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EXACT SOLUTIONS OF SOME COMPLEX PARTIAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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ABSTRACT. In this paper some exact solutions of fractional Schrödinger and Eckhaus equations and numerical solutions of massive Thirring equations are obtained. These equations can be reduced to fractional ordinary differential equations by using a suitable transformation. The fractional derivatives are described in the Caputo sense.

1. INTRODUCTION

In recent years fractional partial differential equations have been recieved considerable interest by many researchers in mathematics and some of the others fields in science. Fractional partial differential equations (FPDE) arise from widely expanded areas such as fluid mechanics, viscoelasticity, biology and engineering [13]. Several methods have been introduced to obtain exact and numerical solutions of these type of equations such as the Adomian decomposition method, variational iteration method, differential transform method and so on ([4], [8],[9],[11],[16] and [18]). In literature generally numerical techniques have been proposed for FPDE, so we are motivated to investigate exact solutions of some important complex FPDE's.

In this paper we consider the fractional Schrödinger equation

$$i\frac{\partial^{\alpha}u}{\partial t^{\alpha}} + p\frac{\partial^2 u}{\partial x^2} + q\left|u\right|^2 u = 0, \tag{1}$$

and Eckhaus equation

$$i\frac{\partial^{\alpha}u}{\partial t^{\alpha}} + \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial(\left|u\right|^2)}{\partial x}u + \left|u\right|^4 u = 0 \tag{2}$$

of order $0 < \alpha \le 1$, with the initial condition

$$u(x,0) = e^{ix}u_0.$$

By using the variable transformation

$$u(x,t) = e^{ix}U(t), \tag{3}$$

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these equations are reduced to ordinary fractional differential equations which provide us to find some exact solutions of these equations by using the Demirci and Ozalp's aproach ([2], [3]). Demirci and Ozalp introduce a new technique to find the exact solutions of fractional differential equations by using the solutions of integer order differential equations.Transformation (3) is a special form of transformation $u(x,t) = e^{i\theta}U(\xi)$ where $\theta = \alpha x + \beta t$ and $\xi = x - ct$. This transformation is a well known transformation and is especially used to reduce the complex nonlinear partial differential equations to ordinary differential equations. For example, nonlinear Schrödinger equation (NLS) and Eckhaus equations are admits a traveling wave solution of the form $u(x,t) = e^{i\theta}U(\xi)$ ([14], [15]).

Also we consider the fractional massive Thirring equations

$$i(\frac{\partial^{\alpha}u}{\partial t^{\alpha}} + \frac{\partial u}{\partial x}) + v + u\left|v\right|^2 = 0$$

$$i(\frac{\partial^{\alpha}v}{\partial t^{\alpha}} - \frac{\partial v}{\partial x}) + u + v\left|u\right|^2 = 0$$

$$(4)$$

with the following conditions

$$u(x,0) = e^{ix}u_0$$
$$v(x,0) = e^{ix}v_0$$

$$(5)$$

By using the same method, we obtain numerical solutions of this equation. Finally graphics of some solutions are presented.

Now we give some definitions of the fractional calculus theory which will be needed in this paper ([5], [7] and [10]).

Definition 1. *The Riemann-Liouville fractional integration of order α, of a function f is defined as*

$$J_a^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad \alpha > 0, \quad x > 0.$$

$$(6)$$

Definition 2. *The fractional derivative of $f(x)$ in the Caputo sense is defined as*

$$^C D_a^{\alpha}f(x) = J_a^{m-\alpha}D^m f(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1}f^{(m)}(t)dt$$

where $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, $x > 0$.

We note that, for the fractional order differentiation, we will use the Caputo's definition due to its convenience for initial conditions of the differential equations.

Definition 3. *For m to be the smallest integer that exceeds α, the Caputo timefractional derivative operator of order $\alpha > 0$ is defined as*

$$D_t^{\alpha}u(x,t) = \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_a^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial \tau^m}d\tau & ; m-1 < \alpha < m \\ \frac{\partial^m u(x,t)}{\partial t^m} & ; \alpha = m \in \mathbb{N} \end{cases}$$

Here and elsewhere Γ denotes the Gamma function.

2. Exact Solutions

In this section first we give some results obtained from Demirci and Ozalp's work [2]. Then we introduced some exact solutions of Schrödinger and Eckhaus equations of fractional order and numerical solutions of massive-Thirring equation of fractional order.

Consider the initial value problem (IVP) with Caputo type FDE given by

$$D^\alpha x(t) = f(t, x(t))$$
$$x(0) = x_0 \tag{7}$$

where $f \in C([0,T]xR, R),\ 0 < \alpha < 1$.

Theorem 1. *Assume that $f \in C[R_0, R]$ where $R_0 = [(t,x) : 0 \leq t \leq \alpha$ and $|x - x_0| \leq b]$ and let $|f(t,x)| \leq M$ on R_0. Then there exist at least one solution for the IVP (7) on $0 \leq t \leq \gamma$ where $\gamma = \min(a, [\frac{b}{M}\Gamma(\alpha + 1)^{1/\alpha}])$.*

Theorem 2. *Consider the IVP given by (7). Let*

$$g(v, x_*(v)) = f(t - (t^\alpha - v\Gamma(\alpha+1))^{1/\alpha}, x(t - (t^\alpha - v\Gamma(\alpha+1))^{1/\alpha})$$

and assume that the conditions of Therorem 1 hold. Then a solution of (7), x(t), is given by

$$x(t) = x_*(t^\alpha/\Gamma(\alpha+1))$$

where $x_*(v)$ is a solution of the integer order differential equation

$$\frac{d(x_*(v))}{dv} = g(v, x_*(v))$$

with the initial condition $x_*(0) = x_0$.

2.1 Exact solutions of fractional Schrödinger equation

First we consider the

$$i\frac{\partial^\alpha u}{\partial t^\alpha} + p\frac{\partial^2 u}{\partial x^2} + q|u|^2 u = 0 \tag{8}$$

fractional Schrödinger equation with the initial condition

$$u(x, 0) = e^{ix}u_0. \tag{9}$$

([6], [12] and [17]). Where p, q and u_0 are non-zero real constants and $u = u(x,t)$ is a complex-valued function of two real variables x, t. The physical model (8) have been introduced in various area of physics such as nonlinear optics, plasma physics, superconductivity and quantum mechanics for $\alpha = 1$. Substituting (3) into (8) yields,

$$D^\alpha U(t) = i(qU^3(t) - pU(t)) \tag{10}$$

with

$$U(0) = u_0. \tag{11}$$

Now we apply the method which is introduced by Demirci and Ozalp [2]. According to this method

$$g(\nu, U_*(\nu)) = i(qU^3 - pU)$$

and the solution of the corresponding integer order IVP given in Theorem 2 is

$$U_*(\nu) = \pm\sqrt{\frac{p}{q + (\frac{p}{u_0^2} - q)e^{2p i \nu}}}.$$

So the solution of the (10)-(11) is

$$U(t) = U_*(\frac{t^\alpha}{\Gamma(\alpha+1)}) = \pm\sqrt{\frac{p}{q + (\frac{p}{u_0^2} - q)e^{2p i t^\alpha/\Gamma(\alpha+1)}}}.$$

Finally, from (3), the exact solution of (8)-(9) is obtained as follows

$$u(x,t) = \pm e^{ix}\sqrt{\frac{p}{q + (\frac{p}{u_0^2} - q)e^{2p i t^\alpha/\Gamma(\alpha+1)}}}. \tag{12}$$

For a special choice of p, q, u_0 and α the graphics are presented (Fig. 1-3).
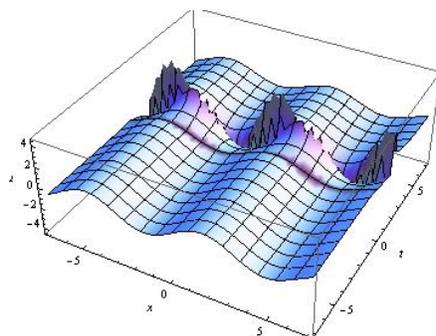

Fig. 1 Imaginary component of $u(x,t)$ (12) for values $p = q = 1$, $u_0 = 1/\sqrt{2}$ and $\alpha = 0.5$
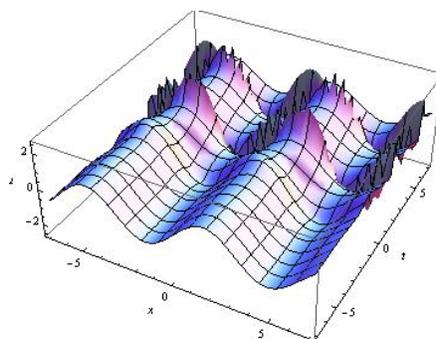

Fig. 2 Imaginary component of $u(x,t)$ (12) for values $p = q = 1$, $u_0 = 1/\sqrt{2}$ and $\alpha = 0.9$
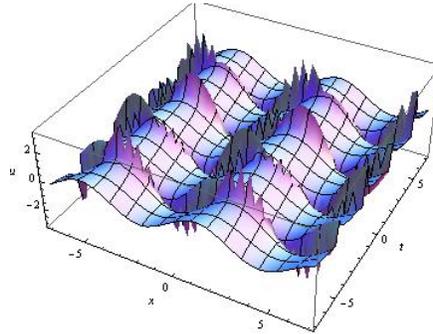
Fig. 3 Imaginary component of $u(x,t)$ (12) for values $p = q = 1$, $u_0 = 1/\sqrt{2}$ and $\alpha = 1$

2.2 Exact solutions of fractional Eckhaus equation

In this part we consider the

$$i\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial(|u|^2)}{\partial x}u + |u|^4\, u = 0 \tag{13}$$

fractional Eckhaus equation with the initial condition (9). In [1], many of the properties of the Eckhaus equation were investigated. By applying the transformation $u(x,t) = e^{ix}U(t)$ to the fractional Eckhaus equation, we obtain an ordinary differential equation of fractional order as follows

$$D^\alpha U(t) = i(U^5(t) - U(t)) \tag{14}$$

with (11). Then from Theorem 2, an exact solution of (14)-(11) is obtained as

$$U(t) = \pm\frac{1}{(1 - (1 - \frac{1}{u_0^4})e^{4it^\alpha/\Gamma(\alpha+1)})^{1/4}}. \tag{15}$$

Thus it follows from (3) that the exact solution of fractional Eckhaus equation is

$$u(x,t) = \pm e^{ix}\frac{1}{(1 + (\frac{1}{u_0^4} - 1)e^{4it^\alpha/\Gamma(\alpha+1)})^{1/4}}. \tag{16}$$

See Fig. 4-6.
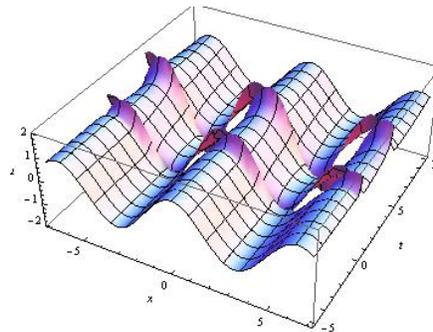
Fig. 4 Real component of $u(x,t)$ (16) for $u_0 = (1/2)^{1/4}$ and $\alpha = 0.5$
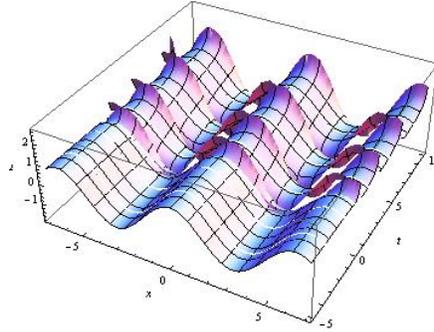
Fig. 5 Real component of $u(x,t)$ (16) for $u_0 = (1/2)^{1/4}$ and $\alpha = 0.7$
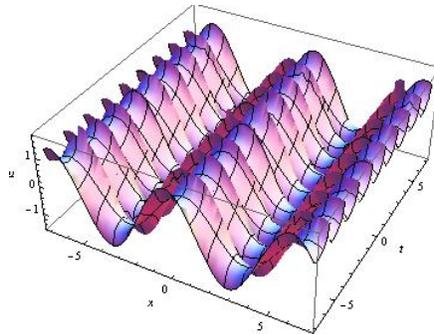

Fig. 6 Real component of $u(x,t)$ (16) for $u_0 = (1/2)^{1/4}$ and $\alpha = 1$.

2.3 Numerical solutions of massive Thirring equation of fractional order

Finally we consider the massive Thirring equation of fractional order

$$i(\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x}) + v + u\left|v\right|^2 = 0$$

$$i(\frac{\partial^\alpha v}{\partial t^\alpha} - \frac{\partial v}{\partial x}) + u + v\left|u\right|^2 = 0 \tag{17}$$

with the initial conditions

$$u(x,0) = e^{ix}u_0$$
$$v(x,0) = e^{ix}v_0. \tag{18}$$

(17) was considered in [19] for $\alpha = 1$. If we use the following transformation

$$u(x,t) = e^{ix}U(t) \tag{19}$$
$$v(x,t) = e^{ix}V(t)$$

then (17)-(18) can be reduced to a system of fractional ordinary differential equation,

$$i\frac{d^\alpha U}{dt^\alpha} = U - V - UV^2$$

$$i\frac{d^\alpha V}{dt^\alpha} = -U - V - VU^2 \tag{20}$$

with the initial conditions

$$
\begin{aligned}
u(0) &= u_0 \\
v(0) &= v_0.
\end{aligned}
\tag{21}
$$

The corresponding integer order system of (20)-(21) that was given in Theorem 2 is

$$
\begin{aligned}
i\frac{dU^*}{dt} &= U^* - V^* - U^*V^{*2} \\[2mm]
i\frac{dV^*}{dt} &= -U^* - V^* - V^*U^{*2}
\end{aligned}
\tag{22}
$$

If the solution of this integer order system is $(U^*(t), V^*(t))$, the solution of the IVP (20)-(21) is $(U^*(t^\alpha/\Gamma(\alpha+1)), V^*(t^\alpha/\Gamma(\alpha+1)))$. In the Fig. 7-10, real components of numerical solution of IVP (17)-(18) are presented for the parameter values $u_0 = v_0 = 1$. Also in Fig. 11 a-d, numerical solutions of system (20) are illustrated for different values of α; $\alpha = 1$ (black line), $\alpha = 0.9$ (green line), $\alpha = 0.7$ (blue line) and $\alpha = 0.5$ (orange line).
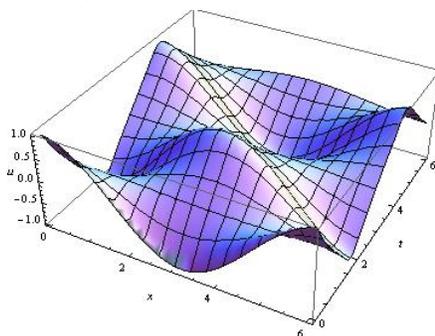
Fig. 7 Numerical solution $u(x,t)$ of (17)-(18) for $\alpha = 0.8$
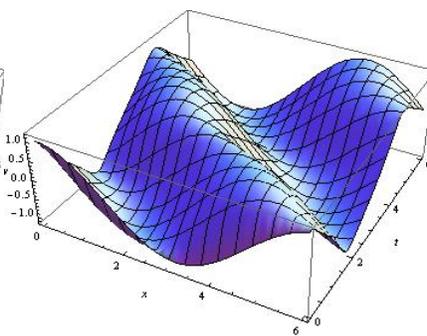
Fig. 8 Numerical solution $v(x,t)$ of (17)-(18) for $\alpha = 0.8$
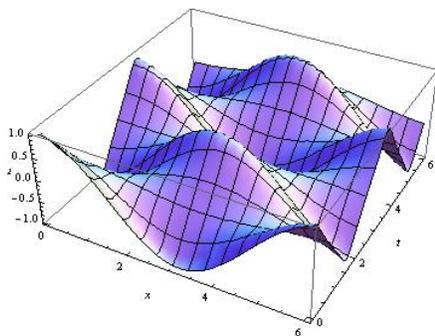
Fig. 9 Numerical solution $u(x,t)$ of (17)-(18) for $\alpha = 1$
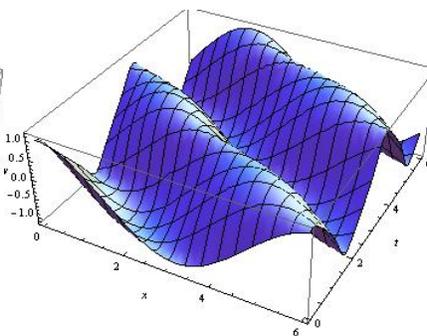
Fig. 10 Numerical solution $v(x,t)$ of (17)-(18) for $\alpha = 1$
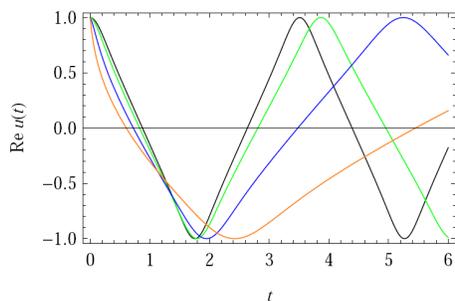
Fig. 11 a. Numerical solutions of system
(20) for $u_0 = v_0 = 1$
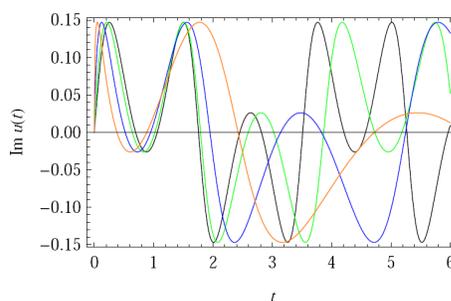
Fig. 11 b. Numerical solutions of system
(20) for $u_0 = v_0 = 1$.
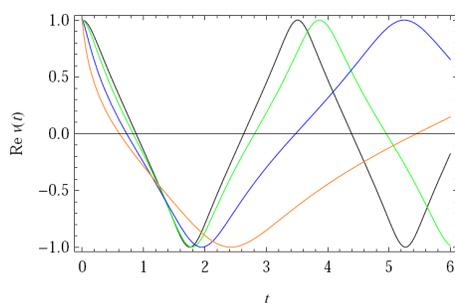
Fig. 11 c. Numerical solutions of system
(20) for $u_0 = v_0 = 1$.
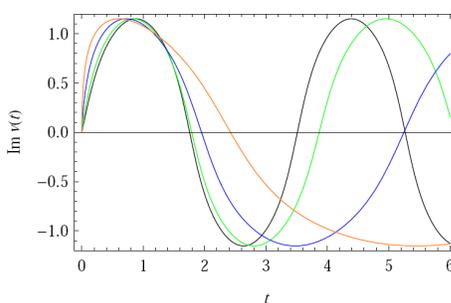
Fig. 11 d. Numerical solutions of system
(20) for $u_0 = v_0 = 1$

Conclusion

In this paper we consider the Schrödinger, Eckhaus and massive Thirring equations of fractional order. These equations can be reduced to fractional ordinary differential equations by transformation $u(x, t) = e^{ix}U(t)$. We know that, especially numerical mehods have been introduced in literature to find the solution of this type fractional partial differential equations. However in 2011, Demirci and Ozalp proposed an efficient method to find the exact solutions of fractional differential eqautions [2], [3]. With the help of this method, we give some exact solutions of Schrödinger and Eckhaus equations of fractional order. Also we give a numerical solution for fractional massive Thirring equation.

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