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# SUBORDINATION RESULTS FOR FRACTIONAL INTEGRAL ASSOCIATED WITH DZIOK-SRIVASTAVA OPERATOR

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ABSTRACT. In this paper, we have discussed differential subordination properties associated with the fractional integral by using Dziok Srivastava operator.

## 1. Introduction

Let $H(U)$ denote the space of analytic functions in the open unit disk

$$U = \{z \in C : |z| < 1\}.$$

Let $A_n = \left\{f \in H(U), f(z) = z + a_{n+1}z^{n+1} + \cdots, z \in U\right\}$ with $A_1 = A$.
If $f$ and $g$ are analytic functions in $U$, then we say that $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w$ analytic in $U$, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)), (z \in U)$. In particular, if the function $g$ is univalent in $U$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.
Let $\psi(z) : C^3 \times U \to C$ and let $h$ be univalent function in $U$.
If $p$ is analytic in $U$ and satisfies the (second-order) differential subordination:

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1}$$

then $p$ is called a solution of the differential subordination. The univalent function $q$ is called a dominant of the solutions of the differential subordination, or more simply dominant if $p \prec q$ for all $p$ satisfying (1).
**Definition 1** (see [6]) For $f \in A$. The Dziok-Srivastava operator is defined by

$$H_m^l(\alpha_1, \alpha_2, \cdots, \alpha_l; \beta_1, \beta_2, \cdots, \beta_m)f(z) = z+\sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1} \cdots (\alpha_l)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1} \cdots (\beta_m)_{n-1}(n-1)!} a_n z^n, \tag{2}$$

$\alpha_i \in C, i = 1, 2, \cdots, l, \beta_j \in C\backslash\{0, -1, -2, \cdots\}, j = 1, 2, \cdots, m,$
where $(x)_n$ is the Pochhammer symbol defined, in terms of the Gamma function,

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$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 if n = 0 and x \in C\backslash\{0\}, \\ x(x+1)\cdots(x+n-1) if n \in N and x \in C. \end{cases}$$

For simplicity, we write

$$H_m^l[\alpha_1] f(z) = H_m^l(\alpha_1, \alpha_2, \cdots, \alpha_l; \beta_1, \beta_2, \cdots, \beta_m) f(z). \tag{3}$$

**Definition 2** (see [1]) The fractional integral of order $\lambda(\lambda > 0)$ is defined for a function $f$ by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} \, dt, \tag{4}$$

where $f$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$.
From Definition (1) and Definition 2, we get

$$D_z^{-\lambda} H_m^l[\alpha_1] f(z) = \frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\lambda))} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}(\alpha_l)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}(\beta_m)_{n-1}(n-1)!} \times a_n z^{n+\lambda}. \tag{5}$$

We note from (5) that, we have

$$z(D_z^{-\lambda} H_m^l[\alpha_1] f(z))' = \alpha_1 D_z^{-\lambda} H_m^l[\alpha_1 + 1] f(z) - [\alpha_1 - (1+\lambda)] D_z^{-\lambda} H_m^l[\alpha_1] f(z). \tag{6}$$

**Lemma 1** (see [5]) Let $g$ be a convex function in $U$ and let $h(z) = g(z) + n\alpha g'(z)$, for $z \in U$, where $\alpha > 0$ and $n$ is a positive integer. If $p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \cdots$, for $z \in U$, is analytic in $U$ and

$$p(z) + \alpha z p'(z) \prec h(z),$$

for $z \in U$, then $p(z) \prec g(z)$ and this result is sharp.
Such type of study was carried out by various authors for another classes (see [2], [3], [4]).

## 2. Main Results

**Theorem 1** Let $g$ be a convex function such that $g(0) = 1$ and let $h$ be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $f \in A$ satisfies the differential subordination:

$$\frac{\alpha_1(1-\lambda)\lambda!}{z^{1+\lambda}} D_z^{-\lambda} H_m^l[\alpha_1 + 1] f(z) - \frac{[\lambda(\lambda - \alpha_1) + \alpha_1 - 1]\lambda!}{z^{1+\lambda}} D_z^{-\lambda} H_m^l[\alpha_1] f(z) + \frac{\lambda!}{z^{-1+\lambda}} (D_z^{-\lambda} H_m^l[\alpha_1] f(z))'' \prec h(z), \tag{7}$$

then

$$\frac{\lambda!(D_z^{-\lambda} H_m^l[\alpha_1] f(z))'}{z^\lambda} \prec g(z).$$

**Proof.** Suppose that

$$p(z) = \frac{\lambda!(D_z^{-\lambda} H_m^l[\alpha_1] f(z))'}{z^\lambda}. \tag{8}$$

Then $p(0) = 1$.
Differentiating both sides of (8) with respect to $z$ and using (7), we have

$$\frac{\alpha_1(1-\lambda)\lambda!}{z^{1+\lambda}} D_z^{-\lambda} H_m^l [\alpha_1+1] f(z) - \frac{[\lambda(\lambda-\alpha_1)+\alpha_1-1]\lambda!}{z^{1+\lambda}} D_z^{-\lambda} H_m^l [\alpha_1] f(z) + \frac{\lambda!}{z^{-1+\lambda}} (D_z^{-\lambda} H_m^l [\alpha_1] f(z))'' = p(z) + zp'(z) \prec h(z). \quad (9)$$

By using Lemma 1, we obtain $p(z) \prec g(z)$. By (8), we get

$$\frac{\lambda!(D_z^{-\lambda} H_m^l [\alpha_1] f(z))'}{z^{\lambda}} \prec g(z).$$

**Theorem 2** Let $g$ be a convex function such that $g(0) = 1$ and let $h$ be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $f \in A$ satisfies the differential subordination:

$$\frac{1}{\alpha_1-(1+\lambda)} \left[ \frac{(\alpha_1-1)(1+\lambda)!}{z^{\lambda}} (D_z^{-\lambda} H_m^l [\alpha_1] f(z))' - \frac{\alpha_1\lambda(1+\lambda)!}{z^{1+\lambda}} D_z^{-\lambda} H_m^l [\alpha_1+1] f(z) \right] \prec h(z), \quad (10)$$

then

$$\frac{(1+\lambda)!(D_z^{-\lambda} H_m^l [\alpha_1] f(z))}{z^{1+\lambda}} \prec g(z).$$

**Proof.** Suppose that

$$p(z) = \frac{(1+\lambda)!(D_z^{-\lambda} H_m^l [\alpha_1] f(z))}{z^{1+\lambda}}. \quad (11)$$

Then $p(0) = 1$.
Differentiating both sides of (11) with respect to $z$ and using (10), we have

$$\frac{1}{\alpha_1-(1+\lambda)} \left[ \frac{(\alpha_1-1)(1+\lambda)!}{z^{\lambda}} (D_z^{-\lambda} H_m^l [\alpha_1] f(z))' - \frac{\alpha_1\lambda(1+\lambda)!}{z^{1+\lambda}} D_z^{-\lambda} H_m^l [\alpha_1+1] f(z) \right] = p(z) + zp'(z) \prec h(z). \quad (12)$$

By using Lemma 1, we obtain $p(z) \prec g(z)$. By (11), we get

$$\frac{(1+\lambda)! D_z^{-\lambda} H_m^l [\alpha_1] f(z)}{z^{1+\lambda}} \prec g(z).$$

**Theorem 3** Let $g$ be a convex function such that $g(0) = 1$ and let $h$ be the function $h(z) = g(z) + zg'(z)$, for $z \in U$. If $f \in A$ satisfies the differential subordination:

$$\left[ \frac{z D_z^{-\lambda} H_m^l [\alpha_1+1] f(z)}{D_z^{-\lambda} H_m^l [\alpha_1] f(z)} \right]' \prec h(z), \quad (13)$$

then

$$\frac{D_z^{-\lambda} H_m^l [\alpha_1+1] f(z)}{D_z^{-\lambda} H_m^l [\alpha_1] f(z)} \prec g(z).$$

**Proof.** Suppose that

$$p(z) = \frac{D_z^{-\lambda} H_m^l [\alpha_1+1] f(z)}{D_z^{-\lambda} H_m^l [\alpha_1] f(z)}. \quad (14)$$

Note

$$p(z) = \frac{\frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\lambda)} \frac{(\alpha_1+1)_{n-1}(\alpha_2)_{n-1}\cdots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\cdots(\beta_m)_{n-1}(n-1)!} a_n z^{n+\lambda}}{\frac{1}{\Gamma(2+\lambda)} z^{1+\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\lambda)} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\cdots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\cdots(\beta_m)_{n-1}(n-1)!} a_n z^{n+\lambda}}$$

$$= \frac{\frac{1}{\Gamma(2+\lambda)} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\lambda)} \frac{(\alpha_1+1)_{n-1}(\alpha_2)_{n-1}\cdots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\cdots(\beta_m)_{n-1}(n-1)!} a_n z^{n-1}}{\frac{1}{\Gamma(2+\lambda)} + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\lambda)} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\cdots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\cdots(\beta_m)_{n-1}(n-1)!} a_n z^{n-1}}$$

Then $p(0) = 1$.

Differentiating both sides of (14) with respect to $z$ and using (13), we have

$$\left[\frac{zD_z^{-\lambda} H_m^l [\alpha_1+1] f(z)}{D_z^{-\lambda} H_m^l [\alpha_1] f(z)}\right]' = \frac{D_z^{-\lambda} H_m^l [\alpha_1+1] f(z)}{D_z^{-\lambda} H_m^l [\alpha_1] f(z)}$$

$$+z\frac{D_z^{-\lambda} H_m^l [\alpha_1] f(z) \left(D_z^{-\lambda} H_m^l [\alpha_1+1] f(z)\right)' - D_z^{-\lambda} H_m^l [\alpha_1+1] f(z) \left(D_z^{-\lambda} H_m^l [\alpha_1] f(z)\right)'}{\left(D_z^{-\lambda} H_m^l [\alpha_1] f(z)\right)^2}$$

$$= p(z) + zp'(z) \prec h(z). \tag{15}$$

By using Lemma 1, we obtain $p(z) \prec g(z)$. By (14), we get

$$\frac{D_z^{-\lambda} H_m^l [\alpha_1+1] f(z)}{D_z^{-\lambda} H_m^l [\alpha_1] f(z)} \prec g(z).$$

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