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SUBCLASSES OF STARLIKE FUNCTIONS ASSOCIATED WITH FRACTIONAL CALCULUS OPERATORS

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Abstract. Making use of generalized fractional integral operator, we introduce a new subclass of $k-$uniformly starlike functions based on Fox's H- functions and determine coefficient estimates, extreme points, closure theorem, distortion bounds, radii of starlikeness and convexity. Furthermore subordination theorem, an integral transform results,neighborhood results and integral means inequalities are also discussed.

1. Introduction

Denote by $\mathcal{A}$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic and univalent in the open disc $U = \{z : |z| < 1\}$. For functions $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in U. \tag{2}$$

Also denote by $\mathcal{T}$, a subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, z \in U \tag{3}$$

is introduced and studied by Silverman [20]. A function $f(z) \in \mathcal{T}$ is starlike of order $\gamma(0 \leq \gamma < 1)$ denoted by $\mathcal{T}^*(\gamma)$, if $\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma$ and it is convex of order $\gamma(0 \leq \gamma < 1)$ denoted by $C(\gamma)$, if $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma$.

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The study of operators plays an important role in the geometric function theory and its related fields. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better.

Now we briefly recall the definitions of the special functions and operators of fractional calculus used in this paper discussed in [10, 11, 14, 16]. The generalized hypergeometric function defined by the Mellin-Barnes type contour integral

$$\mathcal{H}^{l,m}_{p,q}\left[\sigma \mid \begin{array}{c}(a_n,A_n)_1^p\\(b_n,B_n)_1^q\end{array}\right]=\frac{1}{2\pi i}\int_L \frac{\prod\limits_{n=1}^{l}\Gamma(b_n-B_ns)\prod\limits_{j=1}^{m}\Gamma(1-a_j+sA_j)}{\prod\limits_{n=l+1}^{q}\Gamma(1-b_n+sB_n)\prod\limits_{j=m+1}^{p}\Gamma(a_j-sA_j)}\sigma^s ds. \tag{4}$$

Here L is a suitable contour in C and the orders $(l;m;p;q)$ are integers $0\leq l\leq q$, $0\leq m\leq p$. For $a_j\in\mathbb{R}$, $A_j>0$ $(j=1,\dots,p)$, $b_n\in\mathbb{R}$, $B_n>0$ $(n=1,\dots,q)$ and the types of contours, for existence and analyticity of function (4) in disks $\subset\mathbb{C}$, for $A_1=\cdots=A_p=1$, $B_1=\cdots=B_q=1$, (4) turns into the more popular *Meijer's G-function*. The *Mittag-Leffler function,* and the so-called Wright's generalized hypergeometric functions ${}_p\Psi_q$ with irrational $A_j,B_n>0$, give rather general and typical examples of $H-$functions, (not reducible to $G-$functions):

$$\begin{aligned}{}_p\Psi_q\left(\begin{array}{c}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_q,B_q)\end{array}:\sigma\right)&=\sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\dots\Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\dots\Gamma(b_q+nB_q)}\ \frac{\sigma^n}{n!}\\ =\ \mathcal{H}^{1,p}_{p,q+1}&\left[-\sigma\middle|\begin{array}{c}(1-a_1,A_1),\dots,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),\dots,(1-b_q,B_q)\end{array}\right],\end{aligned} \tag{5}$$

when $A_1=\cdots=A_p=B_1=\cdots=B_q=1$, they turn into

$${}_p\Psi_q\left(\begin{array}{c}(a_1,1),\dots,(a_p,1)\\(b_1,1),\dots,(b_q,q)\end{array}:\sigma\right)\ =\ \left[\frac{\prod\limits_{j=1}^{p}\Gamma(a_j)}{\prod\limits_{i=1}^{q}\Gamma(b_j)}\right]{}_pF_q(a_1,\dots,a_p:b_1,\dots,b_q:\sigma)$$

these functions were extensively studied in [3, 4]. All the *classical Fractional Calculus operators* [15], and most of their generalizations by different authors, fall in the Generalized Fractional Calculus operators as very special cases, by taking multiplicities $l=1,2,....$ and some specific parameters.

Let $l\geq 1$ be an integer; $\delta_i\geq 0$, $\alpha_i\in\mathbb{R}$, $\beta_j>0$, $i=1,\dots l$. We consider $\delta=(\delta_1,\dots,\delta_l)$ as a *multi-order* of *fractional integration;* $\alpha=(\alpha_1,\dots,\alpha_l)$ as multi-weight; $\beta=(\beta_1,\dots,\beta_l)$ as additional parameter. The integral operators defined as given below:

$$\begin{aligned}\mathcal{I}f(z)\ &=\ \mathcal{I}^{(\alpha_i),(\delta_i)}_{(\beta_i),l}f(z)\\ &=\ \int_0^1 \mathcal{H}^{l,0}_{l,l}\left[\sigma\middle|\begin{array}{c}(\alpha_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i})_1^l\\(\alpha_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i})_1^l\end{array}\right]f(z\sigma)d\sigma\end{aligned} \tag{6}$$

if $\sum\limits_{i=1}^{l} \delta_i > 0$, and as $\mathcal{I}f(z) = f(z)$ if $\delta_1 = \delta_2 = \cdots = \delta_l = 0$, are said to be multiple (m-tuple) *Erdélyi-Kober* fractional integration operators [11]. More generally, all the operators of the form $\widetilde{\mathcal{I}}f(z) = z^{\delta_0}\mathcal{I}_{(\beta_i),l}^{(\alpha_i),(\delta_i)} f(z)$ with $\delta_0 \geq 0$, are briefly called as generalized *(m-tuple)* fractional integrals. The corresponding generalized fractional derivatives are denoted by $D_{(\beta_i),l}^{(\alpha_i),(\delta_i)}$ and defined by means of explicit differintegral expressions, similarly to the idea for the classical Riemann-Liouville derivative (see [11, 15]).

We recall the following lemma to represent the generalized fractional calculus operator for functions in $f \in \mathcal{A}$ and of the form (6) which are analytic in the unit disk $U = \{z : |z| < 1\}$ based on the $H-$function theory.

Lemma 1. [12, 13] For $\delta_i \geq 0$, $\alpha_i \in \mathbb{R}$, $\beta_i > 0$ $(i = 1, \ldots, l)$, and each $p > \max\limits_i[-\beta_i(\alpha_i + 1)]$,

$$\mathcal{I}_{(\beta_i),l}^{(\alpha_i),(\delta_i)}\{z^p\} = \Omega_p z^p, \tag{7}$$

where

$$\Omega_p = \prod_{i=1}^{l} \frac{\Gamma(\alpha_i + 1 + p/\beta_i)}{\Gamma(\alpha_i + \delta_i + 1 + p/\beta_i)} > 0.$$

Then the conditions $\delta_i \geq 0,\ \ \alpha_i \geq -1,\ \ \beta_i > 0,\ \ i = 1, \ldots, l,$ it has been ensured that (7) holds for each $p \geq 0$. Further, the (normalized) GFC operators is given by

$$\widetilde{I}_{(\beta_i),l}^{(\alpha_i),(\delta_i)} f(z) := \Omega^{-1} I_{(\beta_i),l}^{(\alpha_i),(\delta_i)} f(z). \tag{8}$$

where

$$\Omega^{-1} = \prod_{i=1}^{l} \frac{\Gamma(\alpha_i + \delta_i + 1 + 1/\beta_i)}{\Gamma(\alpha_i + 1 + 1/\beta_i)},\ \ (p = 1)$$

Theorem 1. [11] Under the parameters conditions $\delta_i \geq 0,\ \ \alpha_i > -1,\ \ \beta_i > 0\ \ (i = 1, \ldots, l)$ the generalized fractional integral $\widetilde{\mathcal{I}}_{(\beta_i),l}^{(\alpha_i),(\delta_i)}$ maps the class $\mathcal{A}$ into itself, and the image of a power series (1) has the form

$$\widetilde{\mathcal{I}}f(z) = \widetilde{\mathcal{I}}_{(\beta_i),l}^{(\alpha_i),(\delta_i)} \left\{ z + \sum_{n=2}^{\infty} a_n z^n \right\} = z + \sum_{n=2}^{\infty} \Theta(n) a_n z^n \in \mathcal{A}, \tag{9}$$

with multipliers' sequence (n=2, 3, ...):

$$\Theta(n) = \prod_{i=1}^{l} \frac{\Gamma(\alpha_i + 1 + n/\beta_i)\Gamma(\alpha_i + \delta_i + 1 + 1/\beta_i)}{\Gamma(\alpha_i + \delta_i + 1 + n/\beta_i)(\alpha_i + 1 + 1/\beta_i)} > 0. \tag{10}$$

Remark 1. In the class $\mathcal{A}$ the generalized fractional integral (6) can be represented by the Hadamard product $\widetilde{\mathcal{I}}_{(\beta_i),l}^{(\alpha_i),(\delta_i)} f(z) = h(z) * f(z)$ where $h(z) \in \mathcal{A}$ is expressed by the Wright's hypergeometric function (5) studied in [3, 4].

It is of interest to note that if we specialize that $\alpha_i = -1$ and $\delta_i = 1$ with $l = 1$, $\beta_i = 1$, in (9) gives the Biernacki operator

$$\widetilde{\mathcal{I}}_{1,1}^{-1,1} = \mathcal{B}f(z) = -log(1 - z) * f(z). \tag{11}$$

Further by taking $\alpha_i = 0$ and $\delta_i = 1$ with $l = 1$, $\beta_i = 1$, in (9) gives the Libera operator

$$\mathcal{L}f(z) = 2\mathcal{I}_{1,1}^{0,1} f(z) = z{}_2F_1(1, 2; 3; z) * f(z) \tag{12}$$

and generalized Libera operator

$$B_c f(z) = (c+1)\mathcal{I}_{1,1}^{c-1,1} f(z) = z^{c+1}{}_2F_1(1, c+1; c+2; z) * f(z). \tag{13}$$

By taking $\alpha_i = a-2$ and $\delta_i = c-a$ with $l=1$, $\beta_i = 1$, in (9)we get

$$L(a,c)f(z) := z_2F_1(1, a; c; z) * f(z) = \frac{\Gamma(c)}{\Gamma(c)}\mathcal{I}_{1,1}^{a-2,c-a} f(z) \tag{14}$$

called Carlson-Shaffer operator [2].

For $l=1$ and $\beta_i = 1$, by (9) we have

$$D^\eta f(z) := z_2F_1(1, \eta+1; 1; z) * f(z) = z + \sum_{n=2}^{\infty} \begin{pmatrix} \eta+n-1 \\ \eta-1 \end{pmatrix} a_k z^k \tag{15}$$

called Ruscheweyh Derivative operator [18].

Motivated by the earlier works of Goodman [7] and (also see [1, 17, 20, 23] and using the techniques employed in [5], in this paper we introduce a new subclass of $k-$uniformly starlike functions of order γ based on generalized fractional integral operator operator .

For $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$ and $k \geq 0$, we let $T\mathcal{G}(\lambda, \gamma, k)$ be the subclass of $\mathcal{T}$ consisting of functions of the form (3) and satisfying the analytic criterion

$$\Re\left(\frac{G_\lambda(z)}{zG'_\lambda(z)} - \gamma\right) > k\left|\frac{G_\lambda(z)}{zG'_\lambda(z)} - 1\right| \tag{16}$$

where

$$\frac{G_\lambda(z)}{zG'_\lambda(z)} = \frac{(1-\lambda)\widetilde{\mathcal{I}}f(z) + \lambda z(\widetilde{\mathcal{I}}f(z))'}{z(\widetilde{\mathcal{I}}f(z))' + \lambda z^2(\widetilde{\mathcal{I}}f(z))''} \tag{17}$$

$z \in U$, and $\widetilde{\mathcal{I}}f(z)$ is given by (9).

For different choices of λ we state some special cases of subclasses of starlike and convex functions involving the generalized integral operators. As illustrations, we present few following examples:

Example 1. *If* $\lambda = 0$, *then*

$$\mathcal{T}\mathcal{G}_p(\gamma, k) := \left\{ f \in \mathcal{T} : \Re\left(\frac{\widetilde{\mathcal{I}}f(z)}{z(\widetilde{\mathcal{I}}f(z))'} - \gamma\right) > k\left|\frac{\widetilde{\mathcal{I}}f(z)}{z(\widetilde{\mathcal{I}}f(z))'} - 1\right|, \ \ z \in U \right\} \tag{18}$$

Example 2. *If* $\lambda = 1$, *then*

$$\mathcal{T}\mathcal{G}(\gamma, k) := \left\{ f \in \mathcal{T} : \Re\left(\frac{(\widetilde{\mathcal{I}}f(z))'}{(\widetilde{\mathcal{I}}f(z))' + z(\widetilde{\mathcal{I}}f(z))''} - \gamma\right) > k\left|\frac{(\widetilde{\mathcal{I}}f(z))'}{(\widetilde{\mathcal{I}}f(z))' + z(\widetilde{\mathcal{I}}f(z))''}\right| z \in U \right\} \tag{19}$$

In this paper we determine the coefficient estimate, extreme points, closure theorem, distortion bounds, radii of starlikeness and convexity results for functions in $T\mathcal{G}(\lambda, \gamma, k)$. Further more we discuss subordination theorem, an integral transform results, neighborhood results and integral means inequalities for functions in $T\mathcal{G}(\lambda, \gamma, k)$.

2. Lemmas and Their Proofs

We recall the following lemmas, to prove our main results.

Lemma 2. If γ is a real number and w is a complex number , then

$$\Re(w) \geq \gamma \Leftrightarrow |w + (1-\gamma)| - |w - (1+\gamma)| \geq 0.$$

Lemma 3. If w is a complex number and γ, k are real numbers, then

$$\Re(w) \geq k|w-1| + \gamma \Leftrightarrow \Re\{w(1+ke^{i\theta}) - ke^{i\theta}\} \geq \gamma, \ \ -\pi \leq \theta \leq \pi.$$

Lemma 4. Let $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$, $k \geq 0$ and suppose that the parameters $\alpha_1, \dots, \alpha_l$ and $\beta_1, \dots, \beta_m$ are positive real numbers. Then a function $f \in T\mathcal{G}(\lambda, \gamma, k)$ if and only if

$$\sum_{n=2}^{\infty}(1+n\lambda-\lambda)|(1+k) - n(\gamma+k)|\Theta(n)\ |a_n| \leq 1-\gamma, \tag{20}$$

where $\Theta(n)$is given by (10) .

Proof. Let a function f of the form (3) in T satisfy the condition (20). We will show that (16) is satisfied and so $f \in T\mathcal{G}(\lambda, \gamma, k)$. Using Lemma 3, it is enough to show that

$$\Re\left(\frac{G_\lambda(z)}{zG'_\lambda(z)}(1+ke^{i\theta}) - ke^{i\theta}\right) > \gamma, \ \ -\pi \leq \theta \leq \pi. \tag{21}$$

That is, suppose $f \in T\mathcal{G}(\lambda, \gamma, k)$ then by Lemma 3, and by choosing the values of z on the positive real axis inequality (21) reduces to

$$\Re\left(\frac{(1-\gamma) - \sum\limits_{n=2}^{\infty}\left[(1+ke^{i\theta}) - n(\gamma+ke^{i\theta})\right](1+\lambda n-\lambda)\Theta(n)|a_n|z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} n(1+n\lambda-\lambda)\Theta(n)a_n z^{n-1}}\right) \geq 0.$$

Since $\Re\ (-e^{i\theta}) \geq -e^{i0} = -1$, the above inequality reduces to

$$\Re\left(\frac{(1-\gamma) - \sum\limits_{n=2}^{\infty}(1+n\lambda-\lambda)[(k+1) - n(\gamma+k)]\Theta(n)a_n r^{n-1}}{1 - \sum\limits_{n=2}^{\infty} n(1+n\lambda-\lambda)\Theta(n)a_n r^{n-1}}\right) \geq 0.$$

Letting $r \to 1^-$ and by the mean value theorem we get desired inequality (20).

Conversely, let (20) hold we will show that (16) is satisfied and so $f \in \mathcal{TG}(\lambda, \gamma, k)$. In view of Lemma 2, $\Re(w) > \gamma \Leftrightarrow |w - (1+\gamma)| < |w + (1-\gamma)|$, it is enough to show that

$$\left|\frac{A(z)}{B(z)} - \left(1 + k\left|\frac{A(z)}{B(z)} - 1\right| + \gamma\right)\right| < \left|\frac{A(z)}{B(z)} + \left(1 - k\left|\frac{A(z)}{B(z)} - 1\right| - \gamma\right)\right|,$$

where

$$A(z) := [(1-\lambda)\widetilde{\mathcal{I}}f(z) + \lambda z(\widetilde{\mathcal{I}}f(z))'] = \ z - \sum_{n=2}^{\infty}(1+\lambda n-\lambda)\Theta(n)|a_n|z^n;$$

$$B(z) := [z(\widetilde{\mathcal{I}}f(z))' + \lambda z^2(\widetilde{\mathcal{I}}f(z))''] = \ z - \sum_{n=2}^{\infty} n(1+\lambda n-\lambda)\Theta(n)|a_n|z^n.$$

Hence,

$$\begin{aligned} L &= \left| \frac{A(z)}{B(z)} - \left(1 + k\left|\frac{A(z)}{B(z)} - 1\right| + \gamma\right)\right| \\ &< \frac{|z|}{|B(z)|}\left|\gamma + \sum_{n=2}^{\infty}(1+n\lambda-\lambda)\left[n-1-\gamma+n(\gamma+k)\right]\Theta(n)a_n z^n\right| \\ &< \frac{|z|}{|B(z)|}\left|(2-\gamma) - \sum_{n=2}^{\infty}(1+n\lambda-\lambda)\left[n+1+\gamma-n(\gamma+k)\right]\Theta(n)a_n z^n\right| \\ &< R = \left|\frac{A(z)}{B(z)} + \left(1 - k\left|\frac{A(z)}{B(z)} - 1\right| - \gamma\right)\right|, \end{aligned}$$

and it is easy to show that $R - L > 0$, by the given condition (20) and the proof is complete.

For the sake of brevity we let

$$(i) \quad \Upsilon(\lambda,\gamma,k,n) = (1+n\lambda-\lambda)|(1+k)-n(\gamma+k)|\Theta(n), \tag{22}$$

$$(ii) \quad \Upsilon(\lambda,\gamma,k,2) = (1+\lambda)(1-k-2\gamma)\Theta(2), \tag{23}$$

and

$$(iii) \quad \Theta(2) = \prod_{i=1}^{l}\frac{\Gamma(\alpha_i+1+2/\beta_i)\Gamma(\alpha_i+\delta_i+1+1/\beta_i)}{\Gamma(\alpha_i+\delta_i+1+2/\beta_i)(\alpha_i+1+1/\beta_i)} > 0 \tag{24}$$

unless otherwise stated.

Corollary 1. If $f \in T\mathcal{G}(\lambda,\gamma,k)$, then

$$|a_n| \le \frac{1-\gamma}{\Upsilon(\lambda,\gamma,k,n)}, \quad 0 \le \lambda \le 1, \ 0 \le \gamma < 1, k \ge 0.$$

Equality holds for the function

$$f(z) = z - \frac{(1-\gamma)}{\Upsilon(\lambda,\gamma,k,n)}z^n.$$

3. Distortion Bounds, Extreme Points and Closure theorem

By a routine procedure one can prove the distortion property and extreme points for function f in the class $T\mathcal{G}(\lambda,\gamma,k)$ so we state the results without proof.

Theorem 2. Let the function $f(z)$ defined by (3) belong to $T\mathcal{G}(\lambda,\gamma,k)$, then

$$r - \frac{1-\gamma}{\Upsilon(\lambda,\gamma,k,2)}r^2 \le |f(z)| \le r + \frac{1-\gamma}{\Upsilon(\lambda,\gamma,k,2)}r^2, \ |z| = r \tag{25}$$

and

$$1 - \frac{2(1-\gamma)}{\Upsilon(\lambda,\gamma,k,2)}r \le |f'(z)| \le 1 + \frac{2(1-\gamma)}{\Upsilon(\lambda,\gamma,k,2)}r, \ |z| = r. \tag{26}$$

Equalities are sharp for the function $f(z) = z - \frac{1-\gamma}{\Upsilon(\lambda,\gamma,k,2)}z^2$, where $\Upsilon(\lambda,\gamma,k,2)$ is obtained from (23)

Theorem 3. The extreme points of $T\mathcal{G}(\lambda,\gamma,k)$ are

$$f_1(z) = z \text{ and } f_n(z) = z - \frac{1-\gamma}{\Upsilon(\lambda,\gamma,k,n)}z^n, \quad \text{for } n = 2,3,4,\ldots. \tag{27}$$

where $\Upsilon(\lambda,\gamma,k,n)$ is defined in (22). Then $f \in T\mathcal{G}(\lambda,\gamma,k)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=1}^{\infty}\omega_n f_n(z), \qquad \omega_n \geq 0, \quad \sum_{k=1}^{\infty}\omega_n = 1. \tag{28}$$

Theorem 4. Let the functions $f_j(z)$ $(j=1,2,\dots m)$ defined by

$$f_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}\ z^n \text{ for } a_{n,j}\geq 0,\ z\in U. \tag{29}$$

be in the classes $T\mathcal{G}(\lambda,\gamma_j,k)$ $(j=1,2,\dots m)$ respectively. Then the function

$$h(z)=z-\frac{1}{m}\sum_{n=2}^{\infty}\left(\sum_{j=1}^{m}a_{n,j}\right)z^n$$

is in the class $T\mathcal{G}(\lambda,\gamma,k)$, where $\gamma=\min\limits_{1\leq j\leq m}\{\gamma_j\}$ where $-1\leq\gamma_j<1$.

Proof. Since $f_j(z)\in T\mathcal{G}(\lambda,\gamma_j,k)$ $(j=1,2,3,\dots m)$ by applying Lemma 4, to (29) we observe that

$$\begin{aligned}&\sum_{n=2}^{\infty}\Upsilon(\lambda,\gamma,k,n)\left(\frac{1}{m}\sum_{j=1}^{m}a_{n,j}\right)\\ &\quad=\frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=2}^{\infty}\Upsilon(\lambda,\gamma,k,n)a_{n,j}\right)\\ &\quad\leq\frac{1}{m}\sum_{j=1}^{m}(1-\gamma_j)\leq 1-\gamma\end{aligned}$$

where $\Upsilon(\lambda,\gamma,k,n)$ is defined in in (22) and which in view of Lemma 4, again implies that $h(z)\in T\mathcal{G}(\lambda,\gamma,k)$ and so the proof is complete.

4. Integral Transform of the class $T\mathcal{G}(\lambda,\gamma,k)$

In this section we prove that the class $T\mathcal{G}(\lambda,\gamma,k)$ is closed under integral transform.

For $f\in\mathcal{A}$ we define the integral transform

$$V_\nu(f)(z)=\int_0^1\nu(t)\frac{f(tz)}{t}dt,$$

where ν is a real valued, non-negative weight function normalized so that $\int_0^1\nu(t)dt=1$. Since special cases of $\nu(t)$ are particularly interesting such as $\nu(t)=(1+c)t^c$, $c>-1$, for which V_ν is known as the Bernardi operator, and

$$\nu(t)=\frac{(c+1)^\delta}{\nu(\delta)}t^c\left(\log\frac{1}{t}\right)^{\delta-1}, \ \ c>-1, \quad \delta\geq 0,$$

which gives the Komatu operator. For more details see [8].

First we show that the class $T\mathcal{G}(\lambda,\gamma,k)$ is closed under $V_\nu(f)(z)$.

Theorem 5. Let $f(z)\in T\mathcal{G}(\lambda,\gamma,k)$. Then $V_\nu(f)(z)\in T\mathcal{G}(\lambda,\gamma,k)$.

Proof. By definition, we have

$$\begin{aligned} V_\nu(f)(z) = & \frac{(c+1)^\delta}{\nu(\delta)} \int_0^1 (-1)^{\delta-1} t^c (\log t)^{\delta-1} \left(z - \sum_{n=2}^{\infty} a_n z^n t^{n-1} \right) dt \\ &= \frac{(-1)^{\delta-1}(c+1)^\delta}{\nu(\delta)} \lim_{r\to 0^+} \left[\int_r^1 t^c (\log t)^{\delta-1} \left(z - \sum_{n=2}^{\infty} a_n z^n t^{n-1} \right) dt \right]. \end{aligned}$$

By simple computation, we get

$$V_\nu(f)(z) = z - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right)^\delta a_n z^n.$$

We need to prove that

$$\sum_{n=2}^{\infty} \frac{\Upsilon(\lambda,\gamma,k,n)}{1-\gamma} \left(\frac{c+1}{c+n} \right)^\delta a_n \leq 1. \tag{30}$$

On the other hand by Lemma 4, $f \in T\mathcal{G}(\lambda,\gamma,k)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Upsilon(\lambda,\gamma,k,n)}{1-\gamma} a_n \leq 1,$$

where $\Upsilon(\lambda,\gamma,k,n)$ is defined in (22). Hence $\frac{c+1}{c+n} < 1$, therefore (30) holds and the proof is complete.

The above theorem yields the following results.

Theorem 6. (i) If $f(z)$ is starlike of order γ then $V_\nu(f)(z)$ is also starlike of order γ.
(ii) If $f(z)$ is convex of order γ then $V_\nu(f)(z)$ is also convex of order γ.

Theorem 7. Let $f \in T\mathcal{G}(\lambda,\gamma,k)$. Then $V_\nu(f)(z)$ is starlike of order $0 \leq \xi < 1$ in $|z| < R_1$ where

$$R_1 = \inf_n \left[\left(\frac{c+n}{c+1} \right)^\delta \frac{(1-\xi)\Upsilon(\lambda,\gamma,k,n)}{(n-\xi)(1-\gamma)} \right]^{\frac{1}{n-1}} \qquad (n \geq 2),$$

where $\Upsilon(\lambda,\gamma,k,n)$ is defined in (22).

Proof. It is sufficient to prove

$$\left| \frac{z(V_\nu(f)(z))'}{V_\nu(f)(z)} - 1 \right| < 1 - \xi. \tag{31}$$

For the left hand side of (31) we have,

$$\begin{aligned} \left| \frac{z(V_\nu(f)(z))'}{V_\nu(f)(z)} - 1 \right| &= \left| \frac{\sum_{n=2}^{\infty} (1-n) \left(\frac{c+1}{c+n} \right)^\delta a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right)^\delta a_n z^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} (1-n) \left(\frac{c+1}{c+n} \right)^\delta a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n} \right)^\delta a_n |z|^{n-1}}. \end{aligned}$$

The last expression is less than $1-\xi$ since,

$$|z|^{n-1} < \left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\xi)\Upsilon(\lambda,\gamma,k,n)}{(n-\xi)(1-\gamma)}.$$

Therefore, the proof is complete.

Using the fact that $f(z)$ is convex if and only if $zf'(z)$ is starlike, we obtain the following.

Theorem 8. Let $f \in T\mathcal{G}(\lambda,\gamma,k)$. Then $V_\nu(f)(z)$ is convex of order $0 \leq \xi < 1$ in $|z| < R_2$ where

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\xi)\Upsilon(\lambda,\gamma,k,n)}{n(n-\xi)(1-\gamma)}\right]^{\frac{1}{n-1}} \quad (n \geq 2),$$

where $\Upsilon(\lambda,\gamma,k,n)$ is defined in (22).

5. Neighbourhood Results

In this section we discuss neighbourhood results of the class $T\mathcal{G}(\lambda,\gamma,k)$. Following [6, 19], we define the $\delta-$ neighbourhood of function $f(z) \in \mathcal{T}$ by

$$N_\delta(f) := \left\{h \in \mathcal{T} : h(z) = z - \sum_{n=2}^{\infty} d_n z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - d_n| \leq \delta\right\}. \tag{32}$$

Particulary for the identity function $e(z) = z$, we have

$$N_\delta(e) := \left\{h \in \mathcal{T} : g(z) = z - \sum_{n=2}^{\infty} d_n z^n \text{ and } \sum_{n=2}^{\infty} n|d_n| \leq \delta\right\}. \tag{33}$$

Theorem 9. If

$$\delta := \frac{2(1-\gamma)}{\Upsilon(\lambda,\gamma,k,2)} \tag{34}$$

then $T\mathcal{G}(\lambda,\gamma,k) \subset N_\delta(e)$, where $\Upsilon(\lambda,\gamma,k,2)$ is defined in (23).

Proof. For $f \in T\mathcal{G}(\lambda,\gamma,k)$, Lemma 4 immediately yields

$$\Upsilon(\lambda,\gamma,k,2)\sum_{n=2}^{\infty} a_n \leq 1-\gamma,$$

so that

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\gamma}{\Upsilon(\lambda,\gamma,k,2)}. \tag{35}$$

On the other hand, from (20) and (35) that

$$\begin{aligned} -(k+\gamma)(1+\lambda)\Theta(2)\sum_{n=2}^{\infty} na_n &\leq (1-\gamma) - (1+\lambda)(1+k)\Theta(2)\sum_{n=2}^{\infty} a_n \\ &\leq (1-\gamma) - \frac{(1+\lambda)(1+k)\Theta(2)(1-\gamma)}{(1+\lambda)(1-k-2\gamma)\Theta(2)} \\ &\leq \frac{-2(1-\gamma)(k+\gamma)}{(1-k-2\gamma)}, \end{aligned}$$

that is

$$\sum_{n=2}^{\infty} na_n \leq \frac{2(1-\gamma)}{(1+\lambda)(1-k-2\gamma)\Theta(2)} := \frac{2(1-\gamma)}{\Upsilon(\lambda,\gamma,k,2)} := \delta \tag{36}$$

which, in view of the definition (33) proves Theorem 9.

Now we determine the neighborhood for the class $T\mathcal{G}(\rho,\lambda,\gamma,k)$ which we define as follows. A function $f \in \mathcal{T}$ is said to be in the class $T\mathcal{G}(\rho,\lambda,\gamma,k)$ if there exists a function $h \in T\mathcal{G}(\rho,\lambda,\gamma,k)$ such that

$$\left|\frac{f(z)}{h(z)} - 1\right| < 1-\rho, \ \ (z \in U, \ 0 \leq \rho < 1). \tag{37}$$

Theorem 10. If $h \in T\mathcal{G}(\rho,\lambda,\gamma,k)$ and

$$\rho = 1 - \frac{\delta\Upsilon(\lambda,\gamma,k,2)}{2[(\Upsilon(\lambda,\gamma,k,2)-(1-\gamma)]} \tag{38}$$

then

$$N_\delta(h) \subset T\mathcal{G}(\rho,\lambda,\gamma,k) \tag{39}$$

where $\Upsilon(\lambda,\gamma,k,2)$ is defined in (23).

Proof. Suppose that $f \in N_\delta(h)$ we then find from (32) that

$$\sum_{n=2}^{\infty} n|a_n - d_n| \leq \delta$$

which implies that the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - d_n| \leq \frac{\delta}{2}.$$

Next, since $h \in T\mathcal{G}(\lambda,\gamma,k)$, we have

$$\sum_{n=2}^{\infty} d_n = \frac{1-\gamma}{\Upsilon(\lambda,\gamma,k,2)}$$

so that

$$\begin{aligned}
\left|\frac{f(z)}{h(z)} - 1\right| &< \frac{\sum\limits_{n=2}^{\infty}|a_n-d_n|}{1-\sum\limits_{n=2}^{\infty} d_n} \\
&\leq \frac{\delta}{2} \times \frac{\Upsilon(\lambda,\gamma,k,2)}{\Upsilon(\lambda,\gamma,k,2)-(1-\gamma)} \\
&\leq \frac{\delta\Upsilon(\lambda,\gamma,k,2)}{2[(\Upsilon(\lambda,\gamma,k,2)-(1-\gamma)]} \\
&= 1-\rho,
\end{aligned}$$

provided that ρ is given precisely by (39). Thus by definition, $f \in T\mathcal{G}(\rho,\lambda,\gamma,k)$ for ρ given by (39), which completes the proof.

6. Integral Means

In [20], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$. He applied this function to resolve his integral means inequality, conjectured in [21] and settled in [22], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^\eta d\theta,$$

for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < r < 1$. In [22], he also proved his conjecture for the subclasses $\mathcal{T}^*(\gamma)$ the class of starlike functions and $C(\gamma)$ the class of convex functions with negative coefficients.

We recall the following definition and the lemma to prove our result on Integral means inequality.

Definition 1. (Subordination Principle)[9]*:* For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h, denoted by $g \prec h$, if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$ and $g(z) = h(w(z))$, for all $z \in U$.

Lemma 5. [9] If the functions f and g are analytic in U with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta. \tag{40}$$

Applying Lemma 5, Lemma 4 and Theorem 3, we prove the Silverman's conjecture for the functions in the family $T\mathcal{G}(\lambda, \gamma, k)$.

Theorem 11. Suppose $f \in T\mathcal{G}(\lambda, \gamma, k)$, $\eta > 0$, $0 \leq \lambda \leq 1$, $0 \leq \gamma < 1$, $k \geq 0$ and $f_2(z)$ is defined by

$$f_2(z) = z - \frac{1-\gamma}{\Upsilon(\lambda, \gamma, k, 2)} z^2,$$

where $\Upsilon(\lambda, \gamma, k, 2)$ is defined in (23). Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta d\theta. \tag{41}$$

Proof. For $f(z) \in \mathcal{T}$, (41) is equivalent to proving that

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} |a_n| z^{n-1}\right|^\eta d\theta \leq \int_0^{2\pi} \left|1 - \frac{1-\gamma}{\Upsilon(\lambda, \gamma, k, 2)} z\right|^\eta d\theta.$$

By Lemma 5, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1-\gamma}{\Upsilon(\lambda, \gamma, k, 2)} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{1-\gamma}{\Upsilon(\lambda, \gamma, k, 2)} w(z), \tag{42}$$

and using (20), we obtain

$$\begin{aligned}|w(z)| &= \left|\sum_{n=2}^{\infty}\frac{\Upsilon(\lambda,\gamma,k,n)}{1-\gamma}|a_n|z^{n-1}\right| \\ &\leq |z|\sum_{n=2}^{\infty}\frac{\Upsilon(\lambda,\gamma,k,n)}{1-\gamma}|a_n| \\ &\leq |z|.\end{aligned}$$

This completes the proof .

7. Subordination Results

Now we recall the following results due to Wilf [24], which are very much needed for our study.

Definition 2. (Subordinating Factor Sequence) A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z)=\sum\limits_{n=1}^{\infty}a_nz^n$, $a_1=1$ is regular, univalent and convex in U, we have

$$\sum_{n=1}^{\infty}b_na_nz^n \prec f(z), \quad z\in U. \tag{43}$$

Lemma 6. The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if

$$\Re\left(1+2\sum_{n=1}^{\infty}b_nz^n\right)>0, \quad z\in U. \tag{44}$$

Theorem 12. Let $f\in T\mathcal{G}(\lambda,\gamma,k)$ and $g(z)$ be any function in the usual class of convex functions C, then

$$\frac{\Upsilon(\lambda,\gamma,k,2)}{2[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}(f*g)(z)\prec g(z) \tag{45}$$

where $0\leq\gamma<1$; $k\geq 0$ and $0\leq\lambda\leq 1$, and

$$\Re\left(f(z)\right)>-\frac{[1-\gamma+\Upsilon(\lambda,\gamma,k,2)}{\Upsilon(\lambda,\gamma,k,2)}, \quad z\in U. \tag{46}$$

The constant factor $\frac{\Upsilon(\lambda,\gamma,k,2)}{2[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}$ in (45) cannot be replaced by a larger number.

Proof. Let $f\in T\mathcal{G}(\lambda,\gamma,k)$ and suppose that $g(z)=z+\sum\limits_{n=2}^{\infty}b_nz^n\in C$. Then

$$\begin{aligned}&\frac{\Upsilon(\lambda,\gamma,k,2)}{2[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}(f*g)(z) \\ &\qquad=\frac{\Upsilon(\lambda,\gamma,k,2)}{2[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}\left(z+\sum_{n=2}^{\infty}b_na_nz^n\right).\end{aligned} \tag{47}$$

Thus, by Definition 2, the subordination result holds true if

$$\left\{\frac{\Upsilon(\lambda,\gamma,k,2)}{2[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 6, this is equivalent to the following inequality

$$\Re\left(1+\sum_{n=1}^{\infty}\frac{\Upsilon(\lambda,\gamma,k,2)}{[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}a_n z^n\right)>0,\quad z\in U. \tag{48}$$

By noting the fact that $\frac{\Upsilon(\lambda,\gamma,k,n)}{(1-\gamma)}$ is increasing function for $n \geq 2$ and in particular

$$\frac{\Upsilon(\lambda,\gamma,k,2)}{1-\gamma}\leq\frac{\Upsilon(\lambda,\gamma,k,n)}{1-\gamma},\quad n\geq 2,$$

therefore, for $|z| = r < 1$, we have

$$\begin{aligned}
&\Re\left(1+\frac{\Upsilon(\lambda,\gamma,k,2)}{[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}\sum_{n=1}^{\infty}a_n z^n\right)\\
&=\Re\left(1+\frac{\Upsilon(\lambda,\gamma,k,2)}{[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}z+\frac{\sum\limits_{n=2}^{\infty}\Upsilon(\lambda,\gamma,k,2)a_n z^n}{[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}\right)\\
&\geq 1-\frac{\Upsilon(\lambda,\gamma,k,2)}{[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}r-\frac{\sum\limits_{n=2}^{\infty}|\Upsilon(\lambda,\gamma,k,n)a_n|\,r^n}{[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}\\
&\geq 1-\frac{\Upsilon(\lambda,\gamma,k,2)}{[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}r-\frac{1-\gamma}{[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}r\\
&>0,\quad |z|=r<1,
\end{aligned}$$

where we have also made use of the assertion (20) of Lemma 4. This evidently proves the inequality (48) and hence also the subordination result (45) asserted by Lemma 4.

The inequality (46) follows from (45) by taking

$$g(z)=\frac{z}{1-z}=z+\sum_{n=2}^{\infty}z^n\in C.$$

Next we consider the function

$$F(z):=z-\frac{1-\gamma}{\Upsilon(\lambda,\gamma,k,2)}z^2$$

where $0 \leq \gamma < 1$, $k \geq 0, 0 \leq \lambda < 1$ and $\Upsilon(\lambda,\gamma,k,2)$ is given by (23). Clearly $F \in T\mathcal{G}(\lambda,\gamma,k)$. For this function (45)becomes

$$\frac{\Upsilon(\lambda,\gamma,k,2)}{2[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}F(z)\prec\frac{z}{1-z}.$$

It is easily verified that

$$\min\left\{\Re\left(\frac{\Upsilon(\lambda,\gamma,k,2)}{2[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}F(z)\right)\right\}=-\frac{1}{2},\quad z\in U.$$

This shows that the constant $\frac{\Upsilon(\lambda,\gamma,k,2)}{2[1-\gamma+\Upsilon(\lambda,\gamma,k,2)]}$ cannot be replaced by any larger one.

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