Journal of Fractional Calculus and Applications,
Vol. 5(2) July 2014, pp. 38-43.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

FRACTIONAL MODELING OF TEMPERATURE DISTRIBUTION AND HEAT FLUX IN THE SEMI INFINITE SOLID

JYOTINDRA C. PRAJAPATI AND KRUNAL B. KACHHIA

Abstract. The work carried out in this paper is an interdisciplinary study of Fractional Calculus and Mechanical engineering. The aim of this paper is to introduce new model of temperature distribution and heat flux by transformed the problem into fractional partial differential equation and solved it by using fractional calculus and special functions approach.

1. Introduction

Fractional calculus is now considered as a practical technique in many branches of science including physics (Oldham and Spainier [7]). A growing number of works in sciences and engineering deal with dynamical system described by fractional order equations that involve derivatives and integrals of non-integer order (Benson, Wheatcraft and Meerschaert [1], Metzler and Klafter [5], Zaslavsky [13]). These new models are more adequate than the previously used integer order models, because fractional order derivatives and integrals describe the memory and hereditary properties of different substances (Podulbny [8]). This is the most significant advantage of the fractional order models in comparison with integer order models in which such effects are neglected.

A semi-infinite solid is an idealized body that has a single plane surface and extends to infinity in all directions. This idealized body is used to indicate that the temperature change in the part of the body in which we are interested (the region close to the surface) is due to the thermal conditions on a single surface. The earth, for example, can be considered to be a semi-infinite medium in determining the variation of temperature near its surface. Also, a thick wall can be modeled as a semi-infinite medium if all we are interested in is the variation of temperature in the region near one of the surfaces, and the other surface is too far to have any impact on the region of interest during the time of observation.

2010 *Mathematics Subject Classification.* 76S05, 35R11, 33E12.
Key words and phrases. Caputo Derivative, Fractional Differential Equation, Wright Function, Integral Transforms, Heat Flux.
Submitted Oct. 19, 2013 Revised Jan. 24, 2014.

Consider semi-infinite solid initially at temperature T_0. The left face of the solid is suddenly raised to temperature T_s at time zero. Defining $\theta = \frac{T-T_0}{T_s-T_0}$. If we assume constant thermal conductivity, no internal heat generation and negligible temperature variations in the y and z directions. The applicable differential equation is given by classical non-homogenous heat equation defined in Mills and Ganesan [6]:

$$\frac{\partial \theta}{\partial t} = C \frac{\partial^2 \theta}{\partial x^2} \tag{1}$$

where C is the thermal diffusivity. Subject to boundary conditions

$$t = 0 : \ \theta = 0 \tag{2}$$

$$x = 0 : \ \theta = 1 \tag{3}$$

$$x \to \infty : \ \theta \to 0 \tag{4}$$

The following well-known facts are consider to study the temperature distribution and heat flux in the semi infinite solid.
The Laplace Transform (Sneddon [12]) is defined as

$$L\{f(x)\} = \int_0^\infty e^{-st} f(t)\, dt \ (Re(s) > 0) \tag{5}$$

The Fourier sine transform (Debnath [2]) is defined as

$$u(n,t) = \int_0^\infty u(x,t) \sin nx \, dx \tag{6}$$

The Caputo fractional derivative of order α is given by Podulbny [8]

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{n-\alpha-1}}\, d\tau \ (n-1 < \alpha < n) \tag{7}$$

The Laplace transform of the Caputo fractional deriative is given by Podulbny [8]

$$\int_0^\infty e^{-st} D_t^\alpha f(t)\, dt = s^\alpha f(s) - \sum_{j=0}^{n-1} s^{\alpha-j-1} f^j(0) \ (n-1 < \alpha < n) \tag{8}$$

The Mittag-Leffler function $E_{\alpha,\beta}(z)$ (Podulbny [8]) defined by series representation

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)} \ (\alpha > 0, \beta > 0) \tag{9}$$

Some remarkable properties of generalized Mittag-Leffler function studied by Singh and Rawat [11].
The Wright function $W(\alpha, \beta; z)$ (Podulbny [8]) defined by series representation

$$W(\alpha, \beta; z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)\, k!} \tag{10}$$

Generalized k-Wright function is an interesting generalization of Wright function (10). Some interesting properties of generalized k-Wright function obtained by

Gehlot and Prajapati [4].
Following integral (El-Shahed and Salem [3]) is required for simplification

$$\int_0^\infty n\ \sin nx\ E_{\alpha,\alpha+1}(-n^2Ct^\alpha)\ dn = \frac{\pi}{2Ct^\alpha}\ W\left(\frac{-\alpha}{2}, 1; \frac{-x}{\sqrt{Ct^\alpha}}\right) \tag{11}$$

The error function $erf(x)$ is defined (Rainville [10]) as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x\ e^{-t^2}\ dt \tag{12}$$

The complementary error function $erf_c(x)$ is defined (Rainville [10]) as

$$erf_c(x) = 1 - erf(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty\ e^{-t^2}\ dt \tag{13}$$

The relationship between the Wright function and the complementary error function is given by

$$W\left(\frac{-1}{2}, 1; z\right) =\ erf_c\left(\frac{z}{2}\right) \tag{14}$$

2. Formulation of fractional partial differential equation

Now, consider a new model in the form of fractional partial differential equation

$$\frac{\partial^\alpha \theta}{\partial t^\alpha} = C\frac{\partial^2 \theta}{\partial x^2} \tag{15}$$

where

$$0 < \alpha \leq 1,\ t > 0,\ x \in \mathbb{R} \text{ and } \theta = \frac{T - T_0}{T_s - T_0} \tag{16}$$

The relevant boundary conditions are as follows:

$$\theta(x, 0) = 0 \tag{17}$$

$$\theta(0, t) = 1 \tag{18}$$

$$\lim_{x\to\infty} \theta(x, t) = 0 \tag{19}$$

If we consider $\alpha = 1$ then equation (15) reduces in classical heat equation (1).

3. Solution of Problem

Applying Fourier transform (6) on (15), yields

$$\frac{\partial^\alpha \theta(n,t)}{\partial t^\alpha} = C\sqrt{\frac{2}{\pi}}\ \int_0^\infty \frac{\partial^2 \theta}{\partial x^2} \sin nx\ dx \tag{20}$$

Integrating by parts gives

$$\begin{aligned}\frac{\partial^\alpha \theta(n,t)}{\partial t^\alpha} &= C\sqrt{\frac{2}{\pi}}\left[\sin nx \frac{\partial \theta}{\partial x}\right]_0^\infty - nC\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial \theta}{\partial x}\cos nx\ dx \\ &= C\sqrt{\frac{2}{\pi}}\left[\sin nx\ \frac{\partial \theta}{\partial x}\right]_0^\infty - nC\sqrt{\frac{2}{\pi}}\left[\theta\ \cos nx\ \right]_0^\infty - n^2C\sqrt{\frac{2}{\pi}}\int_0^\infty \theta\ \sin nx\ dx\end{aligned} \tag{21}$$

Now, applying boundary conditions (17) to (19) on (21), we get

$$\frac{\partial^\alpha \theta(n,t)}{\partial t^\alpha} = C\sqrt{\frac{2}{\pi}}(0) - nC\sqrt{\frac{2}{\pi}}(-1) - n^2 C\theta(n,t) \tag{22}$$

Therefore,

$$\frac{\partial^\alpha \theta(n,t)}{\partial t^\alpha} = nC\sqrt{\frac{2}{\pi}} - n^2 C\theta(n,t) \tag{23}$$

Using (8), Laplace transform of (23) gives

$$s^\alpha \theta(n,s) - \sum_{j=0}^{n-1} s^{\alpha-j-1}\theta^j(n,0) = nC\sqrt{\frac{2}{\pi}}L\{1\} - n^2 CL\{\theta(n,t)\} \tag{24}$$

This reduces to

$$s^\alpha \theta(n,s) + n^2 C\theta(n,s) = nC\sqrt{\frac{2}{\pi}}\frac{1}{s} \tag{25}$$

i.e.

$$\theta(n,s) = nC\sqrt{\frac{2}{\pi}}\frac{1}{s(s^\alpha + n^2 C)} \tag{26}$$

The inverse Laplace transform of (26) is given by (Prajapati *et al* [9])

$$\theta(n,t) = nC\sqrt{\frac{2}{\pi}}\, L^{-1}\left\{\frac{1}{s(s^\alpha + n^2 C)}\right\} = nC\sqrt{\frac{2}{\pi}}\, t^\alpha E_{\alpha,\alpha+1}(-n^2 C t^\alpha) \tag{27}$$

Now, inverse Fourier sine transform of equation (27) gives,

$$\theta(x,t) = \frac{2C}{\pi}\, t^\alpha \int_0^\infty n\, \sin nx\, E_{\alpha,\alpha+1}(-n^2 C t^\alpha)\, dn \tag{28}$$

Using (11), equation (28) can be written in the form of Wright function as

$$\theta(x,t) = W\left(\frac{-\alpha}{2}, 1; \frac{-x}{\sqrt{Ct^\alpha}}\right) \tag{29}$$

If we consider $\alpha = 1$, then equation (29) reduces to

$$\theta(x,t) = erf_c\left(\frac{x}{2\sqrt{Ct}}\right) \tag{30}$$

4. The Surface Heat flux

The heat flux at the surface is given by

$$\begin{aligned}
q_s &= -k\left[\frac{\partial T}{\partial x}\right]_{x=0} \\
&= -k\frac{\partial}{\partial x}\left\{T_0 + (T_s - T_0)\, W\left(\frac{-\alpha}{2}, 1; \frac{-x}{\sqrt{Ct^\alpha}}\right)\right\}_{x=0} \\
&= -k(T_s - T_0)\frac{\partial}{\partial x}\left(\sum_{k=0}^{\infty}\frac{\left(\frac{-x}{\sqrt{Ct^\alpha}}\right)^k}{\Gamma(\frac{-\alpha k}{2}+1)k!}\right)_{x=0} \\
&= -k(T_s - T_0)\frac{\partial}{\partial x}\left(1 - \frac{\left(\frac{x}{\sqrt{Ct^\alpha}}\right)}{\Gamma(\frac{-\alpha}{2}+1)} + \frac{\left(\frac{-x}{\sqrt{Ct^\alpha}}\right)^2}{\Gamma(-\alpha+1)2!} - ......\right)_{x=0} \\
&= -k(T_s - T_0)\left(-\frac{\left(\frac{1}{\sqrt{Ct^\alpha}}\right)}{\Gamma(\frac{-\alpha}{2}+1)}\right)
\end{aligned}$$

Finally, we get

$$q_s = \frac{k(T_s - T_0)}{\sqrt{Ct^\alpha}\Gamma(1-\frac{\alpha}{2})} \tag{31}$$

5. Concrete Example

A 15 cm thick concrete firewall has a black silicone paint surface. The wall is approximated as a black body at 1000 K. It will take 2 minutes for the surface to reach 500 K if the initial temperature of the wall is 300 K. Find the surface heat flux. **Solution:** We have given

$$T_s = 500\ K,\ T_0 = 300\ K$$

The required concrete properties are

$$k = 1.4\ W/m\ K,\ C = 0.75\times 10^{-6}\ m^2/s\ t = 2\text{ sec} = 120\text{ seconds}$$

In particular for $\alpha = 0.5$, the heat flux is obtain by using equation (31) as follow

$$q_s = \frac{1.4\times(500-300)}{\sqrt{0.75\times 10^{-6}\times\sqrt{120}}\ \Gamma(\frac{3}{4})} = \frac{1.4\times 200}{0.002866\times 0.6102} = 160109.7896\ W/m^2 \tag{32}$$

For different values of α, different values of heat flux q_s for above problem are shown in following table

α	q_s
0.50	160109.7896
0.625	142849.3595
0.67	121370.7964
0.75	105904.157

6. Conclusion

The Fractional Calculus approach introduced in this paper for study new constitutive model of temperature distribution and heat flux in semi infinite solid. In conventional method (considered only for $\alpha = 1$), equation (15) reduces to classical heat equation (1) whose solution obtained in the form of complementary error function. In this paper, authors obtained exact solution of fractional partial differential equation (15) by using Integral transform and Special functions for $0 < \alpha < 1$, i.e. this new method introduced in paper is useful than conventional method.

Acknowledgement: Authors are indeed extremely grateful to the referees for their valuable suggestions for the betterment of paper.

References

[1] A. D. Benson, W. S. Wheatcraft and M. M. Meerschaert, Application of a fractional advection-dispersion equation, Water Resources Research, 36(6), 1403-1412 (2000).
[2] L. Debnath, Integral Transforms and their Applications, CRC Press, New-York-London-Tokyo, (1995).
[3] El-Shahed Moustafa and Salem ahmed, Decay of Vortex velocity and diffusion of the temperature for fractional viscoelastic fluid through porous medium, International J. of Comm. in Heat and Mass Transfer, Vol. 33, 240-248 (2006).
[4] K. S. Gehlot and J. C. Prajapati, Fractional Calculus of Genearlized K-Wright Function, Journal of Fractional Calculus and Applications, Vol. 4(2) July 2013, pp. 283-289.
[5] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamic approach, Physics Reports, Vol. 339, 1-77(2000).
[6] A. F. Mills and V. Ganesan, Heat Transfer, Dorling Kindersley(India) Pvt. Ltd., (2009).
[7] K. B. Oldham and J. Spanier, The fractional calculus, Achedemic press, New york, (1974).
[8] I. J. Podulbny, Fractional differential equations, Academic Press, New York, (1999).
[9] J. C. Prajapati, A. D. Patel, K. N. Pathak and A. K. Shukla, Fractional calculus approach in the study of instability phenomenon in fluid dynamics, Palestine Journal of Mathematics, Vol. 1(2), (2012), 95-103.
[10] E. D. Rainville, Special functions, The Macmillan company, New York, (1960).
[11] K. D. Singh and R. Rawat, Integrals involving genearlized Mittag-Leffler function, Journal of Fractional Calculus and Applications, Vol. 4(2) July 2013, pp. 234-244.
[12] I. N. Sneddon, Fourier Transform, MacGraw-Hill, New York, (1951).
[13] M. G. Zaslavsky, Choas fractional kinemetics and anomalous transport, Physics Repots, 371(6), 451-580 (2002).

J.C. Prajapati and K. B. Kachhia
Faculty of applied sciences, Charotar University of Science and Technology (Charusat), Changa, Anand-388421, Gujart, India
E-mail address: jyotindra18@rediffmail.com, krunalmaths@hotmail.com