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ON EXISTENCE OF SOLUTIONS FOR FRACTIONAL DIFFERENTIAL EQUATION WITH P-LAPLACIAN OPERATOR

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Abstract. In this paper, we study existence, uniqueness and non existence of solutions for fractional differential equation with boundary conditions and p-Laplacian operator

$$\begin{cases} D^{\beta}(\phi_p(D^{\alpha}u(t))) + a(t)f(t) = 0, \quad 3 < \alpha, \beta \leq 4, \\ u(0) = 0, \quad u'(0) = 0, \quad u'''(0) = 0 \quad \xi u''(1) = u''(0) \\ \phi_p(D^{\alpha}u(0)) = (\phi_p(D^{\alpha}u(\eta)))', \quad (\phi_p(D^{\alpha}u(1)))' = 0, \\ \quad (\phi_p(D^{\alpha}u(0)))'' = 0 = (\phi_p(D^{\alpha}u(0)))'''. \end{cases}$$

where $0 < \xi < 1$, $0 < \eta \leq 1$ and D^{α}, D^{β} are Caputo's fractional derivative of orders α, β respectively. Our results are based on Schauder fixed point theorem. We have added examples for the applications of our results.

1. Introduction

Now a days fractional differential equations are considered an area of interest for researchers in many fields like engineering, mathematics, physics, chemistry, etc [1, 2, 3, 4, 5, 6]. One of the most important area of research in the field of fractional order differential equations is the theory of existence and uniqueness of solutions of fractional order differential equations. This area is rich for research work and many aspects are to be explored and developed. In particular, for the study of boundary value problems for fractional order differential equations, we refer the readers to [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] and the references therein. In literature one can see many articles on fractional differential equations with boundary conditions and p-Laplacian operator. For example, Z. Han et.al [24] studied positive solutions to boundary value problems of p-Laplacian fractional differential equations

$$\begin{cases} D^{\beta}_{0^+}(\phi_p(D^{\alpha}u(t))) + a(t)f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = \gamma(\xi) + \lambda, \quad \phi_p(D^{\alpha}_{0^+}u(0)) = (\phi_p(D^{\alpha}_{0^+}u(1)))' = (\phi_p(D^{\alpha}_{0^+}u(0)))'' = 0, \end{cases} \tag{1}$$

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where $0 < \alpha \leq 1$, $2 < \beta \leq 3$ are real numbers and D^{α}_{0+}, D^{β}_{0+} are standard Caputo fractional fractional derivatives. J. J. Zhang et.al in [25] studied multiple periodic solutions of p-Laplacian equation with one side Nagumo condition

$$\begin{cases} (\phi_p(u'))' = f(t,u,u'), \quad t \in [0,T], \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases} \tag{2}$$

by the help of degree theory and upper lower solution method. X. Xu and B. Xu in [26] studied Sign changing solutions of p-Laplacian equation with a sub-linear nonlinearity at infinity

$$\begin{cases} (\phi_p(u'(t)))' + f(t,u(t),u'(t)) = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{3}$$

by the use of upper and lower solutions method and Leray-Schauder degree theory. In this paper we study existence, uniqueness and nonexistence of solution for fractional differential equation with p-Laplacian operator

$$\begin{cases} D^{\beta}(\phi_p(D^{\alpha}u(t))) + a(t)f(t) = 0, \quad 3 < \alpha, \beta \leq 4, \\ u(0) = 0, \quad u'(0) = 0, \quad u'''(0) = 0 \quad \xi u''(1) = u''(0) \\ \phi_p(D^{\alpha}u(0)) = (\phi_p(D^{\alpha}u(\eta)))', \quad (\phi_p(D^{\alpha}u(1)))' = 0, \\ (\phi_p(D^{\alpha}u(0)))'' = 0 = (\phi_p(D^{\alpha}u(0)))'''. \end{cases} \tag{4}$$

where D^{α}, D^{β} stand for the Caputo's fractional derivative, f is continuous and may be nonlinear and the parameters satisfy $0 < \xi < 1$, $0 < \eta \leq 1$ and $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $1/p + 1/q = 1$.

We recall some basic definitions and results. For $\alpha > 0$, choose $n = [\alpha] + 1$ in case α in not an integer and $n = \alpha$ in case α is an integer. bf definition The fractional order integral of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is given by

$$I^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided the integral converges.

Definition 1 For a function $f \in C^n[0,1]$, the Caputo fractional derivative of order α is define by

$$(D^{\alpha}_{0+})f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2 A cone P in a real Banach space X is called soid if interior if its interior P^o is not empty.

Definition 3 Let P be a solid cone in a real Banach space X, $T : P^o \to P^o$ be an operator and $0 < \theta < 1$. Then T is called a $\theta-$concave operator if $T(ku) \geq k^{\theta}T(u)$ for any $0 < k < 1$ and $u \in P^o$. The following Lemmas gives some properties of fractional integrals.

Lemma 1[27] Assume that P is a normal solid cone in a real Banach space X, $0 < \theta < 1$ and $T : P^o \to P^o$ is a $\theta-$concave increasing operator. Then T has only one fixed point in P^o.

Lemma 2 [2] For $\alpha, \beta > 0$, the following relation hold:

$$D^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha-1}, \beta > n \text{ and } D^{\alpha}t^k = 0, k = 0, 1, 2, ..., n-1.$$

Lemma 3 [9] Let $p, q \geq 0$ $f \in L_1[a,b]$. Then $I^p_{0+}I^q_{0+}f(t) = I^{p+q}_{0+}f(t) = I^q_{0+}I^p_{0+}f(t)$ and ${}^cD^q_{0+}I^q_{0+}f(t) = f(t)$, for all $t \in [a,b]$.
Lemma 4 [2] Fort $\beta \geq \alpha > 0$ and $f \in L_1[a,b]$, the following

$$D^\alpha I^\beta_{a+}f(t) = I^{\beta-\alpha}_{a+}f(t) \text{ holds almost everywhere on } [a,b]$$

and it is valid at any point $t \in [a,b]$ if $f \in C[a,b]$.
Lemma 5 [2] For $g(t) \in C(0,1)$, the homogenous fractional order differential equation $D^\alpha_{0+}g(t) = 0$ has a solution

$$g(t) = c_1 + c_2 t + c_3 t^2 + ... + c_n t^{n-1},\ c_i \in R, i = 1,2,3,...,n. \tag{5}$$

Lemma 6 For $y \in C[0,1]$. The unique solution of the fractional differential equation

$$\begin{cases} D^\alpha u(t) = y(t) \quad 3 < \alpha \leq 4, \\ u(0) = 0, \quad u'(0) = 0, \quad u'''(0) = 0, \quad \xi u''(1) = u''(0) \end{cases} \tag{6}$$

is given by

$$u(t) = \int_0^1 G(t,s)y(s)ds, \tag{7}$$

where

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} + \frac{\xi t^2}{2(1-\xi)\Gamma(\alpha-2)}(1-s)^{\alpha-3} \quad 0 \leq s \leq t \leq 1 \\ \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} \quad 0 \leq s \leq t \leq 1 \end{cases} \tag{8}$$

Proof Applying the operator I^α_0 on (6) and using lemma 1, we obtain

$$u(t) = I^\alpha y(t) + C_1 + C_2 t + C_3 t^2 + C_4 t^3. \tag{9}$$

The boundary conditions $u(0) = u'(0) = u'''(0) = 0$, implies $C_1 = C_2 = C_4 = 0$. By the help of boundary condition $\xi u''(1) = u''(0)$ we have $C_3 = \frac{\xi}{2(1-\xi)}I^{\alpha-2}y(1)$. Hence, (9) takes the form

$$u(t) = I^\alpha y(t) + \frac{t^2\xi}{2(1-\xi)}I^{\alpha-2}y(1) \tag{10}$$

which can be rewritten as

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t^2\xi}{2(1-\xi)}\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds. \end{aligned}$$

Lemma 7 For $y \in C[0,1]$. The unique solution of the fractional differential equation

$$\begin{cases} D^\beta(\phi_p(D^\alpha u(t))) + a(t)f(t) = 0, \quad 3 < \alpha, \beta \leq 4, \\ u(0) = 0, \quad u'(0) = 0, \quad u'''(0) = 0 \quad \xi u''(1) = u''(0) \\ \phi_p(D^\alpha u(0)) = (\phi_p(D^\alpha u(\eta)))', \quad (\phi_p(D^\alpha u(1)))' = 0, \\ (\phi_p(D^\alpha u(0)))'' = 0 = (\phi_p(D^\alpha u(0)))'''. \end{cases} \tag{11}$$

is given by

$$u(t) = \int_0^1 G(t,s)\phi_q(\int_0^1 \mathcal{H}(s,\tau)y(\tau)d\tau)ds, \tag{12}$$

where

$$\mathcal{H}(t,s)=\begin{cases}-\frac{1}{\Gamma(\beta)}(t-s)^{\beta-1}+\frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2}-\frac{1}{\Gamma(\beta-1)}(\eta-s)^{\beta-2}\\ +\frac{t}{\Gamma(\beta-1)}(1-s)^{\beta-2}\quad 0<s\leq t\leq\eta<1,\\ \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2}-\frac{1}{\Gamma(\beta-1)}(\eta-s)^{\beta-2}+\frac{t}{\Gamma(\beta-1)}(1-s)^{\beta-2}\\ 0<t\leq s\leq\eta<1,\\ \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2}+\frac{t}{\Gamma(\beta-1)}(1-s)^{\beta-2}\\ 0<t\leq\eta\leq s<1,\end{cases}\tag{13}$$

and $G(t,s)$ defined by (8).

proof Applying integral I^{β} on fractional differential equation with boundary conditions (11) and using lemma (1) we have

$$\phi_p(D^{\alpha}u(t))=-I^{\beta}y(t)+C_1+C_2t+C_3t^2+c_4t^3\tag{14}$$

By the boundary conditions $(\phi_p(D^{\alpha}u(0)))''=0=(\phi_p(D^{\alpha}u(0))''')$, we have $C_3=0=C_4$. Thus we get the following equation

$$\phi_p(D^{\alpha}u(t))=-I^{\beta}y(t)+C_1+C_2t\tag{15}$$

Now by the boundary condition $\phi_p(D^{\alpha}u(0))=(\phi_p(D^{\alpha}u(\eta)))'$, we have $C_1=I^{\beta-1}y(1)-I^{\beta-1}y(\eta)$, $c_2=I^{\beta-1}y(1)$. By substituting the values of C_1, C_2 in (15) we have

$$\phi_p(D^{\alpha}u(t))=-I^{\beta}y(t)+I^{\beta-1}y(1)-I^{\beta-1}y(\eta)+t(I^{\beta-1}y(1)))\tag{16}$$

Which can be written as

$$\begin{aligned}\phi_p(D^{\alpha}u(t))=&-\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}y(s)ds+\frac{1}{\Gamma(\beta-1)}\int_0^1(1-s)^{\beta-2}y(s)ds\\ &-\frac{1}{\Gamma(\beta-1)}\int_0^{\eta}(\eta-s)^{\beta-2}y(s)ds+\frac{t}{\Gamma(\beta-1)}\int_0^1(1-s)^{\beta-2}y(s)ds\\ &=\int_0^1\mathcal{H}(t,s)y(s)ds\end{aligned}\tag{17}$$

Consequently $D^{\alpha}u(t)=\phi_q(\int_0^1\mathcal{H}(t,s)y(s)ds)$. Thus the differential equation (11) is equivalent to

$$\begin{aligned}&D^{\alpha}u(t)=\phi_q(\int_0^1\mathcal{H}(t,s)y(s)ds)\\ &u(0)=0,\quad u'(0)=0,\quad u'''(0)=0\quad \xi u''(1)=u''(0)\end{aligned}\tag{18}$$

Thus by the help of lemma (6), we have the unique solution of the differential equation (11) as under

$$\begin{aligned}u(t)=&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi_q(\int_0^1\mathcal{H}(s,\tau)y(\tau))d\tau)ds\\ &+\frac{t^2\xi}{2(1-\xi)\Gamma(\alpha-2)}\int_0^1(1-s)^{\alpha-3}\phi_q(\int_0^1\mathcal{H}(s,\tau)y(\tau)d\tau)ds\\ &=\int_0^1G(t,s)\phi_q(\int_0^1\mathcal{H}(s,\tau)a(\tau)f(u(\tau))d\tau)ds\end{aligned}\tag{19}$$

Lemma 8 Let $3 < \alpha, \beta \leq 4$ the function $\mathcal{H}(t,s)$ is continuous on $[0,1]\times[0,1]$ and satisfies

(A) $\mathcal{H}(t,s) \geq 0$, $\mathcal{H}(t,s) \leq \mathcal{H}(1,s)$, for $t,s \in [0,1]$
(B) $\mathcal{H}(t,s) \geq t^{\beta-1}\mathcal{H}(1,s)$ for $t,s \in (0,1)$

Proof For $0 < s \leq t \leq \xi < 1$, we have the following estimates

$$\begin{aligned}
&\mathcal{H}(t,s) = -\frac{1}{\Gamma(\beta)}(t-s)^{\beta-1} + \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2} - \frac{1}{\Gamma(\beta-1)}(\eta-s)^{\beta-2} \\
&+ \frac{t}{\Gamma(\beta-1)}(1-s)^{\beta-2} \\
&= -\frac{t^{\beta-1}}{\Gamma(\beta)}(1-\frac{s}{t})^{\beta-1} + \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2} - \frac{\eta^{\beta-2}}{\Gamma(\beta-1)}(1-\frac{s}{\eta})^{\beta-2} \\
&+ \frac{t}{\Gamma(\beta-1)}(1-s)^{\beta-2} \\
&\geq -\frac{t^{\beta-1}}{\Gamma(\beta)}(1-s)^{\beta-1} + \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2} - \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2} \\
&+ \frac{t^{\beta-1}}{\Gamma(\beta-1)}(1-s)^{\beta-2} \\
&= \frac{t^{\beta-1}}{\Gamma(\beta)}\{-(1-s)^{\beta-1} + (\beta-1)(1-s)^{\beta-2}\} \geq 0
\end{aligned}$$

In other cases the proof is similar so we omit it. Also from (13) we have the following estimates

$$\mathcal{H}'(t,s) = \begin{cases} -\frac{1}{\Gamma(\beta-1)}(t-s)^{\beta-2} + \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2} & 0 < s \leq t \leq \eta < 1 \\ \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2} & 0 < t \leq s \leq \eta < 1 \\ \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2} & 0 < t \leq \eta \leq s < 1 \end{cases} \tag{20}$$

from (20), it is clear that $\mathcal{H}'(t,s) > 0$. That is, $\mathcal{H}(t,s)$ is an increasing function. Thus $\mathcal{H}(t,s) \leq \mathcal{H}(1,s)$. For (B) we have the following estimates

$$\begin{aligned}
&\frac{\mathcal{H}(t,s)}{\mathcal{H}(1,s)} = \\
&\frac{-\frac{1}{\Gamma(\beta)}(t-s)^{\beta-1} + \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2} - \frac{1}{\Gamma(\beta-1)}(\eta-s)^{\beta-2} + \frac{t}{\Gamma(\beta-1)}(1-s)^{\beta-2}}{-\frac{1}{\Gamma(\beta)}(1-s)^{\beta-1} + \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2} - \frac{1}{\Gamma(\beta-1)}(\eta-s)^{\beta-2} + \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2}} \\
&\geq \frac{-\frac{t^{\beta-1}}{\Gamma(\beta)}(1-s)^{\beta-1} + \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2} - \frac{1}{\Gamma(\beta-1)}(\eta-s)^{\beta-2} + \frac{t^{\beta-1}}{\Gamma(\beta)}(1-s)^{\beta-2}}{-\frac{1}{\Gamma(\beta)}(1-s)^{\beta-1} + \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2} - \frac{1}{\Gamma(\beta-1)}(\eta-s)^{\beta-2} + \frac{1}{\Gamma(\beta-1)}(1-s)^{\beta-2}} \\
&= t^{\beta-1}
\end{aligned}$$

This complete the proof. For the proof of our main result we use the following assumptions

(W1) $0 < \int_0^1 \mathcal{H}(1,\tau)a(\tau)d\tau < +\infty$
(W2) There exist $0 < \delta < 1$ and $\rho > 0$ such that

$$f(x) \leq \delta L\phi_p(x), \text{ for } 0 \leq x \leq \rho, \tag{21}$$

where L satisfies

$$0 < L \leq (\phi_p(\varpi_1)\delta \int_0^1 \mathcal{H}(1,s)a(s)ds)^{-1} \tag{22}$$

for $\varpi_1 = \frac{1}{\Gamma(\alpha+1)} + \frac{\xi}{2(1-\xi)\Gamma(\alpha-1)}$.

(W3) There exist $b > 0$, such that

$$f(x) \leq M\phi_p(x), \text{ for } b < x < +\infty \tag{23}$$

where M satisfies

$$0 < M < (\phi_p(\varpi_1 2^{q-1}) \int_0^1 \mathcal{H}(1,\tau)a(\tau)d\tau)^{-1} \tag{24}$$

(W4) There exist $0 < \mu < 1$ and $e > 0$ such that

$$f(x) \geq N\phi_p(x), \text{ for } e < x < +\infty, \tag{25}$$

where N satisfies

$$N > (\phi_p(c_\varrho \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q(s^{\beta-1})ds)(\int_\varrho^1 \mathcal{H}(s,\tau)a(\tau)f(u(\tau))d\tau))^{-1} \tag{26}$$

$$c_\varrho = \int_0^t \alpha(t-s)^{\alpha-1}\phi_q(s^{\beta-1})ds \in (0,1) \tag{27}$$

(W5) $f(x)$ is non-decreasing in x;

(W6) There exist $0 \leq \theta < 1$ such that

$$f(kx) \geq (\phi_p(k^\theta))f(x), \text{ for any } 0 < k < 1 \text{ and } 0 < x < +\infty \tag{28}$$

2. Existence

Theorem 1 Assume that $(W1)$, $(W2)$ hold. Then the fractional differential equation with boundary conditions (11) has at least one positive solution.

Proof Let $\rho > 0$ which is given in (H_2) Define $K_1 = \{u \in C[0,1] : 0 \leq u(t) \leq \rho \text{ on } [0,1]\}$ and an operator $\mathcal{T} : K_1 \to C[0,1]$ by

$$\begin{aligned}\mathcal{T}u(t) = &\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q(\int_0^1 \mathcal{H}(s,\tau)y(\tau))d\tau)ds \\ &+ \frac{t^2\xi}{2(1-\xi)}\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}\phi_q(\int_0^1 \mathcal{H}(s,\tau)y(\tau)d\tau)ds\end{aligned} \tag{29}$$

K_1 is closed convex set [24]. From lemma (1), $u(t)$ is the solution of fractional differential equation (11) if and only if $u(t)$ is a fixed point of $\mathcal{T}$. Moreover a standard argument can be used to show that $\mathcal{T}$ is compact. For any $u \in K_1$ (21), (22) implies that $f(u(t)) \leq \delta L\phi_p(u(t)) \leq \delta L\phi_p(\rho)$ on $[0,1]$ and also we have the following estimates

$$\begin{aligned}0 \leq \mathcal{T}u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q(\int_0^1 \mathcal{H}(s,\tau)y(\tau))d\tau)ds \\ &+ \frac{t^2\xi}{2(1-\xi)}\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}\phi_q(\int_0^1 \mathcal{H}(s,\tau)y(\tau)d\tau)ds \\ &\leq \frac{1}{\Gamma(\alpha+1)}\phi_q(\int_0^1 \mathcal{H}(1,\tau)y(\tau))d\tau)ds \\ &+ \frac{t^2\xi}{2(1-\xi)}\frac{1}{\Gamma(\alpha-1)}\phi_q(\int_0^1 \mathcal{H}(s,\tau)y(\tau)d\tau)ds \\ &\leq (\frac{1}{\Gamma(\alpha+1)} + \frac{\xi}{2(1-\xi)\Gamma(\alpha-1)})\phi_q(\int_0^1 \mathcal{H}(s,\tau)y(\tau)d\tau)ds \\ &= \varpi_1\phi_q(\int_0^1 \mathcal{H}(t,s)a(s)ds)\phi_q(\delta)\phi_q(L)\rho \leq \rho\end{aligned}$$

Thus $\mathcal{T}(K_1) \subseteq K_1$. By Schauder fixed point theorem $\mathcal{T}$ has a fixed point in K_1. That is, the fractional differential equation (11) has at least one positive solution.
Theorem 2 Assume that (W1), (W3) hold. Then the fractional differential equation (11) has at least one positive solution.
Proof Let $b > 0$ as given in (W3). Define $\chi = max_{0\leq x\leq b}f(x)$. Then $f(x) \leq \chi$ for $0 \leq x \leq b$. From (24), we have

$$\varpi_1 2^{q-1}\phi_q(M)\phi_q(\int_0^1 \mathcal{H}(1,\tau)a(\tau)d\tau) < 1$$

There exist $b^* > b$ so large that

$$\varpi_1 2^{q-1}(\phi_q(\chi) + \phi_q(M)b^*)\phi_q(\int_0^1 \mathcal{H}(1,\tau)a(\tau))d\tau) < b^*. \tag{30}$$

Let $K_2 = \{u \in C[0,1] : 0 \leq u(t) \leq b^* \text{ on } [0,1]\}$. For $u \in K_2$, define $S_1 = \{t \in [0,1] : 0 \leq u(t) \leq b\}$, $S_2 = \{t \in [0,1] : b < u(t) \leq b^*\}$. Then we have $S_1 \cup S_2 = [0,1]$ and $S_1 \cap S_2 = \varphi$. From (23) we have that

$$f(u(t)) \leq M\phi_p(u(t)) \leq M\phi_p(b^*) \text{ for } t \in S_2.$$

Let the compact operator $\mathcal{T}$ be defined by (29). Then from Lemma (1) and (23) we have the following estimates

$$\begin{aligned}\mathcal{T}u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q(\int_0^1 \mathcal{H}(s,\tau)y(\tau))d\tau)ds \\ &+ \frac{t^2\xi}{2(1-\xi)}\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}\phi_q(\int_0^1 \mathcal{H}(s,\tau)y(\tau)d\tau)ds \\ &\leq \varpi_1\phi_q(\int_0^1 \mathcal{H}(1,\tau)a(\tau)f(u(\tau))d\tau) \\ &= \varpi_1\phi_q(\int_{S_1} \mathcal{H}(1,\tau)a(\tau)f(u(\tau))d\tau + \int_{S_2} \mathcal{H}(1,\tau)a(\tau)f(u(\tau))d\tau)\end{aligned}$$

$$\leq \varpi_1 \phi_q(\chi \int_{S_1} \mathcal{H}(1,\tau)a(\tau))d\tau + M\phi_p(b^*) \int_{S_2} \mathcal{H}(1,\tau)a(\tau))d\tau)$$
$$\leq \varpi_1 \phi_q(\chi + M\phi_p(b^*))\phi_q(\int_0^1 \mathcal{H}(1,\tau)a(\tau))d\tau)$$

From (30) and by the help of inequality $(a+b)^r \leq 2^r(a^r + b^r)$ for any $a, b, r > 0$. We have

$$0 \leq \mathcal{T}u(t) \leq \varpi_1 2^{q-1}(\phi_q(\chi) + \phi_q(M)b^*)\phi_q(\int_0^1 \mathcal{H}(1,\tau)a(\tau))d\tau) \leq b^*$$

Thus $\mathcal{T}(K_2) \subseteq K_2$. And hence by Schauder fixed point theorem $\mathcal{T}$ has a fixed point $u \in K_2$, thus the fractional differential equation (11) has at least one positive solution.

Example 1

$$\begin{aligned}
&D^{7/2}(\phi_p(D^{7/2}u(t))) + tu(t) = 0,\\
&u(0) = 0, \quad u'(0) = 0, \quad u'''(0) = 0 \quad 1/2u''(1) = u''(0)\\
&\phi_p(D^{7/2}u(0)) = (\phi_p(D^{7/2}u(1/2)))', \quad (\phi_p(D^{7/2}u(1)))' = 0,\\
&(\phi_p(D^{7/2}u(0)))'' = 0 = (\phi_p(D^{7/2}u(0)))'''.
\end{aligned} \tag{31}$$

For the existence of solution of fractional differential equation with boundary conditions and p-Laplacian operator (31) we apply theorem (2). In fractional differential equation (31), we have $\alpha = \beta = 7/2$, $\xi = \eta = 1/2$, $a(t) = t$, $f(u(t)) = u(t)$ and by simple computation we get that $0 < L \leq 9.3560$ and thus considering $L = 9$ also $\delta = 1/2$. Thus we have (31) satisfy (W1), (W2). So by theorem (2), we have fractional differential equation (31) has at least one positive solution.

Example 2

$$\begin{aligned}
&D^{7/2}(\phi_p(D^{7/2}u(t))) + t\sqrt{t(u)} = 0,\\
&u(0) = 0, \quad u'(0) = 0, \quad u'''(0) = 0 \quad 1/2u''(1) = u''(0)\\
&\phi_p(D^{7/2}u(0)) = (\phi_p(D^{7/2}u(1/2)))', \quad (\phi_p(D^{7/2}u(1)))' = 0,\\
&(\phi_p(D^{7/2}u(0)))'' = 0 = (\phi_p(D^{7/2}u(0)))'''.
\end{aligned} \tag{32}$$

For the existence of solution of (32), we apply theorem (2). In equation (32), we have $\alpha = \beta = 7/2$, $\xi = \eta = 1/2$, $a(t) = t$, $f(u(t)) = \sqrt{u}(t)$ and by simple computation we get that $M < 2.3390$ and thus considering $M = 2$ also $b = 1$ and $q = 2$. Thus we have (32) satisfy (W1), (W3). So by theorem (32), we have fractional differential equation (31) has at least one positive solution.

3. UNIQUENESS

Theorem 3 Assume that (W1), (W5) and (W6) hold. Then the fractional differential equation (11), has a unique positive solution.

Proof Define $P = \{u \in C[0,1] : u(t) \geq 0 \text{ on } [0,1]\}$. Then P is a normal solid cone in $C[0,1]$ with $P^o = \{u \in C[0,1] : u(t) > 0 \text{ on } [0,1]\}$ let $\mathcal{T} : P \to C[0,1]$ be defined by (29)

$$\begin{aligned}\mathcal{T}u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q(\int_0^1 \mathcal{H}(s,\tau)y(\tau))d\tau)ds\\ &+ \frac{t^2\xi}{2(1-\xi)}\frac{1}{\Gamma(\alpha-2)}\int_0^1 (1-s)^{\alpha-3}\phi_q(\int_0^1 \mathcal{H}(s,\tau)y(\tau)d\tau)ds\end{aligned}$$

Clearly $\mathcal{T}: P \to C[0,1]$ .Now we prove that $\mathcal{T}$ is $\theta-$concave increasing operator. For u_1,$u_2 \in P$ with $u_1(t) \geq u_2(t)$ on $[0,1]$ we obtain $\mathcal{T}(u_1(t)) \geq \mathcal{T}(u_2(t))$ and for $f(Ku) \geq \phi_p(k^\theta)f(u)$ we have the following estimates

$$\begin{aligned}\mathcal{T}(ku(t)) &\geq \int_0^1 G(t,s)\phi_q(\int_0^1 H(t,s)\phi_q(k^\theta)f(u)ds)ds\\ &= k^\theta\int_0^1 G(t,s)\phi_q(\int_0^1 H(t,s)f(u)ds)ds\\ &= k^\theta\mathcal{T}(u(t))\end{aligned}$$

This implise that $\mathcal{T}$ is $\theta-$concave operator. Thus $\mathcal{T}$ has a unique fixed point.

Example 3

$$\begin{aligned}&D^{7/2}(\phi_p(D^{7/2}u(t))) + t\sqrt{t(u)} = 0,\\ &u(0) = 0, \quad u'(0) = 0, \quad u'''(0) = 0 \quad 1/2u''(1) = u''(0)\\ &\phi_p(D^{7/2}u(0)) = (\phi_p(D^{7/2}u(1/2)))', \quad (\phi_p(D^{7/2}u(1)))' = 0,\\ &(\phi_p(D^{7/2}u(0)))'' = 0 = (\phi_p(D^{7/2}u(0)))'''.\end{aligned} \tag{33}$$

For the uniqueness of solution of fractional differential equation with boundary conditions (33), we apply theorem (3). In equation (32), we have $\alpha = \beta = 7/2$, $\xi = \eta = 1/2$, $q = 2$ $a(t) = t$, $f(u(t)) = \sqrt{u}(t)$. It is clear (32) satisfy (W1), (W5). Also by considering $\theta = 1/2$, (W6) is satisfied. Thus by theorem (33), we have fractional differential equation (31) has a unique solution.

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