Journal of Fractional Calculus and Applications,
Vol. 5(2) July 2014, pp. 78-83.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

FRACTIONAL CALCULUS OPERATORS INVOLVING GENERALIZED M-SERIES

DHARMENDRA KUMAR SINGH

ABSTRACT. The principal aims of this paper is the study of fractional calculus operators due to Saxena and Kumbhat [7] and generalized M-series [6].

1. Introduction

H-function. The H function [3] and [1] in terms of Mellin-Barnes type contour integral, is defined by

$$H_{p,q}^{m,n}[z] = H\left[\begin{array}{c}(a_p, A_p)\\(b_q, B_q)\end{array}|z\right] = H_{p,q}^{m,n}\left[\begin{array}{c}(a_1, A_1), ..., (a_p, A_p)\\(b_1, B_1), ..., (b_q, B_q)\end{array}|z\right] = \frac{1}{2\pi i}\int_L \chi(s) z^s\, ds, \tag{1}$$

where

$$\chi(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j + B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + A_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + B_j s)\prod_{j=n+1}^{p}\Gamma(a_j + A_j s)}, \tag{2}$$

$m, n, p, q \in \mathbb{N}_0$ with $0 \le n \le p$, $1 \le m \le q$, $A_i, B_j \in \mathbb{R}_+$, $a_i, b_j \in \mathbb{R}$ or $\mathbb{C}$ $(i = 1, 2, ..., p; j = 1, 2, ..., q)$ such that

$$A_j(b_j + k) \ne B_j(a_i - l - 1)(k, l \in \mathbb{N}_0; i = 1, 2, ..., n; j = 1, ..., m),$$

where we use the usual notation: $\mathbb{N}_0 = 0, 1, 2, ..., \mathbb{R} = (-\infty, \infty), \mathbb{R}_+ = (0, \infty)$ and $\mathbb{C}$ being the complex number field. $L = L_{i\tau\infty}$ is a contour starting at the point $\tau - i\infty$, terminating at the point $\tau + i\infty$ with $\tau \in \mathbb{R} = (-\infty, \infty)$. Here z may be real or complex but is not equal to zero and an empty product is interpreted as unity. The H-function make sense, if

$$\Omega = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j > 0, |\arg z| < \frac{1}{2}\pi\Omega, z \ne 0$$

2010 *Mathematics Subject Classification.* 26A33, 33C60, 33E12.
Key words and phrases. H-function, generalized M-series, Saxena and Kumbhat operators.
Submitted Dec. 23, 2013. Revised March 5, 2014.

$$\Omega = 0, \triangle\tau + \Re(\mu) < -1, \arg z = 0, z \neq 0$$

, where

$$\triangle = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j ; \mu = \sum_{j=1}^{q} b_j + \sum_{j=1}^{p} a_j + \frac{p-q}{2}.$$

Saxena and Kumbhat Operators. Fractional integral operators involving Fox's H-function where defined and studied by Saxena and Kumbhat [7] in the following form

$$R_{x,r}^{\eta,\alpha}[f(x)] = rx^{-\eta-r\alpha-1} \int_0^x t^{\eta}(x^r - t^r)^{\alpha} H_{p,q}^{m,n} \left[kU \mid \begin{array}{c} (a_p, A_p) \\ (b_q, B_q) \end{array} \right] f(t)dt \quad (3)$$

and

$$K_{x,r}^{\delta,\alpha}[f(x)] = rx^{\delta} \int_x^{\infty} t^{-\delta-r\alpha-1}(t^r - x^r)^{\alpha} H_{p,q}^{m,n} \left[kV \mid \begin{array}{c} (a_p, A_p) \\ (b_q, B_q) \end{array} \right] f(t)dt, \quad (4)$$

where U and V represent the expressions

$$\left(\frac{t^r}{x^r}\right)^m \left(1 - \frac{t^r}{x^r}\right)^n and \left(\frac{x^r}{t^r}\right)^m \left(1 - \frac{x^r}{t^r}\right)^n,$$

respectively and r, m, n are positive integers. The condition of the validity of the operators (3) and (4) are given below:
(i)$1 \leq p, q < \infty; p^{-1} + q^{-1} = 1$;
(ii) $\Re\left[\eta + \left(rm\frac{b_j}{\beta_j}\right)\right] > -q^{-1}$;
$\Re\left[\alpha + \left(rn\frac{b_j}{\beta_j}\right)\right] > -q^{-1}$;
$\Re\left[\alpha + \delta + \left(rm\frac{b_j}{\beta_j}\right)\right] > p^{-1}(j = 1, ..., m)$
(iii)$f(x) \in L_p(0, \infty)$,
where $[L_{p_j}(0; 1)]$ denotes the space of Lebesgue integrable functions of n variables exactly with p_j-th power integrable with respect to each of the variables x_j. This is the space with mixed norm and was considered in [8].
Generalized M-series. The generalized M-series function is introduced by Sharma and Renu [6],

$$\begin{aligned} {}_pM_q^{\alpha,\beta}(z) &:= {}_pM_q^{\alpha,\beta}(a_1, ..., a_p; b_1, ..., b_q; z) \\ &= \sum_{r=0}^{\infty} \frac{(a_1)...(a_p)}{(b_1)...(b_q)} \frac{z^r}{\Gamma(\alpha r + \beta)}. \end{aligned} \quad (5)$$

Here, $(a)_r := \frac{\Gamma(a+r)}{\Gamma(a)}, \alpha, \beta \in C, \Re(\alpha) > 0$. From the ratio test it is evident that the series in (5) is convergent for all z if $q \geq p$, it is convergent for $|z| < 1$ if $p = q + 1$ and divergent if $p > q + 1$. When $p = q + 1$ and $|z| = 1$ the series can converge in some case. Let

$$\beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j.$$

It can be shown that when $p = q + 1$ the series is absolutely convergent for $|z| = 1$ if $\Re(\beta) < 0$, conditionally convergent for $z = -1$ if $0 \leq \Re(\beta) \leq 1$ and divergent for $|z| = 1$ if $1 \leq \Re(\beta)$.

2. Theorems

Theorem 1 If $\alpha, \beta \in C, \Re(\alpha) > 0 f(x) \in L_p(0,\infty), 1 \leq p \leq 2$ or $[f(x) \in M_p(0,\infty)$ and $p > 2]$, $|\arg(k)| < \frac{1}{2}\pi\Omega, \Omega > 0, \Re\left[\eta + \left(rm\frac{b_j}{\beta_j}\right)\right] > -q^{-1}, \Re\left[\alpha + \left(rn\frac{b_j}{\beta_j}\right)\right] > -q^{-1}, \Re\left[\alpha + \delta + \left(rm\frac{b_j}{\beta_j}\right)\right] > p^{-1} (j = 1, ..., m)$ and $p^{-1} + q^{-1} = 1$, then

$$R_{x,r}^{\eta,\alpha}[{}_l\overset{\alpha,\beta}{M}_m(a_1,...,a_l;b_1,...,b_m;x)]$$

$$= {}_l\overset{\alpha,\beta}{M}_m(x) H_{P+2,Q+1}^{M,N+2}\left[k \,\Big|_{(b_Q,B_Q),\left(-\alpha-\frac{\eta}{r}-\frac{k}{r}-\frac{1}{r},m+n\right)}^{(a_P,A_P),(-\alpha,n),\left(1-\left(\frac{\eta}{r}+\frac{k}{r}+\frac{1}{r}\right),m\right)}\right], \tag{6}$$

where $l \leq m+1$.

Proof. With equation (1), (3) and (5), we have

$$R_{x,r}^{\eta;\alpha}[{}_l\overset{\alpha,\beta}{M}_m(a_1,...,a_l;b_1,...,b_m;x)]$$

$$= rx^{-\eta-r\alpha-1}\int_0^x t^\eta (x^r - t^r)^\alpha \frac{1}{2\pi i}\int_L \chi(s)(kU)^s ds$$

$$\times {}_l\overset{\alpha,\beta}{M}_m(a_1,...,a_l;b_1,...,b_m;t)dt. \tag{7}$$

Changing the order of the integration valid under the condition given with the theorem, we obtain

$$= rx^{-\eta-r\alpha-1}\sum_{k=0}^{\infty}\frac{(a_1)_k,...,(a_l)_k}{(b_1)_k,...,(b_m)_k}\frac{1}{\Gamma(\alpha k+\beta)}\frac{1}{2\pi i}\int_L \chi(s)k^s$$

$$\times x^{r\alpha - rms}\left\{\int_0^x\left(1-\frac{t^r}{x^r}\right)^{\alpha+ns} t^{\eta+k+rms}dt\right\}ds. \tag{8}$$

Put $\frac{t^r}{x^r} = u$ then $t = xu^{\frac{1}{r}}$ in (8) and using beta function formula defined as

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^1 (1-x)^{m-1}x^{n-1}dx, \tag{9}$$

where m and n are positive i.e.,$m > 0, n > 0$. We arrive at the desired result

$$= \sum_{k=0}^{\infty}\frac{(a_1)_k,...,(a_l)_k}{(b_1)_k,...,(b_m)_k}\frac{z^k}{\Gamma(\alpha k+\beta)}$$

$$\times \frac{1}{2\pi i}\int_L \chi(s)\frac{\Gamma(1+\alpha+ns)\Gamma((\eta+k+1)\frac{1}{r}+ms)}{\Gamma(1+\alpha+ns+(\eta+k+1)\frac{1}{r}+ms)}k^s ds. \tag{10}$$

Theorem 2 If $\alpha, \beta \in C, \Re(\alpha) > 0 f(x) \in L_p(0,\infty), 1 \leq p \leq 2$ or $[f(x) \in M_p(0,\infty)$ and $p > 2]$, $|\arg(k)| < \frac{1}{2}\pi\Omega, \Omega > 0, \Re\left[\eta + \left(rm\frac{b_j}{\beta_j}\right)\right] > -q^{-1}, \Re\left[\alpha + \left(rn\frac{b_j}{\beta_j}\right)\right] > -q^{-1}, \Re\left[\alpha + \delta + \left(rm\frac{b_j}{\beta_j}\right)\right] > p^{-1} (j = 1, ..., m)$ and $p^{-1} + q^{-1} = 1$, then

$$k_{x,r}^{\delta,\alpha}[{}_l\overset{\alpha,\beta}{M}_m(a_1,...,a_l;b_1,...,b_m;x)]$$

$$= {}_l\overset{\alpha,\beta}{M}_n(x) H_{P+2,Q+1}^{M,N+2}\left[k \,\Big|_{(b_Q,B_Q),(k/r-\delta/r-\alpha,m+n)}^{(a_P,A_P),(1-\delta/r+k/r,m),(-\alpha,n))}\right], \tag{11}$$

where $l \leq m+1$.

Proof. Using (1), (4) and (5), we get

$$K_{x,r}^{\delta,\alpha}[{}_l\overset{\alpha,\beta}{M}_m(a_1,...,a_l;b_1,...,b_m;x)] = rx^{\delta}\int_x^{\infty} t^{-\delta-r\alpha-1}(t^r-x^r)^{\alpha}\frac{1}{2\pi i}\int_L \chi(s)(kV)^s ds$$
$$\times {}_l\overset{\alpha,\beta}{M}_m(a_1,...,a_l;b_1,...,b_m;t)dt. \tag{12}$$

Changing the order of integration of (12) which is valid under the above stated conditions, we have the left side of (12) as

$$= rx^{\delta}\sum_{k=0}^{\infty}\frac{(a_1)_k...(a_l)_k}{(b_1)_k...(b_q)_k}\frac{1}{\Gamma(\alpha k+1)}\frac{1}{2\pi i}\int_L \chi(s)k^s$$
$$\times x^{rms}\left\{\int_x^{\infty}\left(1-\frac{x^r}{t^r}\right)^{\alpha+ns} t^{-\delta-rms+k-1}dt\right\}ds. \tag{13}$$

Put $\frac{x^r}{t^r}=u$ then $t=\frac{x}{u^{\frac{1}{r}}}$ in above expression and using (9), we get

$$= \sum_{k=0}^{\infty}\frac{(a_1)_k...(a_l)_k}{(b_1)_k...(b_m)_k}\frac{x^k}{\Gamma(\alpha k+1)}$$
$$\times \frac{1}{2\pi i}\int_L \chi(s)\frac{\Gamma(1+\alpha+ns)\Gamma(\delta/r-k/r+ms)}{\Gamma(\delta/r-k/r+\alpha+1+(n+m)s)}k^s ds. \tag{14}$$

Which is the required result.

3. Special Cases

1. Putting l=m=1, a=$\rho \in C$ and b=1, in(6) and (11), we get

$$R_{x,r}^{\eta,\alpha}\left[{}_1\overset{\alpha,\beta}{M}_1(\rho;1;z)\right]$$
$$= \sum_{r=0}^{\infty}\frac{(\rho)_r}{(1)_r}\frac{z^r}{\Gamma(\alpha+\beta)}H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{array}{c}(a_P,A_P),(-\alpha,n),(1-(\eta/r+k/r+1/r),m)\\(b_Q,B_Q),(-\alpha-\eta/r-k/r-1/r,m+n)\end{array}\right]. \tag{15}$$

$$K_{x,r}^{\delta,\alpha}\left[{}_1\overset{\alpha,\beta}{M}_1(\rho;1;z)\right]$$
$$= \sum_{r=0}^{\infty}\frac{(\rho)_r}{(1)_r}\frac{z^r}{\Gamma(\alpha r+\beta)}H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{array}{c}(a_P,A_P),(-\alpha,n),(1-\delta/r+k/r,m)\\(b_Q,B_Q),(k/r-\delta/r-\alpha,m+n).\end{array}\right]. \tag{16}$$

where $p \leq q+1$ and the ${}_1\overset{\alpha,\beta}{M}_1(\rho;1;x)$ on the right is known as generalized Mittag-Leffer function, introduced by Prabhakar [9] and Studied by Kilbas et al ([2] p.45).

2. For l=m=0 in (6) and (11), we obtain the following interesting result for Mittag-Leffer function (see [2], [5])defined by

$$R_{x,r}^{\eta,\alpha}\left[{}_0\overset{\alpha,\beta}{M}_0(-;-;z)\right]$$
$$= \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\alpha r+\beta)}H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{array}{c}(a_P,A_P),(-\alpha,n),(1-(\eta/r+k/r+1/r,m)\\(b_Q,B_Q),(-\alpha-\eta/r-k/r-1/r,m+n)\end{array}\right]. \tag{17}$$

$$K^{\delta,\alpha}_{x,r}\left[{}_{0}\overset{\alpha,\beta}{M}_{0}(-;-;x)\right]$$

$$=\sum_{r=0}^{\infty}\frac{z^{r}}{\Gamma(\alpha r+\beta)}H^{M,N+2}_{P+2,Q+1}\left[k\mid\begin{array}{c}(a_P,A_P),(-\alpha,n),(1-\delta/r+k/r,m)\\(b_Q,B_Q),(k/r-\delta/r-\alpha,m+n).\end{array}\right].\quad(18)$$

where $p\leq q+1$.

3. If we set $\alpha=1,\beta=1$ in (6) and (11), we get

$$R^{\eta,\alpha}_{x,r}\left[{}_{l}\overset{1,1}{M}_{m}(a_1,...,a_p;b_1,...,b_q;z)\right]$$

$$={}_{p}F_{q}\left(a_1,...,a_p;b_1,...,b_q;z\right)H^{M,N+2}_{P+2,Q+1}\left[k\mid\begin{array}{c}(a_P,A_P),(-\alpha,n),(1-(\eta/r+k/r+1/r),m)\\(b_Q,B_Q),(-\alpha-\eta/r-k/r-1/r,m+n)\end{array}\right].\quad(19)$$

$$K^{\delta,\alpha}_{x,r}\left[{}_{l}\overset{1,1}{M}_{m}(a_1,...,a_p;b_1,...,b_q;z)\right]$$

$$={}_{p}F_{q}\left(a_1,...,a_p;b_1,...,b_q;z\right)H^{M,N+2}_{P+2,Q+1}\left[k\mid\begin{array}{c}(a_P,A_P),(-\alpha,n),(1-\delta/r+k/r,m)\\(b_Q,B_Q),(k/r-\delta/r-\alpha,m+n).\end{array}\right],\quad(20)$$

where pF_q is generalized hypergeometric function and $p\leq q+1$.

4. In equation (6) and (11) generalized M-series can be represented as a special case of the Wright generalized hypergeometric function ${}_{p}\Psi_{q}(z)$ (for its definition see e. g.[2], p. 56 (1. 11. 14) also[4] and [10])

$$R^{\eta,\alpha}_{x,r}\left[{}_{l}\overset{\alpha,\beta}{M}_{m}(a_1,...,a_p;b_1,...,b_q;z)\right]$$

$$={}_{p+1}\Psi_{q+1}\left[z\mid\begin{array}{c}(a_1,1),...,(a_p,1),(1,1)\\(b_1,1),...,(b_q,1),(\alpha,\beta)\end{array}\right]$$

$$\times H^{M,N+2}_{P+2,Q+1}\left[k\mid\begin{array}{c}(a_P,A_P),(-\alpha,n),(1-(\eta/r+k/r+1/r),m)\\(b_Q,B_Q),(-\alpha-\eta/r-k/r-1/r,m+n)\end{array}\right].\quad(21)$$

$$K^{\delta,\alpha}_{x,r}\left[{}_{l}\overset{\alpha,\beta}{M}_{m}(a_1,...,a_p;b_1,...,b_q;z)\right]$$

$$={}_{p+1}\Psi_{q+1}\left[z\mid\begin{array}{c}(a_1,1),...,(a_p,1),(1,1)\\(b_1,1),...,(b_q,1),(\alpha,\beta)\end{array}\right]$$

$$\times H^{M,N+2}_{P+2,Q+1}\left[k\mid\begin{array}{c}(a_P,A_P),(-\alpha,n),(1-\delta/r+k/r,m)\\(b_Q,B_Q),(k/r-\delta/r-\alpha,m+n).\end{array}\right].\quad(22)$$

or equations (6) and (11) can be written in the following form

$$R^{\eta,\alpha}_{x,r}\left[{}_{l}\overset{\alpha,\beta}{M}_{m}(a_1,...,a_p;b_1,...,b_q;x)\right]$$

$$=\kappa H^{1,P+1}_{P+1,Q+2}\left[-x\mid\begin{array}{c}(1-\alpha_j,1,1)_1^p,(0,1)\\(0,1),(1-\beta_j,1)_1^q(1-\beta,\alpha)\end{array}\right].$$

$$\times H^{M,N+2}_{P+2,Q+1}\left[k\mid\begin{array}{c}(a_P,A_P),(-\alpha,n),(1-(\eta/r+k/r+1/r),m)\\(b_Q,B_Q),(-\alpha-\eta/r-k/r-1/r,m+n)\end{array}\right].\quad(23)$$

$$K_{x,r}^{\delta,\alpha}\left[{}_l\overset{\alpha,\beta}{M}_m(a_1,...,a_p;b_1,...,b_q;z)\right]$$

$$= \kappa H_{P+1,Q+2}^{1,P+1}\left[-x \mid \begin{array}{c}(1-\alpha_j,1,1)_1^p,(0,1)\\(0,1),(1-\beta_j,1)_1^q(1-\beta,\alpha)\end{array}\right].$$

$$\times H_{P+2,Q+1}^{M,N+2}\left[k \mid \begin{array}{c}(a_P,A_P),(-\alpha,n),(1-\delta/r+k/r,m)\\(b_Q,B_Q),(k/r-\delta/r-\alpha,m+n)\end{array}\right], \quad (24)$$

where $\kappa = \frac{\Pi_{j=1}^{q}\Gamma(b_j)}{\Pi_{j=1}^{p}\Gamma(a_j)}$ and $p \leq q+1$.

References

[1] A. M. Mathai, R. K. Saxena, The H-function with Applications in Statistics and other discipline, Halsred Press, Wiely Eastran Limited, New Delhi: Wiley, New York 1978.

[2] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Euations, Elsevier, North Holland Math. Studies 204. Amsterdam, 2006.

[3] C. Fox, The G and H-function as symmetrical Fourier Kernals, Trans. Aner. Math. Soc. 98, 395-429, 1961.

[4] E. M. Wright, J. London Math. Soc. 10, 287-293, 1935.

[5] I. Podlubny. Fractional Differential Equations, Acad. Press, San Diegon, New York etc 1999.

[6] M. Sharma and Renu Jain, A notes on generalized M-Series as a special function of fractional calculus, Fract. Calc. appl. Anal. 4, no. 12, 449-452, 2009.

[7] R. K. Saxena, R. K. Kumbhat, Integral operators involving H-function, Indian J. Pure and Applied Mathematics, 5, 1-6, 1974.

[8] S. Samko, A. Kilbas and O. I. Marichev, Integrals and derivatives of fractional order and some of their applications (in Russian), Nauka i Technika, Minsk, 1987.

[9] T. R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J. 19, 7-15, 1971.

[10] V. Kriyakova, Some Special functions related to fractional calculus and fractional(noninteger)order control system and equations. Facta Univ. (Sci. J. of Univ. Nsi.), Ser. Automatic Control and Robotics, 7, No. 1, 79-98, 2008..

Dharmendra Kumar Singh
Department of Mathematics
University Institute of Engineering and Technology
C. S. J. M. University, Kanpur
(UP)India
E-mail address: drdksinghabp@gmail.com