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A NOTE ON MATICHEV-SAIGO-MAEDA FRACTIONAL INTEGRAL OPERATOR

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Abstract. In the present investigation, the generalized operators of fractional integration involving Appell's function $F_3(.)$ due to Marichev-Saigo-Maeda are applied to the M-series and the Fox's H-function. M-series is a further extension of both Mittag-Leffler function and generalized hypergeometric function ${}_pF_q$. Both H-function and M-series have recently found essential applications in solving problems in physics, biology, engineering and applied sciences. The results are expressed in terms of generalized Wright function and H-function.

1. Introduction and Preliminaries

In view of the usefulness and importance of the fractional calculus, the properties, applications and different extensions of various fractional integrations operators are studied by Kalla and Saxena [14], McBride [24], Kumbhat and Khan [20], Kilbas [15], Kilbas and Sebastian [17], Kiryakova [19], Baleanu and Mustafa [4], Baleanu et al. [5], Baleanu et al. [3], Agarwal [1, 2], Chouhan and Saraswat [9, 10] etc.

Marichev [22] introduced a generalization of the hypergeometric fractional integrals for $\alpha, \alpha^{'}, \beta, \beta^{'}, \gamma \in \mathbb{C}$ and $Re(\gamma) > 0$, as follows:

$$\left(I_{0,+}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha^{'}} F_3\left(\alpha,\alpha^{'},\beta,\beta^{'};\gamma;1-\frac{t}{x},1-\frac{t}{x}\right) f(t)\,dt \tag{1}$$

and

$$\left(I_{0,-}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma} f\right)(x) = \frac{x^{-\alpha^{'}}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha,\alpha^{'},\beta,\beta^{'};\gamma;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\,dt \tag{2}$$

In (1) and (2), $F_3(.)$ denotes the 3rd Appell function (also known as Horn function) (Srivastava and Karlson [32]):

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$$ {}_pF_q\left(\alpha,\alpha^{'};\beta,\beta^{'};\gamma;x;y\right)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\alpha^{'})_n(\beta)_m(\beta^{'})_n}{(\gamma)_{m+n}\ m!n!}x^my^n,\ \max\{|x|,|y|\}<1 \tag{3}$$

These operators were discovered and studied by Saigo [27] as generalization of Saigo fractional integral operators [18]. Further their properties were studied by Saigo and Maeda [28], following which the left-hand sided and right-hand sided generalized integration of the type (1) and (2) for a power function are given by:

$$\left(I_{0,+}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma}x^{\rho-1}\right)(x)$$
$$=\frac{\Gamma(\rho)\Gamma\left(\rho+\gamma-\alpha-\alpha^{'}-\beta\right)\Gamma\left(\rho+\beta^{'}-\alpha^{'}\right)}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)}x^{\rho+\gamma-\alpha-\alpha^{'}-1} \tag{4}$$

where $Re(\gamma)>0,\ Re(\rho)>max\ \left\{0,Re\left(\alpha+\alpha^{'}+\beta-\gamma\right),Re\left(\alpha^{'}-\beta^{'}\right)\right\}$ and

$$\left(I_{0,-}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma}x^{\rho-1}\right)(x)$$
$$=\frac{\Gamma\left(1-\rho-\gamma+\alpha+\alpha^{'}\right)\Gamma\left(1-\rho+\alpha+\beta^{'}-\gamma\right)\Gamma(1-\rho-\beta)}{\Gamma(1-\rho)\Gamma(1-\rho+\alpha+\alpha'+\beta'-\gamma)\Gamma(1-\rho+\alpha-\beta)}x^{\rho+\gamma-\alpha-\alpha^{'}-1} \tag{5}$$

where $Re(\gamma)>0,\ Re(\rho)<1+min\ \left\{Re(-\beta),Re\left(\alpha+\alpha^{'}-\gamma\right),Re\left(\alpha+\beta^{'}-\gamma\right)\right\}.$

Wright [33] defined generalized hypergeometric function by means of the series representation in the form

$$ {}_p\psi_q(z)={}_p\psi_q\left[\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p);\\(b_1,B_1),\ldots,(b_q,B_q);\end{matrix}\ z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+nA_i)}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}\frac{z^n}{n!} \tag{6}$$

where $z,a_i,b_j\in\mathbb{C},A_i,B_j\in\mathbb{R}_+,A_i\neq0,B_j\neq0;i=1,\ldots,p;j=1,\ldots,q,$

$$\sum_{j\ =\ 1}^{q}B_j-\sum_{i=\ 1}^{p}A_i>-1$$

Sharma and Jain [31] introduced the generalized M-series as the function defined by means of the power series:

$$ {}_p^{\alpha}M_q^{\beta}(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots b_q;z)={}_p^{\alpha}M_q^{\beta}(z)={}_p^{\alpha}M_q^{\beta}\left((a_j)_1^p;(b_j)_1^q;z\right)$$
$$=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{z^n}{\Gamma(\alpha n+\beta)},z,\alpha,\beta\in\mathbb{C},R(\alpha)>0 \tag{7}$$

where, $(a_j)_n,(b_j)_n$ are the known Pochammer symbols. The series (7) is defined when none of the parameters $b_j's,j=1,2,\ldots,q,$ is a negative integer or zero; if any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial in z. The series in (7) is convergent for all z if $p\leq q$, it is convergent for $|z|<\delta=\alpha^{\alpha}$ if $p=q+1$ and divergent, if $p>q+1$. When $p=q+1$ and $|z|=\delta$, the series can converge on conditions depending on the parameters. Properties of M-series are further studied by Saxena [29], Chouhan and Sarswat [8] etc.

The generalized Mittag-Leffler function [25], is obtained from (7) for $p = q = 1$; $a = \gamma \in \mathbb{C}$; b = 1, as

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{m=0}^{\infty} \frac{(\gamma)_m}{\Gamma(\alpha m + \beta)} \frac{z^m}{m!} = \sum_{m=0}^{\infty} \frac{(\gamma)_m}{(1)_m} \frac{z^m}{\Gamma(\alpha m + \beta)} = {}_1^{\alpha}M_1^{\beta}(\gamma; 1; z) \tag{8}$$

The generalized M-series (7) can be represented as a special case of the Wright generalized hypergeometric function (6), as

$$ {}_p^{\alpha}M_q^{\beta}\left((a_j)_1^p; (b_j)_1^q; z\right) = k \; {}_{p+1}\psi_{q+1}\left[\begin{matrix}(a_1, 1), \ldots, (a_p, 1), (1, 1); \\ (b_1, 1), \ldots, (b_q, 1), (\beta, \alpha);\end{matrix} \; z\right] \tag{9}$$

where, $k = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)}$.

Fox [11] introduced and defined the H-function, via a Mellin-Barnes type integral for integers m, n, p, q such that $0 \leq m \leq q$, $0 \leq n \leq p$, for $a_i, b_j \in \mathbb{C}$ and for $\alpha_i, \beta_j \in \mathbb{R}_+ = (0, \infty)$ $(i = 1, 2, \ldots, p; j = 1, 2 \ldots, q)$, as

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n}\left[z\Big|\begin{matrix}(a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}\end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s)\, z^{-s} ds \tag{10}$$

with

$$\mathcal{H}_{p,q}^{m,n}(s) = \mathcal{H}_{p,q}^{m,n}\left[\begin{matrix}(a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q}\end{matrix}\Big| s\right] = \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)} \tag{11}$$

with all convergence conditions as given by Braaksma [6], Mathai [23], Kilbas and Saigo [16]. On putting $\alpha_j = \beta_j = 1$ in H-function, we obtain the Meijer's G-functions $G_{p,q}^{m,n}(z)$ [11].

The importance of the H-Function and M-Series realized by scientists, engineers and statisticians (Caputo [7], Glockle and Nonnenmacher [12], Mainardi et al. [21], Hilfer [13] etc.) due to its vast potential of their applications in diversified fields of science and engineering such as fluid flow, rheology, diffusion in porous media, propagation of seismic waves, anomalous diffusion and turbulence etc.

In this paper, we apply the integral operators (1) and (2) to the M-series and H-function to express the images in terms of generalized Wright and H-functions again.

2. MAIN RESULTS

In this section, the image formulas for the M-series and Fox's H-function involving Saigo-Maeda fractional integral operators (1) and (2) are obtained. Results are established in term of the generalized Wright function and H-function. These results are given by mean of the following theorems:

Theorem 2.1 Let $\alpha, \alpha', \beta, \beta', \gamma, \upsilon, \delta \in \mathbb{C}$ and $x > 0$ be such that $Re(\gamma) > 0, Re(\upsilon) > 0, Re(\delta) > max\left\{0, Re\left(\alpha + \alpha' + \beta - \gamma\right), Re\left(\alpha' - \beta'\right)\right\}$ then there holds the formula

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\delta-1} {}_p^{\upsilon}M_q^{\delta}(at^{\upsilon})\right)(x) = x^{\delta+\gamma-\alpha-\alpha'-1} \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)}$$

$$\times {}_{p+3}\psi_{q+3}\left[\begin{array}{c}(a_1,1),\ldots,(a_p,1),(1,1),\left(\delta+\gamma-\alpha-\alpha^{'}-\beta,\upsilon\right),\left(\delta+\beta^{'}-\alpha^{'},\upsilon\right);\\(b_1,1),\ldots,(b_q,1),(\delta+\beta',\upsilon),(\delta+\gamma-\alpha-\alpha',\upsilon),(\delta+\gamma-\alpha'-\beta,\upsilon);\end{array} ax^{\upsilon}\right] \tag{12}$$

Proof. Using (1) and (7), and then changing the order of integration and summation, we get

$$\left(I_{0,+}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma}t^{\delta-1}\,{}_{p}^{\upsilon}M_{q}^{\delta}\left(at^{\upsilon}\right)\right)(x)$$
$$=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{a^n}{\Gamma(\upsilon n+\delta)}\left(I_{0,+}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma}t^{\upsilon n+\delta-1}\right)(x)$$

applying (4), under the conditions stated with theorem 2.1, we obtained

$$\left(I_{0,+}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma}t^{\delta-1}\,{}_{p}^{\upsilon}M_{q}^{\delta}\left(at^{\upsilon}\right)\right)(x)=x^{\delta+\gamma-\alpha-\alpha^{'}-1}\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}$$
$$\times\frac{\Gamma\left(\upsilon n+\delta+\gamma-\alpha-\alpha^{'}-\beta\right)\Gamma\left(\upsilon n+\delta+\beta^{'}-\alpha^{'}\right)}{\Gamma(\upsilon n+\delta+\beta')\Gamma(\upsilon n+\delta+\gamma-\alpha-\alpha')\Gamma(\upsilon n+\delta+\gamma-\alpha'-\beta)}(ax^{\upsilon})^n$$

Interpreting the right-hand side of the above equation, in view of the definition (6), we arrive at the result (12).

On setting $p=q=1; a=\eta\in\mathbb{C}; b=1$ in (12), we obtained the following particular case of theorem 2.1:

Corollary 2.1 Let $\alpha,\alpha^{'},\beta,\beta^{'},\gamma,\upsilon,\delta\in\mathbb{C}$ and $x>0$ be such that $Re(\gamma)>0$, $Re(\upsilon)>0$, $Re(\delta)>max\ \left\{0,Re\left(\alpha+\alpha^{'}+\beta-\gamma\right),Re\left(\alpha^{'}-\beta^{'}\right)\right\}$ then there holds the formula

$$\left(I_{0,+}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma}t^{\delta-1}E_{\upsilon,\delta}^{\eta}\left(at^{\upsilon}\right)\right)(x)=\frac{x^{\delta+\gamma-\alpha-\alpha^{'}-1}}{\Gamma(\eta)}$$
$$\times\,{}_{3}\psi_{3}\left[\begin{array}{c}(\eta,1),\left(\delta+\gamma-\alpha-\alpha^{'}-\beta,\upsilon\right),\left(\delta+\beta^{'}-\alpha^{'},\upsilon\right);\\(\delta+\beta',\upsilon),(\delta+\gamma-\alpha-\alpha',\upsilon),(\delta+\gamma-\alpha'-\beta,\upsilon);\end{array} ax^{\upsilon}\right] \tag{13}$$

Theorem 2.2 Let $\alpha,\alpha^{'},\beta,\beta^{'},\gamma,\upsilon,\delta\in\mathbb{C}$ and $x>0$ be such that $Re(\upsilon)>0$, $Re(\gamma)>0, Re(1-\beta-\delta)<1+min\ \left\{Re(-\beta),Re\left(\alpha+\alpha^{'}-\gamma\right),Re\left(\alpha+\beta^{'}-\gamma\right)\right\}$ then there holds the formula

$$\left(I_{0,-}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma}t^{-\beta-\delta}\,{}_{p}^{\upsilon}M_{q}^{\delta}\left(at^{-\upsilon}\right)\right)(x)=x^{-\left(\beta+\delta+\alpha+\alpha'\right)}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}$$

$$\times {}_{p+3}\psi_{q+3}\left[\begin{array}{c}(a_1,1),\ldots,(a_p,1),(1,1),\left(\beta+\delta+\alpha+\alpha^{'}-\gamma,\upsilon\right),\left(\beta+\delta+\alpha+\beta^{'}-\gamma,\upsilon\right);\\(b_1,1),\ldots,(b_q,1),(\beta+\delta,\upsilon),(\beta+\delta+\alpha+\alpha'+\beta'-\gamma,\upsilon),(\delta+\alpha,\upsilon);\end{array} ax^{-\upsilon}\right] \tag{14}$$

Proof. Using (2) and (7), and then changing the order of integration and summation, we get

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}\ t^{-\beta-\delta}\ {}_{p}^{\upsilon}M_{q}^{\delta}\left(at^{-\upsilon}\right)\right)(x)$$

$$=\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{a^n}{\Gamma(\upsilon n+\delta)}\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{-\upsilon n-\beta-\delta}\right)(x)$$

applying (5), under the conditions stated with theorem 2.2, we obtained

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}\ t^{-\beta-\delta}\ {}_{p}^{\upsilon}M_{q}^{\delta}\left(at^{-\upsilon}\right)\right)(x)=x^{-\left(\beta+\delta+\alpha+\alpha'\right)}\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}$$

$$\times\frac{\Gamma\left(\upsilon n+\beta+\delta-\gamma+\alpha+\alpha'\right)\Gamma\left(\upsilon n+\beta+\delta+\alpha+\beta'-\gamma\right)}{\Gamma(\upsilon n+\beta+\delta)\Gamma(\upsilon n+\beta+\delta+\alpha+\alpha'+\beta'-\gamma)\Gamma(\upsilon n+\delta+\alpha)}\left(ax^{-\upsilon}\right)^n$$

In view of the definition of the generalized Wright function given by (6), the above equation leads to the result (14).

On setting $p=q=1; a=\eta\in\mathbb{C}; b=1$ in (14), we obtained the following particular case of theorem 2.2:

Corollary 2.2 Let $\alpha,\alpha',\beta,\beta',\gamma,\upsilon,\delta\in\mathbb{C}$ and $x>0$ be such that $Re(\upsilon)>0$, $Re(\gamma)>0$, $Re(1-\beta-\delta)<1+min\left\{Re(-\beta), Re\left(\alpha+\alpha'-\gamma\right), Re\left(\alpha+\beta'-\gamma\right)\right\}$ then there holds the formula

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{-\beta-\delta}E_{\upsilon,\delta}^{\eta}\left(at^{-\upsilon}\right)\right)(x)=\frac{x^{-\left(\beta+\delta+\alpha+\alpha'\right)}}{\Gamma(\eta)}$$

$$\times{}_3\psi_3\left[\begin{matrix}(\eta,1),\left(\beta+\delta+\alpha+\alpha'-\gamma,\upsilon\right),\left(\beta+\delta+\alpha+\beta'-\gamma,\upsilon\right);\\(\beta+\delta,\upsilon),(\beta+\delta+\alpha+\alpha'+\beta'-\gamma,\upsilon),(\delta+\alpha,\upsilon);\end{matrix}\ ax^{-\upsilon}\right]\quad(15)$$

Theorem 2.3 Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma,\omega,a_i,b_j\in\mathbb{C},\alpha_j,\beta_j\in\mathbb{R}_+$, $(i=1,2,\dots,p; j=1,2,\dots,q), m,n,p,q\in I$ where $0\le m\le q, 0\le n\le p$, and $x>0$ such that $Re(\sigma)>0$, $Re(\gamma)>0, Re(\omega+1)>max\left\{0,\ Re\ \left(\alpha+\alpha'+\beta-\gamma\right), Re(\alpha'-\beta')\right\}$ then there holds the formula

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\omega}H_{p,q}^{m,n}\left(t^{\sigma}\ \right)\right)(x)=x^{\omega+\gamma-\alpha-\alpha'}$$

$$\times H_{p+3,q+3}^{m,n+3}\left[x^{\sigma}\Big|\begin{matrix}(a_j,\alpha_j)_{1,p},\ (-\omega,\sigma),\left(-\omega-\gamma+\alpha+\alpha'+\beta,\sigma\right),(-\omega-\beta'+\alpha',\sigma)\\(b_j,\beta_j)_{1,q},\ (-\omega,-\beta'\ ,)(-\omega-\gamma+\alpha+\alpha',\sigma),(-\omega-\gamma+\alpha'+\beta,\sigma)\end{matrix}\right]\quad(16)$$

Proof. Using (1) and (10), and then changing the order of integrations, we get

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\omega}H_{p,q}^{m,n}\left[t^{\sigma}\Big|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]\right)(x)$$

$$=\frac{1}{2\pi i}\int_{\mathcal{L}}\mathcal{H}_{p,q}^{m,n}\left[\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\Big|s\right]\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\omega-\sigma s}\right)(x)\,ds$$

by using equation (4), under the conditions stated with theorem 2.3, we obtained

$$= \frac{x^{\omega+\gamma-\alpha-\alpha'}}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \left[\begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} | s \right]$$

$$\times \frac{\Gamma(\omega - \sigma s + 1)\Gamma(\omega - \sigma s + 1 + \gamma - \alpha - \alpha' - \beta)\Gamma(\omega - \sigma s + 1 + \beta' - \alpha')x^{-\sigma s} ds}{\Gamma(\omega - \sigma s + 1 + \beta')\Gamma(\omega - \sigma s + 1 + \gamma - \alpha - \alpha')\Gamma(\omega - \sigma s + 1 + \gamma - \alpha' - \beta)}$$

finally, by the virtue of the definition of the H-function (10) and (11), we obtained equation (16).

On Setting $\alpha_j = \beta_j = 1$ we obtain following particular case for Meijer's G-function $G_{p,q}^{m,n}(z)$:

Corollary 2.3 Let $\alpha, \alpha', \beta, \beta', \gamma, \sigma, \omega, a_i, b_j \in \mathbb{C}$, $(i = 1, 2, \ldots .p; j = 1, 2, \ldots ..q)$; $m, n, p, q \in I$ where $0 \le m \le q, 0 \le n \le p$ and $x > 0$ be such that

$$Re(\sigma) > 0, Re(\gamma) > 0, Re(\omega + 1) > max\left\{0, Re\left(\alpha + \alpha' + \beta - \gamma\right), Re(\alpha' - \beta')\right\}$$

then there holds the formula

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\omega} G_{p,q}^{m,n}(t^{\sigma})\right)(x) = x^{\omega+\gamma-\alpha-\alpha'}$$

$$\times G_{p+3,q+3}^{m,n+3} \left[x^{\sigma} | \begin{matrix} (a_j)_{1,p}, (-\omega, \sigma), \left(-\omega - \gamma + \alpha + \alpha' + \beta, \sigma\right), (-\omega - \beta' + \alpha', \sigma) \\ (b_j)_{1,q}, (-\omega, -\beta'), (-\omega - \gamma + \alpha + \alpha', \sigma), (-\omega - \gamma + \alpha' + \beta, \sigma) \end{matrix} \right] \tag{17}$$

Theorem 2.4 Let $\alpha, \alpha', \beta, \beta', \gamma, \sigma, \omega, a_i, b_j \in \mathbb{C}, \alpha_j, \beta_j \in \mathbb{R}_+$, $(i = 1, 2, \ldots, p; j = 1, 2, \ldots, q), m, n, p, q \in I$ where $0 \le m \le q, 0 \le n \le p$, and $x > 0$ be such that $Re(\sigma) > 0, Re(\gamma) > 0$, and

$$Re(1 - \beta - \delta) < 1 + min\left\{(-\beta), Re\left(\alpha + \alpha' - \gamma\right), Re\left(\alpha + \beta' - \gamma\right)\right\}$$

then there holds the formula

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\omega} H_{p,q}^{m,n}(t^{\sigma})\right)(x) = x^{\omega+\gamma-\alpha-\alpha'}$$

$$\times H_{p+3,q+3}^{m+3,n} \left[x^{\sigma} | \begin{matrix} (a_j, \alpha_j)_{1,p}, & (-\omega, \sigma), \left(-\omega + \alpha + \alpha' + \beta' - \gamma, \sigma\right), (-\omega + \alpha - \beta, \sigma) \\ (b_j, \beta_j)_{1,q}, & (-\omega - \gamma + \alpha + \alpha', \sigma), (-\omega + \alpha + \beta' - \gamma, \sigma), (-\omega - \beta, \sigma) \end{matrix} \right] \tag{18}$$

Proof. Using (2) and (10), and then changing the order of integrations, we get

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\omega} H_{p,q}^{m,n}(t^{\sigma})\right)(x)$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \left[\begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} | s \right] \left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\omega-\sigma s}\right)(x) \ ds$$

by using equation (5) under the conditions stated with theorem 2.4, we obtained

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\omega} H_{p,q}^{m,n}(t^{\sigma})\}\right)(x) = x^{\omega+\gamma-\alpha-\alpha'} \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \left[\begin{matrix} (a_j, \alpha_j)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} | s \right]$$

$$\times\frac{\Gamma\left(-\omega+\sigma s-\gamma+\alpha+\alpha^{'}\right)\Gamma\left(\sigma s-\omega+\alpha+\beta^{'}-\gamma\right)\Gamma\left(\sigma s-\omega-\beta\right)}{\Gamma\left(\sigma s-\omega\right)\Gamma\left(\sigma s-\omega+\alpha+\alpha'+\beta'-\gamma\right)\Gamma\left(\sigma s-\omega+\alpha-\beta\right)}x^{-\sigma s}ds$$

finally, by the virtue of the definition of the H-function (10) and (11), we obtained equation (18).

On Setting $\alpha_j = \beta_j = 1$ we obtain following particular case for Meijer's G-functions $G_{p,q}^{m,n}(z)$:

Corollary 2.4 Let $\alpha,\alpha^{'},\beta,\beta^{'},\gamma,\sigma,\omega,a_i,b_j\in\mathbb{C}$, $(i=1,2,\ldots.p;j=1,2,\ldots.q)$; $m,n,p,q\in I$ where $0\le m\le q, 0\le n\le p$ and $x>0$ be such that $Re(\sigma)>0$, $Re(\gamma)>0$, and

$$Re\left(1-\beta-\delta\right)<1+min\left\{\left(-\beta\right),Re\left(\alpha+\alpha^{'}-\gamma\right),Re\left(\alpha+\beta^{'}-\gamma\right)\right\}$$

then there holds the formula

$$\left(I_{0,-}^{\alpha,\alpha^{'},\beta,\beta^{'},\gamma}t^{\omega}G_{p,q}^{m,n}\left(t^{\sigma}\right)\right)(x)=x^{\omega+\gamma-\alpha-\alpha^{'}}$$

$$\times G_{p+3,q+3}^{m+3,n}\left[x^{\sigma}\middle|\begin{matrix}(a_j)_{1,p}, & (-\omega,\sigma),\left(-\omega+\alpha+\alpha^{'}+\beta^{'}-\gamma,\sigma\right),\left(-\omega+\alpha-\beta,\sigma\right)\\ (b_j)_{1,q}, & \left(-\omega-\gamma+\alpha+\alpha',\sigma\right),\left(-\omega+\alpha+\beta'-\gamma,\sigma\right),\left(-\omega-\beta,\sigma\right)\end{matrix}\right] \tag{19}$$

On setting $\alpha^{'}=0$ in the operators (1) and (2), then by the known identities due to Saxena and Saigo [[30], p.93, eqn. (2.15) and (2.16)], we obtained Saigo operators [26]. Further, the Riemann-Liouville,Weyl and Erdélyi-Kober fractional calculus operators are special cases of Saigo's operators [26]. Therefore, the results obtained in this article are useful in deriving certain composition formulas involving Riemann-Liouville, Weyl and Erdélyi-Kober fractional calculus operators with M-series and H-function.

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