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q-HYPERGEOMETRIC REPRESENTATIONS OF THE q-ANALOGUE OF ZETA FUNCTION

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Abstract. In this paper, we give a summary introduction to the ordinary hypergeometric ${}_rF_s$ series and q-hypergeometric ${}_r\varphi_s$ series. We also provide a brief overview of the q-calculus topics which are necessary to understand the main results. Finally, we give some q-hypergeometric representations for the q-analogue of the generalized Zeta function.

1. Introduction

The Hurwitz or generalized Zeta function at integer points

$$\zeta(s,\alpha) \equiv \sum_{n\geq 0} \frac{1}{(n+\alpha)^s}, \quad 0<\alpha\leq 1, \tag{1}$$

has a q-analogue [5, 6, 9], defined by

$$\zeta_q(s,\alpha) \equiv \sum_{n\geq 0} \frac{q^{(n+\alpha)(s-1)}}{[n+\alpha]_q^s}, \quad 0<q<1, \quad 0<\alpha\leq 1, \tag{2}$$

where the q-number $[z]_q$ is defined through

$$[z]_q \equiv \frac{1-q^z}{1-q}, \quad z\in\mathbb{C}. \tag{3}$$

Notice that, the series (1) and (2) are convergent as Re $s>1$.

As it's known, nowadays there is no general rigorous definition of a q-analogues. An intuitive definition of a q-analogues of a mathematical object $\mathcal{G}$ is a family of objects $\mathcal{G}_q$ with $0<q<1$, such that

$$\lim_{q\to 1^-} \mathcal{G}_q = \mathcal{G}.$$

Observe that, it makes sense to call to (2) a q-analogue of (1), since

$$\lim_{q\to 1^-} \zeta_q(s,\alpha) = \zeta(s,\alpha).$$

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A special case of (3) when $z \in \mathbb{N}$ is

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{0\leq j\leq n-1} q^j, \quad n \in \mathbb{N},$$

which is called the q-analogue of $n \in \mathbb{N}$, since

$$\lim_{q\to 1^-} [n]_q = \lim_{q\to 1^-} \sum_{0\leq j\leq n-1} q^j = n.$$

Another manner of represent to (1) it's through

$$\begin{aligned} \zeta(s,\alpha) &= \alpha^{-s} \sum_{n\geq 0} \frac{\alpha^s (\alpha+1)^s (\alpha+2)^s \cdots (\alpha+n-1)^s}{(\alpha+1)^s (\alpha+2)^s \cdots (\alpha+1+n-2)^s (\alpha+1+n-1)^s} \\ &= \alpha^{-s} \sum_{n\geq 0} \frac{(1)_n (\alpha)_n^s}{(\alpha+1)_n^s} \frac{1^n}{n!} = \alpha^{-s} {}_{s+1}F_s \left(\begin{array}{c} 1, \alpha, \ldots, \alpha \\ \alpha+1, \ldots, \alpha+1 \end{array} \middle| 1 \right) \quad (4) \\ &= \alpha^{-s} \sum_{n\geq 0} \overbrace{{}_2F_1 \left(\begin{array}{c} -n, 1 \\ \alpha+1 \end{array} \middle| 1 \right) \cdots {}_2F_1 \left(\begin{array}{c} -n, 1 \\ \alpha+1 \end{array} \middle| 1 \right)}^{s\text{-times}}, \quad (5) \end{aligned}$$

where $(\cdot)_k$ denotes the Pochhammer symbol, also called the shifted factorial, defined by

$$(z)_k \equiv \prod_{0\leq j\leq k-1} (z+j), \quad k \geq 1,$$
$$(z)_0 = 1, \quad (-z)_k = 0, \quad \text{if } z < k,$$

which in terms of the gamma function is given by

$$(z)_k = \frac{\Gamma(z+k)}{\Gamma(z)}, \quad k = 0, 1, 2, \ldots,$$

and ${}_rF_s$ denotes the ordinary hypergeometric series [4, 7, 8] with variable z is defined by

$${}_rF_s \left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| z \right) \equiv \sum_{k\geq 0} \frac{(a_1, \ldots, a_r)_k}{(b_1, \ldots, b_s)_k} \frac{z^k}{k!}, \quad (6)$$

being

$$(a_1, \ldots, a_r)_k \equiv \prod_{1\leq i\leq r} (a_i)_k,$$

with $\{a_i\}_{i=1}^r$ and $\{b_j\}_{j=1}^s$ complex numbers subject to the condition that $b_j \neq -n$ with $n \in \mathbb{N} \backslash \{0\}$ for $j = 1, 2, \ldots, s$.

The equality (5) is justified by the Chu-Vandermonde identity [4, 7], which occur very often in practice, and the same comes given by

$${}_2F_1 \left(\begin{array}{c} -n, b \\ c \end{array} \middle| 1 \right) = \frac{(c-b)_n}{(c)_n}, \quad n = 0, 1, 2, \ldots. \quad (7)$$

Moreover, taking into account that the ordinary hypergeometric ${}_{s+1}F_s$ series is called k-balanced if in ${}_{s+1}F_s$ the sum of denominator parameters is equal to k plus the sum of numerator parameters, i.e.,

$$\sum_{1\leq j\leq s}(b_j-a_j)-a_{s+1}=k,$$

then, from (4) is deduce that $\zeta(s,\alpha)$ is the product of α^{-s} by a ordinary hypergeometric ${}_{s+1}F_s$ series $(s-1)$-balanced.

The structure of the paper is as follows. In Section 2, we compress some necessary definitions and tools. Finally, in Section 3, we give the main results.

2. Basic definitions and notations

Here we will give some usual notions and notations used in q-Calculus, i.e the q-analogues of the usual calculus.

Let the q-analogues of Pochhammer symbol or q-shifted factorial [4, 7] be defined by

$$(a;q)_n\equiv\begin{cases}1, & n=0,\\ \prod_{0\leq j\leq n-1}\left(1-aq^j\right), & n=1,2,\ldots,\end{cases}\tag{8}$$

where

$$\begin{aligned}\left(q^{-n};q\right)_k &= 0, \quad \text{whenever } n<k,\\ (0;q)_n &= 1,\end{aligned}\tag{9}$$

and

$$\lim_{q\to1^-}\frac{(q^z;q)_k}{(1-q)^k}=(z)_k\,.$$

The formula (8) is known as the Watson notation [2, 3]. The q-binomial coefficient [4, 7] is defined by

$$\begin{bmatrix}n\\k\end{bmatrix}_q\equiv\frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}},\quad k,n\in\mathbb{N},$$

and for complex z is defined by

$$\begin{bmatrix}z\\k\end{bmatrix}_q\equiv\frac{(q^{-z};q)_k}{(q;q)_k}(-1)^k\,q^{zk-\binom{k}{2}},\quad k\in\mathbb{N}.\tag{10}$$

In addition, using the above definitions, we have that the binomial theorem

$$(x+y)^n=\sum_{0\leq k\leq n}\binom{n}{k}x^kb^{n-k},\quad n=0,1,2,\ldots,$$

has a q-analogue of the form [1]-[4, pp. 25]

$$\begin{aligned}(xy;q)_n &= \sum_{0\leq k\leq n}\begin{bmatrix}n\\k\end{bmatrix}_q y^k\,(x;q)_k\,(y;q)_{n-k}\\ &= \sum_{0\leq k\leq n}\begin{bmatrix}n\\k\end{bmatrix}_q x^{n-k}\,(x;q)_k\,(y;q)_{n-k}\,.\end{aligned}$$

In particular, when $y=0$ we have that

$$\sum_{0\le k\le n}\begin{bmatrix} n\\ k\end{bmatrix}_q x^{n-k}(x;q)_k=(0;q)_n=1. \tag{11}$$

In comparison with the ordinary hypergeometric ${}_rF_s$ series defined by (6), is present here in a concise manner, the basic hypergeometric or q-hypergeome-tric ${}_r\varphi_s$ series. The details can be found in [4, 7].

Let $\{a_i\}_{i=0}^r$ and $\{b_j\}_{i=0}^s$ be complex numbers subject to the condition that $b_j\neq q^{-n}$ with $n\in\mathbb{N}\backslash\{0\}$ for $j=1,2,\ldots,s$. Then the basic hypergeometric or q-hypergeometric ${}_r\varphi_s$ series with variable z is defined by

$${}_r\varphi_s\left(\begin{array}{c} a_1,\ldots,a_r\\ \\ b_1,\ldots,b_s\end{array}\middle|\, q;z\right)\equiv\sum_{k\ge 0}\frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k}(-1)^{(1+s-r)k}q^{(1+s-r)\binom{k}{2}}\frac{z^k}{(q;q)_k},$$

where

$$(a_1,\ldots,a_r;q)_k\equiv\prod_{1\le j\le r}(a_j;q)_k\,.$$

In addition, for brevity, let us denote by

$$\left[{}_r\varphi_s\left(\begin{array}{c} a_1,\ldots,a_r\\ \\ b_1,\ldots,b_s\end{array}\middle|\, q;z\right)\right]^n={}_r\varphi_s^n\left(\begin{array}{c} a_1,\ldots,a_r\\ \\ b_1,\ldots,b_s\end{array}\middle|\, q;z\right),\quad n=1,2,\ldots.$$

Analogously to the ordinary hypergeometric ${}_{s+1}F_s$ series, the q-hypergeome-tric ${}_{s+1}\varphi_s$ series is called k-balanced if $b_1b_2\cdots b_s=q^ka_1a_2\cdots a_{s+1}$.

The q-hypergeometric ${}_r\varphi_s$ series is a q-analogue of the ordinary hypergeometric ${}_rF_s$ series defined by (6) since

$$\lim_{q\to 1^-}{}_r\varphi_s\left(\begin{array}{c} q^{\tilde a_1},\ldots,q^{\tilde a_r}\\ \\ q^{\tilde b_1},\ldots,q^{\tilde b_s}\end{array}\middle|\, q;z(q-1)^{1+s-r}\right)={}_rF_s\left(\begin{array}{c} \tilde a_1,\ldots,\tilde a_r\\ \\ \tilde b_1,\ldots,\tilde b_s\end{array}\middle|\, z\right).$$

The q-analogue of the Chu-Vandermonde convolution (7) is given by

$${}_2\varphi_1\left(\begin{array}{c} q^{-n},a\\ \\ b\end{array}\middle|\, q;\frac{bq^n}{a}\right)=\frac{(a^{-1}b;q)_n}{(b;q)_n},\quad n=0,1,2,\ldots, \tag{12}$$

$${}_2\varphi_1\left(\begin{array}{c} q^{-n},a\\ \\ b\end{array}\middle|\, q;q\right)=\frac{(a^{-1}b;q)_n}{(b;q)_n}a^n,\quad n=0,1,2,\ldots. \tag{13}$$

The details can be found in [4, 7].

In the last years, into the q-Calculus have been found many applications of the quantum group theory. In particular, the q-hypergeometric ${}_r\varphi_s$ series are applicable nowadays to different subjects of combinatorics, quantum theory, number theory, statistical mechanics, etc....

3. Main Results

In this section we will give the main results.

Lemma 1. *The following relation*

$$ {}_2\varphi_0\left(\begin{array}{c} q^{-n}, z \\ - \end{array}\middle|\, q; q^n z^{-1}\right) = z^{-n}, \quad n = 0, 1, 2, \ldots, $$

holds.

Proof. Since

$$ {}_2\varphi_0\left(\begin{array}{c} q^{-n}, z \\ - \end{array}\middle|\, q; q^n z^{-1}\right) = \sum_{k\geq 0} \frac{(q^{-n};q)_k}{(q;q)_k} (-1)^k q^{nk-\binom{k}{2}} (z;q)_k z^{-k}. $$

Then, from (9) and (10) we have that

$$ {}_2\varphi_0\left(\begin{array}{c} q^{-n}, z \\ - \end{array}\middle|\, q; q^n z^{-1}\right) = z^{-n} \sum_{0\leq k\leq n} \begin{bmatrix} n \\ k \end{bmatrix}_q z^{n-k} (z;q)_k . $$

Finally, using (11) we get the desired result. □

Theorem 2. *Let s be an integer number, with $s > 1$, $|q| < 1$ and $0 < \alpha \leq 1$. Then the q-analogue of the generalized Zeta function (2) admits the following representations*

i.)

$$ \zeta_q(s,\alpha) = q^{\alpha(s-1)} \left(\frac{1-q}{1-q^\alpha}\right)^s {}_{s+1}\varphi_s\left(\begin{array}{c} q, q^\alpha, \ldots, q^\alpha \\ q^{\alpha+1}, \ldots, q^{\alpha+1} \end{array}\middle|\, q; q^{s-1}\right), \tag{14} $$

ii.)

$$ \zeta_q(s,\alpha) = q^{\alpha(s-1)} \left(\frac{1-q}{1-q^\alpha}\right)^s \times \sum_{n\geq 0} {}_2\varphi_0\left(\begin{array}{c} q^{-n}, q \\ - \end{array}\middle|\, q; q^{n-1}\right) {}_2\varphi_1^s\left(\begin{array}{c} q^{-n}, q \\ q^{\alpha+1} \end{array}\middle|\, q; q\right), \tag{15} $$

iii.)

$$ \zeta_q(s,\alpha) = q^{\alpha(s-1)} \left(\frac{1-q}{1-q^\alpha}\right)^s \sum_{n\geq 0} (-1)^n q^{\binom{n}{2}+n\alpha} \times {}_2\varphi_1\left(\begin{array}{c} q^{-n}, q^\alpha \\ q^{\alpha+1} \end{array}\middle|\, q; q^{n+1}\right) {}_2\varphi_1^{s-1}\left(\begin{array}{c} q^{-n}, q \\ q^{\alpha+1} \end{array}\middle|\, q; q\right) \times {}_s\varphi_{s-1}\left(\begin{array}{c} q^{n+1}, q^{n+\alpha}, \ldots, q^{n+\alpha} \\ q^{n+\alpha+1}, \ldots, q^{n+\alpha+1} \end{array}\middle|\, q; q^{s-1}\right). \tag{16} $$

Proof. In fact, firstly let's prove $i.$). For such purpose it is enough to check

$$\begin{aligned}\zeta_q(s,\alpha) &= q^{\alpha(s-1)}\left(\frac{1-q}{1-q^{\alpha}}\right)^s \\ &\times \sum_{n\geq 0}\frac{(1-q^{\alpha})^s(1-q^{\alpha}q)^s\cdots\left(1-q^{\alpha}q^{n-1}\right)^s q^{n(s-1)}}{(1-q^{\alpha+1})^s\cdots(1-q^{\alpha+1}q^{n-2})^s(1-q^{\alpha+1}q^{n-1})^s} \\ &= q^{\alpha(s-1)}\left(\frac{1-q}{1-q^{\alpha}}\right)^s\sum_{n\geq 0}\frac{(q;q)_n(q^{\alpha};q)_n^s}{(q^{\alpha+1};q)_n^s}\frac{q^{n(s-1)}}{(q;q)_n} \\ &= q^{\alpha(s-1)}\left(\frac{1-q}{1-q^{\alpha}}\right)^s {}_{s+1}\varphi_s\left(\begin{array}{c|c} q,q^{\alpha},\ldots,q^{\alpha} & \\ & q;q^{s-1} \\ q^{\alpha+1},\ldots,q^{\alpha+1} & \end{array}\right). \quad (17)\end{aligned}$$

Clearly, according to (17), the function $\zeta_q(s,\alpha)$ is the product of

$$q^{\alpha(s-1)}\left(\frac{1-q}{1-q^{\alpha}}\right)^s,$$

by a q-hypergeometric ${}_{s+1}\varphi_s$ series $(s-1)$-balanced.

Now let's prove $ii.$). Taking into account

$$\zeta_q(s,\alpha) = q^{\alpha(s-1)}\left(\frac{1-q}{1-q^{\alpha}}\right)^s\sum_{n\geq 0}q^{-n}\left[\frac{(q^{\alpha};q)_n}{(q^{\alpha+1};q)_n}q^n\right]^s,$$

and using the lemma 1 as well as the q-Chu-Vandermonde formula (13) we obtain the desired result for (15).

According to the q-Chu-Vandermonde formula (12), we have that

$$\begin{aligned}\frac{(q^{\alpha};q)_n}{(q^{\alpha+1};q)_n} &= {}_2\varphi_1\left(\begin{array}{c|c} q^{-n},q & \\ & q;q^{n+\alpha} \\ q^{\alpha+1} & \end{array}\right) \\ &= \sum_{k\geq 0}\frac{(q^{-n};q)_k(q;q)_k}{(q^{\alpha+1};q)_k}\frac{q^{k(n+\alpha)}}{(q;q)_k} \\ &= \sum_{0\leq k\leq n}\frac{(q^{-n};q)_k}{(q^{\alpha+1};q)_k}q^{k(n+\alpha)}.\end{aligned}$$

Consequently

$$\zeta_q(s,\alpha) = q^{\alpha(s-1)}\left(\frac{1-q}{1-q^{\alpha}}\right)^s\sum_{n\geq 0}\frac{(q^{\alpha};q)_n^{s-1}}{(q^{\alpha+1};q)_n^{s-1}}q^{n(s-1)}\sum_{0\leq k\leq n}\frac{(q^{-n};q)_k}{(q^{\alpha+1};q)_k}q^{k(n+\alpha)}.$$

Since

$$\left(q^{-n};q\right)_k = \frac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}, \quad k\leq n.$$

Then, as result we obtain that

$$\zeta_q(s,\alpha) = q^{\alpha(s-1)}\left(\frac{1-q}{1-q^{\alpha}}\right)^s \times \sum_{k\geq 0}\sum_{n\geq k}\frac{(q;q)_n\,(q^{\alpha};q)_n^{s-1}}{(q^{\alpha+1};q)_n^{s-1}\,(q^{\alpha+1};q)_k\,(q;q)_{n-k}}\left(-q^{\alpha}\right)^k q^{\binom{k}{2}+n(s-1)}$$

$$= q^{\alpha(s-1)}\left(\frac{1-q}{1-q^{\alpha}}\right)^s \times \sum_{k\geq 0}\sum_{n\geq 0}\frac{(q;q)_{n+k}\,(q^{\alpha};q)_{n+k}^{s-1}}{(q^{\alpha+1};q)_{n+k}^{s-1}\,(q^{\alpha+1};q)_k\,(q;q)_n}\left(-q^{s-1+\alpha}\right)^k q^{\binom{k}{2}+n(s-1)}.$$

Using the property

$$(a;q)_{n+k} = (a;q)_n\,(aq^n;q)_k = (a;q)_k\,(aq^k;q)_n,$$

we have

$$\zeta_q(s,\alpha) = q^{\alpha(s-1)}\left(\frac{1-q}{1-q^{\alpha}}\right)^s \times \sum_{k\geq 0}(-1)^k q^{\binom{k}{2}+k\alpha}\frac{(q;q)_k}{(q^{\alpha+1};q)_k}\frac{(q^{\alpha};q)_k^{s-1}}{(q^{\alpha+1};q)_k^{s-1}}q^{k(s-1)} \times \sum_{n\geq 0}\frac{\left(q^{k+1};q\right)_n\left(q^{k+\alpha};q\right)_n^{s-1}}{(q^{k+\alpha+1};q)_n^{s-1}}\frac{q^{n(s-1)}}{(q;q)_n},$$

which coincides with (16). Thus, the proof is completed. □

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