\gamma+\alpha+\beta'+1}{2}\right)_k(v+1)_k k!}\left(\frac{-x^2}{4}\right)^k. \tag{23}$$

Thus interpreting with (11), we get the result (21).

Similarly, from Theorem 2., we can obtain the following result.

Corollary 2. *Let the conditions of Theorem 2. be satisfied, and let* $1-\rho+v+\gamma-\alpha-\alpha', 1-\rho+v-\alpha'-\beta+\gamma, 1-\rho+v+\beta' \neq 0, -1, ...$ *Then the following formula holds:*

$$\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} J_v(1/t)\right)(x) = \frac{x^{\rho-v+\alpha+\alpha'-\gamma-1}\Gamma(1-\rho+v+\gamma-\alpha-\alpha')}{2^v\Gamma(1-\rho+v)}$$
$$\frac{\Gamma(1-\rho+v-\alpha'-\beta+\gamma)(1-\rho+v+\beta')}{\Gamma(1-\rho+v-\alpha-\alpha'-\beta+\gamma)\Gamma(1-\rho+v-\alpha'+\beta')\Gamma(v+1)}$$
$${}_6F_7\left[\begin{array}{c} \frac{1-\rho+v+\gamma-\alpha-\alpha'}{2}, \frac{2-\rho+v+\gamma-\alpha-\alpha'}{2}, \frac{1-\rho+v-\alpha'-\beta+\gamma}{2}, \frac{2-\rho+v-\alpha'-\beta+\gamma}{2}, \\ v+1, \frac{1-\rho+v}{2}, \frac{2-\rho+v}{2}, \frac{1-\rho+v-\alpha-\alpha'-\beta+\gamma}{2}, \frac{2-\rho+v-\alpha-\alpha'-\beta+\gamma}{2}, \end{array}\right.$$
$$\left.\begin{array}{c} \frac{1-\rho+v+\beta'}{2}, \frac{2-\rho+v+\beta'}{2} \\ \frac{1-\rho+v-\alpha'+\beta'}{2}, \frac{2-\rho+v-\alpha'+\beta'}{2} \end{array} \middle| \frac{-1}{4x^2}\right]. \tag{24}$$

3. Special Cases

In this section, we derive certain image formulas for the cosine and sine functions under the fractional derivative operators (1) and (3), in terms of the generalized Wright function.

For $v=-1/2$, the Bessel function $J_v(z)$ given by (7) coincides with the cosine function, excluding the multiplier $(2/\pi z)^{1/2}$, (see, for example, [7, p. 79, Eq. (15)]):

$$J_{(-1/2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos z. \tag{25}$$

Corollary 3. *Let* $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$ *and* $x > 0$ *be such that* $\Re(\gamma) < 0$,

$$\Re(\rho) > max[0, \Re(\gamma-\alpha-\alpha'-\beta), \Re(\beta-\alpha)]. \tag{26}$$

Then the following formula holds:

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \cos t\right)(x)$$
$$= \pi^{1/2} x^{\rho+\alpha+\alpha'-\gamma-1} {}_3\psi_4\left[\begin{array}{c} (\rho,2), (\rho+\alpha+\alpha'+\beta'-\gamma,2), \\ (\rho-\beta,2), (\rho-\gamma+\alpha+\alpha',2), \end{array}\right.$$
$$\left.\begin{array}{c} (\rho+\alpha-\beta,2) \\ (\rho+\alpha+\beta'-\gamma,2), (1/2,1) \end{array} \middle| \frac{-x^2}{4}\right]. \tag{27}$$

Proof. On setting $v=-1/2$ into the result (14) and using (25), we have

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-3/2}\cos t\right)(x)$$
$$= \pi^{1/2} x^{\rho+\alpha+\alpha'-\gamma-3/2} {}_3\psi_4 \left[\begin{array}{c} (\rho-1/2,2),(\rho+\alpha+\alpha'+\beta'-\gamma-1/2,2), \\ (\rho-\beta-1/2,2),(\rho-\gamma+\alpha+\alpha'-1/2,2), \end{array}\right.$$
$$\left.\begin{array}{c} (\rho+\alpha-\beta-1/2,2) \\ (\rho+\alpha+\beta'-\gamma-1/2,2),(1/2,1) \end{array} \middle| \frac{-x^2}{4}\right]. \quad (28)$$

Now if we replace ρ by $\rho+1/2$, then the conditions given by (13) transform to the condition (26) and Equation (28) yields the result (27).

The next statement follows from the Theorem 2. by setting $v=-1/2$ in result (18) and taking (25) and (17) into account.

Corollary 4. *Let $\alpha,\alpha',\beta,\beta',\gamma,v,\rho\in\mathbb{C}$ and $x>0$ be such that $\Re(\gamma)<0$,*

$$\Re(\rho) < min[\Re(\beta'),\Re(\gamma-\alpha-\alpha'),\Re(\gamma-\alpha'-\beta)]. \quad (29)$$

Then the following formula holds:

$$\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-2}\cos(1/t)\right)(x)$$
$$= \pi^{1/2} x^{\rho+\alpha+\alpha'-\gamma-1} {}_3\psi_4 \left[\begin{array}{c} (-\rho+\gamma-\alpha-\alpha',2), \\ (-\rho,2),(-\rho-\alpha-\alpha'-\beta+\gamma,2), \end{array}\right.$$
$$\left.\begin{array}{c} (-\rho-\alpha'-\beta+\gamma,2),(-\rho+\beta',2) \\ (-\rho-\alpha'+\beta',2),(1/2,1) \end{array} \middle| \frac{-1}{4x^2}\right]. \quad (30)$$

The next statements give image formulas for the cosine function under the Saigo-Maeda fractional derivative operators, in terms of the generalized hypergeometric series (11), follow from Corollaries 3 and 4 with $v=-1/2$, respectively.

Corollary 5. *Let the conditions of Corollary 3 be satisfied, and let $\rho,\rho+\alpha+\alpha'+\beta'-\gamma,\rho+\alpha-\beta\neq 0,-1,\ldots$. Then the following formula holds:*

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\cos t\right)(x) = x^{\rho+\alpha+\alpha'-\gamma-1}$$
$$\frac{\Gamma(\rho)\Gamma(\rho+\alpha+\alpha'+\beta'-\gamma)\Gamma(\rho-\beta+\alpha)}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\alpha')\Gamma(\rho-\gamma+\alpha+\beta')}$$
$${}_6F_7\left[\begin{array}{c} \frac{\rho}{2},\frac{\rho+1}{2},\frac{\rho+\alpha+\alpha'+\beta'-\gamma}{2},\frac{\rho+\alpha+\alpha'+\beta'-\gamma+1}{2}, \\ \frac{1}{2},\frac{\rho-\beta}{2},\frac{\rho-\beta+1}{2},\frac{\rho-\gamma+\alpha+\alpha'}{2},\frac{\rho-\gamma+\alpha+\alpha'+1}{2}, \end{array}\right.$$
$$\left.\begin{array}{c} \frac{\rho-\beta+\alpha}{2},\frac{\rho-\beta+\alpha+1}{2} \\ \frac{\rho-\gamma+\alpha+\beta'}{2},\frac{\rho-\gamma+\alpha+\beta'+1}{2} \end{array} \middle| \frac{-x^2}{4}\right]. \quad (31)$$

Corollary 6. *Let the conditions of Corollary 4 be satisfied, and let $-\rho+\gamma-\alpha-\alpha', -\rho-\alpha'-\beta+\gamma, -\rho+\beta' \neq 0, -1, \ldots$. Then the following formula holds:*

$$\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-2}\cos(1/t)\right)(x) = \frac{x^{\rho+\alpha+\alpha'-\gamma-1}\Gamma(-\rho+\gamma-\alpha-\alpha')}{\Gamma(-\rho)} \frac{\Gamma(-\rho-\alpha'-\beta+\gamma)(-\rho+\beta')}{\Gamma(-\rho-\alpha-\alpha'-\beta+\gamma)\Gamma(-\rho-\alpha'+\beta')} {}_6F_7\left[\begin{array}{l} \frac{-\rho+\gamma-\alpha-\alpha'}{2}, \frac{1-\rho+\gamma-\alpha-\alpha'}{2}, \frac{-\rho-\alpha'-\beta+\gamma}{2}, \frac{1-\rho-\alpha'-\beta+\gamma}{2}, \\ \frac{1}{2}, \frac{-\rho}{2}, \frac{1-\rho}{2}, \frac{-\rho-\alpha-\alpha'-\beta+\gamma}{2}, \frac{1-\rho-\alpha-\alpha'-\beta+\gamma}{2}, \end{array} \begin{array}{l} \frac{-\rho+v+\beta'}{2}, \frac{1-\rho+v+\beta'}{2} \\ \frac{-\rho-\alpha'+\beta'}{2}, \frac{1-\rho-\alpha'+\beta'}{2} \end{array} \left| \frac{-1}{4x^2} \right.\right]. \tag{32}$$

Again for $v = 1/2$, the Bessel function $J_v(z)$ coincides with the sine function, excluding the multiplier $(2/\pi z)^{1/2}$ (see, for example, [7, p. 79, Eq. (14)]) that is

$$J_{(1/2)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z. \tag{33}$$

Then from Theorem 1, we obtain the following result:

Corollary 7. *Let $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$ and $x > 0$ be such that the conditions (28) is satisfied. Then*

$$\begin{aligned} &\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-2}\sin t\right)(x) \\ &= \frac{\pi^{1/2}}{2} x^{\rho+\alpha+\alpha'-\gamma-1} {}_3\psi_4\left[\begin{array}{l} (\rho,2), (\rho+\alpha+\alpha'+\beta'-\gamma,2), \\ (\rho-\beta,2), (\rho-\gamma+\alpha+\alpha',2), \end{array} \begin{array}{c} (\rho+\alpha-\beta,2) \\ (\rho+\alpha+\beta'-\gamma,2), (3/2,1) \end{array} \left| \frac{-x^2}{4} \right.\right]. \end{aligned} \tag{34}$$

Proof. On setting $v = 1/2$ into the result (14) and using (33), we have

$$\begin{aligned} &\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-3/2}\sin t\right)(x) \\ &= \frac{\pi^{1/2}}{2} x^{\rho+\alpha+\alpha'-\gamma-1/2} {}_3\psi_4\left[\begin{array}{l} (\rho+1/2,2), (\rho+\alpha+\alpha'+\beta'-\gamma+1/2,2), \\ (\rho-\beta+1/2,2), (\rho-\gamma+\alpha+\alpha'+1/2,2), \end{array} \begin{array}{c} (\rho+\alpha-\beta+1/2,2) \\ (\rho+\alpha+\beta'-\gamma+1/2,2), (3/2,1) \end{array} \left| \frac{-x^2}{4} \right.\right]. \end{aligned} \tag{35}$$

Now, if we replace ρ by $\rho - 1/2$, then the conditions given by (13) transform to the condition (26) and Equation (35) yields the result (34).

The next statement give rise to the image formulas of the sine function under fractional derivative operators, follows from Theorem 2, Corollary 3 and Corollary 4 with $v = 1/2$.

Corollary 8. *Let* $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$ *and* $x > 0$ *be such that* $\Re(\gamma) < 0$,

$$\Re(\rho) < 1 + min[\Re(\beta'), \Re(\gamma - \alpha - \alpha'), \Re(\gamma - \alpha' - \beta)]. \tag{36}$$

Then the following formula holds:

$$\begin{aligned}&\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \sin(1/t)\right)(x)\\ &= \frac{\pi^{1/2}}{2} x^{\rho+\alpha+\alpha'-\gamma} {}_3\psi_4 \left[\begin{matrix} (1-\rho+\gamma-\alpha-\alpha', 2), \\ (1-\rho, 2), (1-\rho-\alpha-\alpha'-\beta+\gamma, 2), \end{matrix} \right.\\ &\qquad \left. \begin{matrix} (1-\rho-\alpha'-\beta+\gamma, 2), (1-\rho+\beta', 2) \\ (1-\rho-\alpha'+\beta', 2), (3/2, 1) \end{matrix} \middle| \frac{-1}{4x^2} \right].\end{aligned} \tag{37}$$

Corollary 9. *Let the conditions of Corollary 7 be satisfied, and let* $\rho, \rho + \alpha + \alpha' + \beta' - \gamma, \rho + \alpha - \beta \neq 0, -1, \ldots$ *. Then the following formula holds:*

$$\begin{aligned}&\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-2} \sin t\right)(x) = \frac{\pi^{1/2}}{2} x^{\rho+\alpha+\alpha'-\gamma-1}\\ &\frac{\Gamma(\rho)\Gamma(\rho+\alpha+\alpha'+\beta'-\gamma)\Gamma(\rho-\beta+\alpha)}{\Gamma(\rho-\beta)\Gamma(\rho-\gamma+\alpha+\alpha')\Gamma(\rho-\gamma+\alpha+\beta')}\\ &{}_6F_7 \left[\begin{matrix} \frac{\rho}{2}, \frac{\rho+1}{2}, \frac{\rho+\alpha+\alpha'+\beta'-\gamma}{2}, \frac{\rho+\alpha+\alpha'+\beta'-\gamma+1}{2}, \\ \frac{3}{2}, \frac{\rho-\beta}{2}, \frac{\rho-\beta+1}{2}, \frac{\rho-\gamma+\alpha+\alpha'}{2}, \frac{\rho-\gamma+\alpha+\alpha'+1}{2}, \end{matrix} \right.\\ &\qquad \left. \begin{matrix} \frac{\rho-\beta+\alpha}{2}, \frac{\rho-\beta+\alpha+1}{2} \\ \frac{\rho-\gamma+\alpha+\beta'}{2}, \frac{\rho-\gamma+\alpha+\beta'+1}{2} \end{matrix} \middle| \frac{-x^2}{4} \right].\end{aligned} \tag{38}$$

Corollary 10. *Let the conditions of Corollary 8 be satisfied, and let* $-\rho + \gamma - \alpha - \alpha', -\rho - \alpha' - \beta + \gamma, -\rho + \beta' \neq 0, -1, \ldots$ *. Then the following formula holds:*

$$\begin{aligned}&\left(D_{0-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \sin(1/t)\right)(x) = \frac{\pi^{1/2}}{2} \frac{x^{\rho+\alpha+\alpha'-\gamma}\Gamma(1-\rho+\gamma-\alpha-\alpha')}{\Gamma(1-\rho)}\\ &\frac{\Gamma(1-\rho-\alpha'-\beta+\gamma)\Gamma(1-\rho+\beta')}{\Gamma(1-\rho-\alpha-\alpha'-\beta+\gamma)\Gamma(1-\rho-\alpha'+\beta')}\\ &{}_6F_7 \left[\begin{matrix} \frac{1-\rho+\gamma-\alpha-\alpha'}{2}, \frac{2-\rho+\gamma-\alpha-\alpha'}{2}, \frac{1-\rho-\alpha'-\beta+\gamma}{2}, \frac{2-\rho-\alpha'-\beta+\gamma}{2}, \\ \frac{3}{2}, \frac{1-\rho}{2}, \frac{2-\rho}{2}, \frac{1-\rho-\alpha-\alpha'-\beta+\gamma}{2}, \frac{2-\rho-\alpha-\alpha'-\beta+\gamma}{2}, \end{matrix} \right.\\ &\qquad \left. \begin{matrix} \frac{1-\rho+\beta'}{2}, \frac{2-\rho+\beta'}{2} \\ \frac{1-\rho-\alpha'+\beta'}{2}, \frac{2-\rho-\alpha'+\beta'}{2} \end{matrix} \middle| \frac{-1}{4x^2} \right].\end{aligned} \tag{39}$$

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