Journal of Fractional Calculus and Applications,
Vol. 5(1) Jan. 2014, pp. 177-187.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

SUFFICIENT CONDITIONS FOR THE OSCILLATION OF NONLINEAR FRACTIONAL DIFFERENCE EQUATIONS

J. O. ALZABUT, T. ABDELJAWAD

ABSTRACT. Sufficient conditions are established for the oscillation of solutions for fractional difference equations of the form

$$\begin{cases} \nabla^{q}_{a(q)-1}x(t)+f_1(t,x(t))=r(t)+f_2(t,x(t)),\ t\in\mathbb{N}_{a(q)}, \\ \nabla^{-(1-q)}_{a(q)-1}x(t)\Big|_{t=a(q)}=x(a(q))=c,\ c\in\mathbb{R}, \end{cases}$$

where $m-1<q<m,\ m\in\mathbb{N},\ \nabla^{q}_{a(q)}$ is the Riemann–Liouville's difference operator of order q and $\nabla^{-q}_{a(q)}$ is the Riemann–Liouville's sum operator where $\mathbb{N}_{a(q)}=\{a(q)+1,a(q)+2,\ldots\},\ a(q)=a+m-1,\ m=[q]+1$ and $[q]$ is the greatest integer less than or equal to q. The main theorems are also stated for fractional difference equation of Caputo's type

$$\begin{cases} {}^{c}\nabla^{q}_{a(q)}x(t)+f_1(t,x(t))=r(t)+f_2(t,x(t)),\ t\in\mathbb{N}_{a(q)}, \\ \nabla^{k}x(a(q))=b_k,\ b_k\in\mathbb{R},\ k=0,1,2,\ldots,m-1, \end{cases}$$

where ${}^{c}\nabla^{q}_{a(q)}$ is the Caputo's difference operator of order q. A couple of numerical examples are constructed to demonstrate the validity of the assumptions of the main theorems.

1. INTRODUCTION

Fractional differential equations are generalization of differential equations of integer order to an arbitrary non integer order. In spite of its old history which is dated back to the 16–th century, fractional differential equations have ambiguously lagged behind differential equations of integer order. In the last two decades, however, fractional differential equations have noticeably started attracting considerable interest because of their ability to provide more adequate description of memory and hereditary properties to various complex phenomena. Indeed, these equations capture nonlocal relations in space and time with power–law memory kernels. Due to extensive applications of these equations, research in this area has grown significantly all around the world. The recent years have witnessed the appearance of many papers and books that have investigated the qualitative aspects of fractional differential equations such as the fractional calculus, the existence and

---

2000 *Mathematics Subject Classification.* 26A33, 39A11.
Key words and phrases. Fractional difference equations, Riemann–Liouville operator, Caputo's operator, Oscillation.
Submitted march 23, 2013 Revised July 4, 2013.

uniqueness of solutions for Cauchy type problems and the stability and periodicity of solutions for these types of equations, we suggest the reader to consult the relevant monographs [1, 2, 3, 4, 5, 6] and some recent papers [7, 8, 9, 10, 11, 12, 13]. The oscillation of fractional differential equations, in particular, has been lately attacked in the new papers [14, 15] in which the authors claimed that their contributions had initiated the subject.

Fractional difference equations, nevertheless, which is the discrete counterpart of the corresponding fractional differential equations have comparably gained less attention among researchers. Indeed, the development of the qualitative features of fractional difference equations are still considered to be at its first stage of progress. There are a few papers which have taken the lead to develop some fundamental concepts concerning the nature of fractional difference equations among them we list the papers [16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

To the best of authors' observation, however, no paper has been published in the literature regarding the oscillation of solutions for fractional difference equations. A primary purpose of this paper is to establish several oscillation criteria for a type of nonlinear fractional difference equations involving the Riemann–Liouville's and Caputo's operators of arbitrary order. Two examples are provided to demonstrate the effectiveness of the main theorems.

2. Preliminary assertions and essential lemmas

Let $\mathbb{N}$ be the set of positive integer numbers, $\mathbb{R}$ the set of real numbers and $\mathbb{R}^+$ the set of nonnegative real numbers. Define the set $\mathbb{N}_{a(q)} = \{a(q)+1, a(q)+2, \dots\}$ where $a(q) = a+m-1$, $m = [q]+1$, $m \in \mathbb{N}$ and $[q]$ is the greatest integer less than or equal to q. Let $\rho(t) = t-1$ and $t^{\overline{q}} = \frac{\Gamma(t+q)}{\Gamma(t)}$, $t \in \mathbb{R} - \{\dots, -2, -1, 0\}$ where $0^{\overline{q}} = 0$. Define the Riemann–Liouville's sum operator $\nabla^{-q}_{a(q)}$ as

$$\nabla^{-q}_{a(q)} x(t) = \frac{1}{\Gamma(q)} \sum_{s=a(q)+1}^{t} \big(t-\rho(s)\big)^{\overline{q-1}} x(s), \quad t \in \mathbb{N}_{a(q)}. \tag{1}$$

By using (1), the Riemann–Liouville's difference operator is defined by

$$\nabla^{q}_{a(q)} x(t) = \nabla^m \nabla^{-(m-q)}_{a(q)} x(t) = \frac{\nabla^m}{\Gamma(m-q)} \sum_{s=a(q)+1}^{t} \big(t-\rho(s)\big)^{\overline{m-q-1}} x(s), \quad t \in \mathbb{N}_{a(q)}. \tag{2}$$

Consider the nonlinear fractional difference equation of the form

$$\begin{cases} \nabla^{q}_{a(q)-1} x(t) + f_1(t, x(t)) = r(t) + f_2(t, x(t)), \ t \in \mathbb{N}_{a(q)}, \\ \nabla^{-(1-q)}_{a(q)-1} x(t)\Big|_{t=a(q)} = x(a(q)) = c, \ c \in \mathbb{R}, \end{cases} \tag{3}$$

where $m-1 < q < m$, $r : \mathbb{N}_{a(q)} \to \mathbb{R}$, $f_i : \mathbb{N}_{a(q)} \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ and $\nabla^{-q}_{a(q)}$ and $\nabla^{q}_{a(q)}$ are defined as in (1) and (2), respectively.

The initial value problem (3) can be expressed by using the Caputo's difference operator. Indeed, equation (3) is replaced by

$$\begin{cases} {}^c\nabla^{q}_{a(q)} x(t) + f_1(t, x(t)) = r(t) + f_2(t, x(t)), \ t \in \mathbb{N}_{a(q)}, \\ \nabla^{k} x(a(q)) = b_k, \ b_k \in \mathbb{R}, \ k = 0, 1, 2, \dots, m-1, \end{cases} \tag{4}$$

where $m-1<q<m$, $r:\mathbb{N}_{a(q)}\to\mathbb{R}$, $f_i:\mathbb{N}_{a(q)}\times\mathbb{R}\to\mathbb{R}$ $(i=1,2)$ and ${}^c\nabla^q_{a(q)}$ is the Caputo's difference operator defined by

$$ {}^c\nabla^q_{a(q)}x(t)=\nabla^{-(m-q)}_{a(q)}\nabla^m x(t)=\frac{1}{\Gamma(m-q)}\sum_{s=a(q)+1}^{t}(t-\rho(s))^{\overline{q-1}}\nabla^m x(s),\quad t\in\mathbb{N}_{a(q)}. \tag{5} $$

Remark 1 It is to be noted that the initial value problem within Riemann–Liouville's operators of form (3) involves one single initial value whereas the initial value problem within Caputo's difference operator of form (4) involves $m-1$ initial values. This surprising conclusion consolidates the fact that fractional derivatives of Caputo's type are much closer than those of Riemann's type to ordinary derivatives of integer order. Further explanation regarding this remark can be found in [24, 25].

By a solution of equation (3) (or (4)), we mean a real–valued sequence $x(t)$ satisfying equation (3) (or (4)) for $t\in\mathbb{N}_{a(q)}$. A solution $x(t)$ of (3) (or (4)) is said to be oscillatory if for every integer $N>0$, there exists $t\geq N$ such that $x(t)x(t+1)\leq 0$; otherwise it is called non oscillatory. An equation is said to be oscillatory if all of its solutions are oscillatory.

Before proceeding to the main results, we set forth some essential lemmas needed in the proofs of the main theorems. We borrow the details and terminologies from the recent papers [24, 25]. The identities are stated without proofs whereas the proof of Lemma 3 is provided for the sake of readers' convenience.

Lemma 1 [24] Let $g:\mathbb{N}_{a(q)}\to\mathbb{R}$. Then, for any real number q and any positive integer m the following equalities hold

I. $\nabla t^{\overline{q}}=qt^{\overline{q-1}}$;
II. $\nabla^{-q}_{a(q)}\nabla^q_{a(q)}g(t)=g(t)$;
III. $\nabla^{-q}_{a(q)}\nabla^m g(t)=\nabla^m\nabla^{-q}_{a(q)}g(t)-\sum_{k=0}^{m-1}\frac{(t-a(q))^{\overline{q-m+k}}}{\Gamma(q+k-m+1)}\nabla^k g(a(q))$.

Lemma 2 [24] Let $q>0$ and $\mu>-1$. Then for $t\in\mathbb{N}_{a(q)}$, we have

$$ \nabla^{-q}_{a(q)}(t-a(q))^{\overline{\mu}}=\frac{\Gamma(\mu+1)}{\Gamma(\mu+q+1)}(t-a(q))^{\overline{q+\mu}}. \tag{6} $$

Lemma 3 If $x(t)$ is a solution of equation (3), then it satisfies the following fractional Volterra sum equation

$$ x(t)=\frac{(t-a(q)+1)^{\overline{q-1}}}{\Gamma(q)}c+\nabla^{-q}_{a(q)}\big[r(t)+f_2(t,x(t))-f_1(t,x(t))\big],\ t\in\mathbb{N}_{a(q)}. \tag{7} $$

Proof. Applying the sum operator $\nabla^{-q}_{a(q)}$ to both sides of equation (3), we obtain

$$ \nabla^{-q}_{a(q)}\Big\{\nabla^q_{a(q)}x(t)+\nabla^m g_q(t)\Big\}=\nabla^{-q}_{a(q)}\big[r(t)+f_2(t,x(t))-f_1(t,x(t))\big], \tag{8} $$

where $g_q(t)=\frac{(t-a(q)+1)^{\overline{m-q-1}}}{\Gamma(m-q)}x(a(q))$. In view of (II) of Lemma 1, it follows that

$$ x(t)+\nabla^{-q}_{a(q)}\nabla^m g_q(t)=\nabla^{-q}_{a(q)}\big[r(t)+f_2(t,x(t))-f_1(t,x(t))\big]. \tag{9} $$

By using relation (III) of Lemma 1, we get

$$\begin{aligned} x(t)+\nabla^m\nabla_{a(q)}^{-q}g_q(t) &- \sum_{k=0}^{m-1}\frac{(t-a(q))^{\overline{q-m+k}}}{\Gamma(q+k-m+1)}\nabla^k g_q(a(q)) \\ &= \nabla_{a(q)}^{-q}\big[r(t)+f_2(t,x(t))-f_1(t,x(t))\big]. \end{aligned} \tag{10}$$

However, we observe that

$$g_q(t)\Big|_{t=a(q)} = \frac{1^{\overline{m-q-1}}}{\Gamma(m-q)}x(a(q)) = \frac{\Gamma(m-q)}{\Gamma(m-q)}x(a(q)) = x(a(q)),$$

$$\begin{aligned} \nabla g_q(t)\Big|_{t=a(q)} &= \frac{(m-q-1)(t-a(q)+1)^{\overline{m-q-2}}}{\Gamma(m-q)}x(a(q))\Big|_{t=a(q)} \\ &= \frac{(t-a(q)+1)^{\overline{m-q-2}}}{\Gamma(m-q-1)}x(a(q))\Big|_{t=a(q)} = x(a(q)), \end{aligned}$$

$$\begin{aligned} \nabla^2 g_q(t)\Big|_{t=a(q)} &= \frac{(m-q-2)(t-a(q)+1)^{\overline{m-q-3}}}{\Gamma(m-q-1)}x(a(q))\Big|_{t=a(q)} \\ &= \frac{(t-a(q)+1)^{\overline{m-q-3}}}{\Gamma(m-q-2)}x(a(q))\Big|_{t=a(q)} = x(a(q)), \end{aligned}$$

and also

$$\nabla^3 g_q(t)\Big|_{t=a(q)} = x(a(q))$$

which inductively implies that

$$\nabla^k g_q(a(q)) = x(a(q)), \quad k=0,1,2,\dots,m-1. \tag{11}$$

The term $\nabla^m\nabla_{a(q)}^{-q}g_q(t)$ in (10) can be expanded by using the power formula (6) as

$$\nabla^m\big\{\nabla_{a(q)}^{-q}g_q(t)\big\} = -\nabla^m\frac{(t-a(q)+1)^{\overline{q-1}}}{\Gamma(q)}x(a(q)). \tag{12}$$

Substituting (11) and (12) back in equation (10) we reach to

$$x(t) = \Phi_q^m(t)x(a(q)) + \nabla_{a(q)}^{-q}\big[r(t)+f_2(t,x(t))-f_1(t,x(t))\big],$$

where

$$\Phi_q^m(t) = \nabla^m\frac{(t-a(q)+1)^{\overline{q-1}}}{\Gamma(q)} + \sum_{k=0}^{m-1}\frac{(t-a(q))^{\overline{q-m+k}}}{\Gamma(q+k-m+1)}.$$

Let $m=1$ and $a(q)=a$. Thus

$$\Phi_q^1(t) = \frac{(t-a+1)^{\overline{q-1}}}{\Gamma(q)}.$$

Let $m=2$ and $a(q)=a+1$. Thus

$$\Phi_q^2(t) = \frac{(t-a)^{\overline{q-3}}}{\Gamma(q-2)} + \frac{(t-a-1)^{\overline{q-2}}}{\Gamma(q-1)} + \frac{(t-a-1)^{\overline{q-1}}}{\Gamma(q)} = \frac{(t-a)^{\overline{q-1}}}{\Gamma(q)}.$$

Proceeding inductively, we end up with

$$x(t) = \frac{(t-a(q)+1)^{\overline{q-1}}}{\Gamma(q)}c + \nabla_{a(q)}^{-q}\big[r(t)+f_2(t,x(t))-f_1(t,x(t))\big],$$

which is the desired result. The proof is complete.

For the case $0 < q < 1$, Lemma 3 is reduced to the following immediate corollary which has been discussed in [26].

Corollary 1 [26] Let $0 < q < 1$. If $x(t)$ is a solution of equation (3), then it satisfies the following Volterra sum equation

$$x(t) = \frac{(t-a+1)^{\overline{q-1}}}{\Gamma(q)} x(a) + \nabla_a^{-q}\big[r(t) + f_2(t, x(t)) - f_1(t, x(t))\big]. \tag{13}$$

By employing the same technique used to obtain formula (7) and in view of Proposition 5.6 in [25], one can derive the corresponding fractional Volterra sum equation for (4).

Lemma 4 If $x(t)$ is a solution of equation (4), then it satisfies the following fractional Volterra sum equation

$$x(t) = \sum_{k=0}^{m-1} \frac{(t-a(q))^{\overline{k}}}{k!} b_k + \frac{1}{\Gamma(q)} \sum_{s=a(q)+1}^{t} (t-\rho(s))^{\overline{q-1}}\big[r(s) + f_2(s, x(s)) - f_1(s, x(s))\big]. \tag{14}$$

The following immediate consequence of Lemma 4 has been investigated in [25].

Corollary 2 [25] Let $0 < q < 1$. If $x(t)$ is a solution of equation (4), then it satisfies the following Volterra sum equation

$$x(t) = b_0 + \frac{1}{\Gamma(q)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{q-1}}\big[r(s) + f_2(s, x(s)) - f_1(s, x(s))\big]. \tag{15}$$

3. The main results

In this section, we establish the main results of this paper. Indeed, several criteria are reported for the oscillation of equations (3) and (4). We should note that the main theorems are obtained without imposing any restriction on the forcing term $r(t)$. The investigations are carried out by employing the following key–tool lemma.

Lemma 5 [27] Let $X \geq 0$ and $Y > 0$. Then, we have

$$X^{\lambda} + (\lambda - 1)Y^{\lambda} - \lambda X Y^{\lambda-1} \geq 0, \ \lambda > 1 \tag{16}$$

and

$$X^{\lambda} - (1 - \lambda)Y^{\lambda} - \lambda X Y^{\lambda-1} \leq 0, \ \lambda < 1, \tag{17}$$

where equality holds if and only if $X = Y$.

3.1. Oscillation of equation (3).

Let

H.1 $x f_i(t, x) > 0, \ (i = 1, 2), \ x \neq 0, \ t \in \mathbb{N}_{a(q)}$.

Define

$$c(T) := M|c| + \sum_{s=a(q)+1}^{T} M_s |F(s)|, \tag{18}$$

where $F(s) = r(s) + f_2(s, x(s)) - f_1(s, x(s))$, $M > 0$ and $0 < M_s$ is a constant depending on s and bounding the term $t^{1-q}(t-s+1)^{\overline{q-1}}$ for all t.

The first theorem is concerned with the oscillation of equation (3) when $f_2 = 0$.

Theorem 1 Let $f_2 = 0$ and condition (H.1) holds. If

$$\liminf_{t\to\infty} t^{1-q} \sum_{s=a(q)}^{t} (t-s+1)^{\overline{q-1}} r(s) = -\infty, \tag{19}$$

and

$$\limsup_{t\to\infty} t^{1-q}\sum_{s=a(q)}^{t}(t-s+1)^{\overline{q-1}}r(s)=\infty, \tag{20}$$

then equation (3) is oscillatory.

Proof. Let $x(t)$ be a non oscillatory solution of equation (3) with $f_2=0$. Suppose that $T\in\mathbb{N}_{a(q)}$ is large enough so that $x(t)>0$ for $t\in\mathbb{N}_T$. In view of formula (7), we obtain

$$\begin{aligned} x(t) &\leq \frac{(t-a(q)+1)^{\overline{q-1}}}{\Gamma(q)}|c| \\ &+ \frac{1}{\Gamma(q)}\sum_{s=a(q)+1}^{T}(t-s+1)^{\overline{q-1}}|F(s)|+\frac{1}{\Gamma(q)}\sum_{s=T+1}^{t}(t-s+1)^{\overline{q-1}}|r(s)|, \end{aligned}$$

or

$$\begin{aligned} \Gamma(q)t^{1-q}x(t) &\leq t^{1-q}(t-a(q)+1)^{\overline{q-1}}|c|+t^{1-q}\sum_{s=a(q)+1}^{T}(t-s+1)^{\overline{q-1}}|F(s)| \\ &+ t^{1-q}\sum_{s=T+1}^{t}(t-s+1)^{\overline{q-1}}|r(s)|,\ \ t\in\mathbb{N}_T. \end{aligned} \tag{21}$$

By using the Stirling's formula $\lim_{n\to\infty}\frac{\Gamma(n)n^{\varepsilon}}{\Gamma(n+\varepsilon)}=1,\ \varepsilon>0$, we observe that

$$\begin{aligned} \lim_{t\to\infty} t^{1-q}(t-a(q)+1)^{\overline{q-1}} &= \lim_{t\to\infty}\frac{t^{1-q}}{(t-a(q))}\frac{\Gamma(t-a(q)+q)}{\Gamma(t-a(q))}\frac{(t-a(q))^q}{(t-a(q))^q} \\ &= \lim_{t\to\infty}\Big(\frac{t}{t-a(q)}\Big)^{1-q}=1. \end{aligned} \tag{22}$$

Hence, there exists $M>0$ such that

$$\Big|t^{1-q}(t-a(q)+1)^{\overline{q-1}}\Big|\leq M,\ t\in\mathbb{N}_T. \tag{23}$$

It follows from (21) that

$$\Gamma(q)t^{1-q}x(t)\leq c(T)+t^{1-q}\sum_{s=T+1}^{t}(t-s+1)^{\overline{q-1}}r(s),\ \ t\in\mathbb{N}_T, \tag{24}$$

where $c(T)$ is defined as in (18). Taking the limit inferior of both sides of (24) as $t\to\infty$, we get a contradiction to condition (19). In case $x(t)$ is eventually negative, one can proceed in the same way and reach to a contradiction with (20). The proof is finished.

Let

H.2 $|f_1(t,x(t))|\geq p_1(t)|x|^{\beta}$ and $|f_2(t,x(t))|\leq p_2(t)|x|^{\gamma}$, $x\neq 0,\ t\geq a(q)$,

where $p_1,p_2:\mathbb{N}_{a(q)}\to\mathbb{R}^+$ and $\beta,\gamma>0$ are real numbers.

Theorem 2 Let conditions (H.1)–(H.2) hold with $\beta>1$ and $\gamma=1$. If

$$\liminf_{t\to\infty} t^{1-q}\sum_{s=a(q)}^{t}(t-s+1)^{\overline{q-1}}[r(s)+H_{\beta}(s)]=-\infty, \tag{25}$$

and

$$\limsup_{t\to\infty} t^{1-q} \sum_{s=a(q)}^{t} (t-s+1)^{\overline{q-1}}[r(s)+H_\beta(s)] = \infty, \tag{26}$$

where

$$H_\beta(s) = (\beta-1)\beta^{\beta/1-\beta} p_1^{1/(1-\beta)}(s) p_2^{\beta/\beta-1}(s),$$

then equation (3) is oscillatory.

Proof. Let $x(t)$ be a non oscillatory solution of equation (3). Suppose that $x(t) > 0$ for $t \geq T > a(q)$. Using condition (H.2) in formula (7) with $\gamma = 1$ and $\beta > 1$, we obtain

$$\begin{aligned} \Gamma(q)t^{1-q}x(t) \quad &\leq \quad c(T) + t^{1-q}\Big[\sum_{s=T+1}^{t}(t-s+1)^{\overline{q-1}}r(s) \\ &+ \sum_{s=T+1}^{t}(t-s+1)^{\overline{q-1}}\big[p_2(s)x(s) - p_1(s)x^\beta(s)\big]\Big]. \end{aligned} \tag{27}$$

In virtue of relation (16) of Lemma 5, we observe that

$$p_2(t)x(t) - p_1(t)x^\beta(t) \leq (\beta-1)\beta^{\beta/1-\beta}p_1^{1/(1-\beta)}(t)p_2^{\beta/\beta-1}(t), \tag{28}$$

where we assign $\lambda = \beta,\ X = p_1^{1/\beta}$ and $Y = \big(\frac{p_2 p_1^{-1/\beta}}{\beta}\big)^{1/(\beta-1)}$.

It follows that

$$\Gamma(q)t^{1-q}x(t) \quad \leq \quad c(T) + t^{1-q}\sum_{s=T+1}^{t}(t-s+1)^{\overline{q-1}}[r(s)+H_\beta(s)],\ t \geq T.$$

The remaining part of the proof is the same as in the proof of Theorem 1. Hence, it is omitted.

Theorem 3 Let conditions (H.1)–(H.2) hold with $\beta = 1$ and $\gamma < 1$. If

$$\liminf_{t\to\infty} t^{1-q} \sum_{s=a(q)}^{t} (t-s+1)^{\overline{q-1}}[r(s)+H_\gamma(s)] = -\infty, \tag{29}$$

and

$$\limsup_{t\to\infty} t^{1-q} \sum_{s=a(q)}^{t} (t-s+1)^{\overline{q-1}}[r(s)+H_\gamma(s)] = \infty, \tag{30}$$

where

$$H_\gamma(s) = (1-\gamma)\gamma^{\gamma/1-\gamma}p_1^{\gamma/(\gamma-1)}(s)p_2^{1/(1-\gamma)}(s),$$

then equation (3) is oscillatory.

Proof. Let $x(t)$ be a non oscillatory solution of equation (3). Suppose that $x(t) > 0$ for $t \geq T > a(q)$. Using condition (H.2) in formula (7) with $\beta = 1$ and $\gamma < 1$, we find

$$\begin{aligned} \Gamma(q)t^{1-q}x(t) \quad &\leq \quad c(T) + t^{1-q}\Big[\sum_{s=a(q)}^{t}(t-s+1)^{\overline{q-1}}r(s) \\ &+ \sum_{s=a(q)}^{t}(t-s+1)^{\overline{q-1}}\big[p_2(s)x^\gamma(s) - p_1(s)x(s)\big]\Big]. \end{aligned} \tag{31}$$

In virtue relation (17), we observe that

$$p_2(t)x^\gamma(t) - p_1(t)x(t) \leq (1-\gamma)\gamma^{\gamma/1-\gamma} p_1^{\gamma/(\gamma-1)}(t) p_2^{1/(1-\gamma)}(t), \tag{32}$$

where we assign $\lambda = \gamma,\ X = p_2^{1/\gamma}$ and $Y = \left(\frac{p_1 p_2^{-1/\gamma}}{\gamma}\right)^{1/(\gamma-1)}$.

It follows that

$$\Gamma(q)t^{1-q}x(t) \quad \leq \quad c(T) + t^{1-q} \sum_{s=T+1}^{t} (t-s+1)^{\overline{q-1}}[r(s) + H_\gamma(s)],\ t \geq T.$$

The remaining part of the proof is the same as in the proof of Theorem 1. Hence, it is omitted.

Theorem 4 Let conditions (H.1)–(H.2) hold with $\beta > 1$ and $\gamma < 1$. If

$$\liminf_{t\to\infty} t^{1-q} \sum_{s=a(q)}^{t} (t-s+1)^{\overline{q-1}}[r(s) + H_{\beta,\gamma}(s)] = -\infty, \tag{33}$$

and

$$\limsup_{t\to\infty} t^{1-q} \sum_{s=a(q)}^{t} (t-s+1)^{\overline{q-1}}[r(s) + H_{\beta,\gamma}(s)] = \infty, \tag{34}$$

where

$$H_{\beta,\alpha}(s) = (\beta-1)\beta^{\beta/1-\beta} p_1^{1/(1-\beta)}(s)\xi^{\beta/\beta-1}(s) + (1-\gamma)\gamma^{\gamma/1-\gamma}\xi^{\gamma/(\gamma-1)}(s) p_2^{1/(1-\gamma)}(s)$$

with $\xi : \mathbb{N}_{a(q)} \to \mathbb{R}^+$, then equation (3) is oscillatory.

Proof. Let $x(t)$ be a non oscillatory solution of equation (3). Suppose that $x(t) > 0$ for $t \geq T > a(q)$. Using condition (H.2) in formula (7), we find

$$\begin{aligned} \Gamma(q)t^{1-q}x(t) \quad &\leq \quad c(T) + t^{1-q} \sum_{s=T+1}^{t} (t-s+1)^{\overline{q-1}} r(s) \\ &+ \quad t^{1-q} \sum_{s=T+1}^{t} (t-s+1)^{\overline{q-1}} \big[\xi(s)x(s) - p_1(s)x^\beta(s)\big] \\ &+ \quad t^{1-q} \sum_{s=T+1}^{t} (t-s+1)^{\overline{q-1}} \big[p_2(s)x^\gamma(s) - \xi(s)x(s)\big],\ t \geq T. \end{aligned} \tag{35}$$

In virtue of the inequalities (28) and (32), we get

$$\xi(t)x(t) - p_1(t)x^\beta(t) \leq (\beta-1)\beta^{\beta/1-\beta} p_1^{1/(1-\beta)}(t) p_2^{\beta/\beta-1}(t),$$

and

$$p_2(t)x^\gamma(t) - \xi(t)x(t) \leq (1-\gamma)\gamma^{\gamma/1-\gamma} p_1^{\gamma/(\gamma-1)}(t) p_2^{1/(1-\gamma)}(t).$$

It follows that

$$\Gamma(q)t^{1-q}x(t) \quad \leq \quad c(T) + t^{1-q} \sum_{s=T+1}^{t} (t-s+1)^{\overline{q-1}}[r(s) + H_{\beta,\gamma}(s)],\ t \geq T.$$

The remaining part of the proof is the same as in the proof of Theorem 1. Hence, it is omitted.

3.2. **Oscillation of equation (4).** This subsection is devoted to oscillation criteria for equation (4). The proofs of the main theorems are similar to those of Subsection 3.1 and hence they are omitted.

Theorem 5 Let $f_2 = 0$ and condition (H.1) hold. If

$$\liminf_{t\to\infty} t^{1-m} \sum_{s=a(q)}^{t} (t-s+1)^{\overline{q-1}} r(s) = -\infty, \tag{36}$$

and

$$\limsup_{t\to\infty} t^{1-m} \sum_{s=a(q)}^{t} (t-s+1)^{\overline{q-1}} r(s) = \infty, \tag{37}$$

then equation (4) is oscillatory.

Theorem 6 Let conditions (H.1)–(H.2) hold with $\beta > 1$ and $\gamma = 1$. If

$$\liminf_{t\to\infty} t^{1-m} \sum_{s=a(q)}^{t} (t-s+1)^{\overline{q-1}} [r(s) + H_\beta(s)] = -\infty, \tag{38}$$

and

$$\limsup_{t\to\infty} t^{1-m} \sum_{s=a(q)}^{t} (t-s+1)^{\overline{q-1}} [r(s) + H_\beta(s)] = \infty, \tag{39}$$

where

$$H_\beta(s) = (\beta-1)\beta^{\beta/1-\beta} p_1^{1/(1-\beta)}(s) p_2^{\beta/\beta-1}(s)$$

then equation (4) is oscillatory.

Theorem 7 Let conditions (H.1)–(H.2) hold with $\beta = 1$ and $\gamma < 1$. If

$$\liminf_{t\to\infty} t^{1-m} \sum_{s=a(q)}^{t} (t-s+1)^{\overline{q-1}} [r(s) + H_\gamma(s)] = -\infty, \tag{40}$$

and

$$\limsup_{t\to\infty} t^{1-m} \sum_{s=a(q)}^{t} (t-s+1)^{\overline{q-1}} [r(s) + H_\gamma(s)] = \infty, \tag{41}$$

where

$$H_\gamma(s) = (1-\gamma)\gamma^{\gamma/1-\gamma} p_1^{\gamma/(\gamma-1)}(s) p_2^{1/(1-\gamma)}(s)$$

then equation (4) is oscillatory.

Theorem 8 Let conditions (H.1)–(H.2) hold with $\beta > 1$ and $\gamma < 1$. If

$$\liminf_{t\to\infty} t^{1-m} \sum_{s=a(q)}^{t} (t-s+1)^{\overline{q-1}} [r(s) + H_{\beta,\gamma}(s)] = -\infty, \tag{42}$$

and

$$\limsup_{t\to\infty} t^{1-m} \sum_{s=a(q)}^{t} (t-s+1)^{\overline{q-1}} [r(s) + H_{\beta,\gamma}(s)] = \infty, \tag{43}$$

where

$$H_{\beta,\gamma}(s) = (\beta-1)\beta^{\beta/1-\beta} p_1^{1/(1-\beta)}(s)\xi^{\beta/\beta-1}(s) + (1-\gamma)\gamma^{\gamma/1-\gamma}\xi^{\gamma/(\gamma-1)}(s) p_2^{1/(1-\gamma)}(s)$$

with $\xi : \mathbb{N}_a \to \mathbb{R}^+$, then equation (4) is oscillatory.

4. Examples

In what follows, two examples are constructed in the text of equations (3) and (4). We conclude that the assumed conditions of Theorem 2 and Theorem 6 can not be dropped.

Example 1 Consider the Riemann–Liouville's fractional difference equation

$$\nabla_0^q x(t) + x^3 e^t = \frac{t^{1-q}}{\Gamma(2-q)} + te^t(t^2-1) + xe^t, \ \ \nabla_0^{-(1-q)} x(t)\big|_{t=1} = x(1) = 1, \quad (44)$$

where $m = 1,\ 0 < q < 1,\ a = 1,\ c = 1,\ f_1(t,x) = x^3 e^t,\ f_2(t,x) = xe^t$ and $r(t) = \frac{t^{1-q}}{\Gamma(2-q)} + te^t(t^2-1)$. It is clear that conditions (H.1)–(H.2) are satisfied for $\beta = 3,\ \gamma = 1$ and $p_1(t) = p_2(t) = e^t$. However, since $r(t) \geq 0$ we may write

$$t^{1-q}\sum_{s=1}^{t}(t-s+1)^{\overline{q-1}}\big[\frac{s^{1-q}}{\Gamma(2-q)} + se^s(s^2-1) + 0.4e^s\big] \geq t^{1-q}\sum_{s=1}^{t}(t-s+1)^{\overline{q-1}}0.4e^s \quad (45)$$

and

$$t^{1-q}\sum_{s=1}^{t}(t-s+1)^{\overline{q-1}}0.4e^s \geq t^{1-q}\sum_{s=1}^{t}(t-s+1)^{\overline{q-1}} = \frac{t^{1-q}\Gamma(t+q)}{q\Gamma(t)} \geq 0. \quad (46)$$

This tells that condition (25) of Theorem 2 does not hold. Indeed, one can easily verify that $x(t) = t$ is a non oscillatory solution of equation (44).

Example 2 Consider the Caputo's fractional difference equation

$${}^c\nabla_1^q x(t) + x^3 e^t = \frac{(t-1)^{\overline{1-q}}}{\Gamma(2-q)} + te^t(t^2-1) + xe^t, \ \ x(1) = 1, \quad (47)$$

where $m = 1,\ 0 < q < 1,\ a = 1,\ b_0 = 1,\ f_1(t,x) = x^3 e^t,\ f_2(t,x) = xe^t$ and $r(t) = \frac{(t-1)^{\overline{1-q}}}{\Gamma(2-q)} + te^t(t^2-1)$. It is clear that conditions (H.1)–(H.2) are satisfied for $\beta = 3,\ \gamma = 1$ and $p_1(t) = p_2(t) = e^t$. However, since $r(t) \geq 0$ we may write

$$\sum_{s=1}^{t}(t-s+1)^{\overline{q-1}}\big[\frac{s^{1-q}}{\Gamma(2-q)} + se^s(s^2-1) + 0.4e^s\big] \geq \sum_{s=1}^{t}(t-s+1)^{\overline{q-1}}0.4e^s \quad (48)$$

and

$$\sum_{s=1}^{t}(t-s+1)^{\overline{q-1}}0.4e^s \geq \sum_{s=1}^{t}(t-s+1)^{\overline{q-1}} = \frac{\Gamma(t+q)}{q\Gamma(t)} \geq 0. \quad (49)$$

Thus, condition (38) of Theorem 6 does not hold. Indeed, one can easily verify that $x(t) = t$ is a non oscillatory solution of equation (47).

References

[1] K. B. Oldham, J. Spanier: The Fractional Calculus, Academic Press, San Diego, 1974.
[2] k. S. Miller, B. Ross: An Introduction to The Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, New York, 1993.
[3] S. G. Samko, A. A. Kilbas, O. I. Marichev: Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, 1993.
[4] I. Podlubny: Fractional Differential Equations. Academic Press, San Diego (1999)
[5] A. A. Kilbas, H. M. Srivastava, J. J. Trujilli: Theory and Applications of Fractional Differential Equations, North–Holland Math. Studies 204, Elsevier, Amsterdam, 2006.
[6] K. Diethelm: The Analysis of Fractional Differential Equations, Springer, Berlin, 2010.

[7] Y. Zhou, F. Jiao, J. Li: Existence and uniqueness for p–type fractional neutral differential equations, Nonlinear Anal. 71 (2009), 2724-2733.
[8] L. Galeone, R. Garrappa: Explicit methods for fractional differential equations and their stability properties, J. Comput. Appl. Math. 228 (2009), 548-560.
[9] A. Saadatmandi, M. Dehghan: A new operational matrix for solving fractional–order differential equations, Comput. Math. Appl. 59 (2010), 1326-1336.
[10] W. Deng: Smoothness and stability of the solutions for nonlinear fractional differential equations, Nonlinear Anal. 72 (2010), 1768-1777.
[11] J. C. Trigeassou, N. Maamri, J. Sabatier, A. Oustaloup: A Lyapunov approach to the stability of fractional differential equations, Signal Process 91 (2011), 437-445.
[12] M. Belmekki, J. J. Nieto, R. R-Lopez: Existence of periodic solution for a nonlinear fractional differential equation Bound. Value Prob. Art. no. 324561, 19 July 2009.
[13] M. M. El–Borai, A. Debbouche: Almost periodic solutions of some nonlinear fractional differential equations, Int. J. Contemp. Math. Sciences 4 (28) (2009), 1373–1387.
[14] D-Xue Chen: Oscillation criteria of fractional differential equations, Adv. Differ. Equ. 2012, 2012:33.
[15] S. R. Grace, R. P. Agarwal, P. J. Y. Wong, A. Zafer: On the oscillation of fractional differential equaions, Fract. Calc. Appl. Anal. 15 (2) (2012), 222–231.
[16] F. M. Atici, P. W. Eloe: Initial value problems in discrete fractional calculus, Proc. Amer. Math. Soc. 137 (3) (2008), 981–989.
[17] F. M. Atici, P. W. Eloe: Linear systems of fractional nabla difference equations, Rocky Mountain J. Math. 41 (2011), 353–370.
[18] C. S. Goodrich: On a discrete fractional three–point boundary value problem, J. Differ. Equ. Appl. 18 (3) (2012), 397–415.
[19] C. S. Goodrich: Existence of a positive solution to a system of discrete fractional boundary value problems, App. Math. Comput. 217 (9) (2011), 4740–4753.
[20] T. Abdeljawad: On Riemann and Caputo fractional differences, Comput. Math. Appl. 62 (3) (2011), 1602–1611.
[21] J-Fa Cheng, Y-Ming Chu: Fractional difference equations with real variable, Abstr. Appl. Anal. Volume 2012 (2012), Article ID 918529.
[22] F. Chen, Z. Liu, Asymptotic stability results for nonlinear fractional difference equations, J. Appl. Math. Volume 2012 (2012), Article ID 879657.
[23] J-Fa Cheng, Y-Ming Chu: On the fractional difference equations of order $(2, q)$, Abstr. Appl. Anal. Volume 2011 (2011), Article ID 497259.
[24] T. Abdeljawad: Dual identities in fractional difference calculus within Riemann, arXiv:1112.5795v2.
[25] T. Abdeljawad: On delta and nabla Caputo fractional differences and dual identities, arXiv:1102.1625v2.
[26] T. Abdeljawad, F. M. Atici: On the definitions of nabla fractional operators, Abstr. Appl. Anal. Volume 2012 (2012), Article ID 406757.
[27] G. H. Hardy, J. E. Littlewood, G. Polya: Inequalities, Cambridge University Press, Cambridge, 1988.

J. O. Alzabut
Department of Mathematics and Physical Sciences, Prince Sultan University, P. O. Box 66388, Riyadh 11586, Saudi Arabia
E-mail address: jalzabut@psu.edu.sa

T. Abdeljawad
Department of Mathematics and Computer Science, Çankaya University, 06530 Ankara, Turkey
E-mail address: thabet@cankaya.edu.tr