Journal of Fractional Calculus and Applications,
Vol. 5(1) Jan. 2014, pp. 156 -164.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

EXISTENCE OF SOLUTION FOR A THREE POINT BOUNDARY VALUE PROBLEM OF FRACTIONAL DIFFERENTIAL EQUATION

R. A. KHAN, H. KHAN

Abstract. In this paper, we study existence and uniqueness of solutions to a class of three-point boundary value problems for nonlinear fractional order differential equation of the form

$$\begin{cases} D^{\alpha}u(t) + f(t, u(t), u'(t)) \ = \ 0, \qquad 1 < \alpha \leq 2, \\ u(0) \ = \ 0, \quad D^{p}u(1) \ = \ \delta D^{p}u(\eta), \quad 0 < p < 1 \end{cases}$$

where $0 < \mu < p < 1$, $0 < \eta \leq 1$ and D^{α} is Caputo's fractional derivative of order α. Our results are based on some classical results from fixed point theory. We impose some growth and continuity conditions on the nonlinear f. For the applications of our results we present an example.

1. Introduction

Due to the development and applications of fractional calculus in many fields such as engineering, mathematics, physics, chemistry, etc (see [1]-[6]), have attracted the attentions of many researchers in a variety of directions. The efforts of mathematicians to develop theory for applied scientists played significant role in this area of mathematics too. Different aspects of fractional differential equations are studied and being developed but one of the most important area of research in the field of fractional order differential equations is the theory of existence and uniqueness of solutions of nonlinear fractional order differential equations. This ares of research gained much interest in the community of mathematicians. In particular, for the study of boundary value problems for fractional order differential equations, we refer the readers to ([7]-[23]) and the references therein.

Here we refer some boundary value problems which motivated us for the present work. By the help of fixed-point theorem, in [25] Bai and Lu investigated the existence and multiplicity of positive solutions for fractional differential equation

$$\begin{cases} D^{q}_{0+}u(t) + f(t, u(t))) \ = \ 0, \quad 0 < t < 1 \\ u(0) \ = \ u(1) \ = \ 0, \end{cases} \tag{1}$$

2000 *Mathematics Subject Classification.* 26A33, 34A08, 34B18, 46N10.

Key words and phrases. Fractional-order differential equations, existence, uniqueness, Green's function.

Submitted march 23, 2013 Revised July 4, 2013.

where $1 < q \leq 2$ and D^q_{0+} is the Riemann-Liouville fractional derivative. Jiang et.al [24] studied existence of positive solutions for two-point boundary value problem of the form

$$\begin{cases} D^{\alpha}_{0+}u(t) + \mu a(t)f(t,u(t)) \ = \ 0 \\ u(0) \ = \ u(1) \ = \ 0, \quad 1 < \alpha < 2, \end{cases} \tag{2}$$

where D^{α}_{0+} is standard Riemann-Liouville fractional derivative. M. El-Shahed in [7] studied sufficient conditions for existence and as well as nonexistence of positive solutions to the two-point BVP

$$\begin{cases} D^{\alpha}u(t) + \lambda a(t)f(u(t)) \ = \ 0, \quad 2 < \alpha < 3 \\ u(0) \ = \ u''(0) \ = \ 0, \quad \gamma u'(1) + \beta u''(1) \ = \ 0, \end{cases} \tag{3}$$

where $0 < t < 1$, where λ is a positive parameter and D^{α} is Caputo's fractional derivative.

In this paper we study existence and uniqueness of solution for nonlinear three-point boundary value problem corresponding to fractional order differential equation of the form

$$\begin{cases} D^{\alpha}u(t) + f(t,u(t),u'(t)) \ = \ 0 \qquad 1 < \alpha \leq 2, \\ u(0) \ = \ 0, \quad D^p u(1) \ = \ \delta D^p u(\eta), \quad 0 < p < 1 \end{cases} \tag{4}$$

where D^{α} is Caputo's fractional derivative of order α, f is continuous and may be nonlinear and the parameters satisfy $0 < \delta < p < 1$, $0 < \eta \leq 1$.

We recall some basic definitions and results. For $\alpha > 0$, choose $n = [\alpha] + 1$ in case α in not an integer and $n = \alpha$ in case α is an integer.

Definition 1 The fractional order integral of order $\alpha > 0$ of a function $f : (0,\infty) \to R$ is given by

$$I^{\alpha}f(t) \ = \ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds, \tag{5}$$

provided the integral converges.

Definition 2 For a function $f \in C^n[0,1]$, the Caputo fractional derivative of order α is define by

$$(D^{\alpha})f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds,$$

provided that the right side is pointwise defined on $(0,\infty)$.

The following Lemmas give some properties of fractional integrals

Lemma 1[2] Fort $\beta \geq \alpha > 0$ and $f \in L_1[a,b]$, the following

$$D^{\alpha}I^{\beta}_{a+}f(t) \ = \ I^{\beta-\alpha}_{a+}f(t)$$

holds almost everywhere on $[a,b]$ and it is valid at any point $t \in [a,b]$ if $f \in C[a,b]$.

Lemma 2 [2] Let $\alpha > 0$ then

$$I^{\alpha}\,D^{\alpha}_0 u(t) \ = \ u(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}, \text{ for } c_i \in R. \tag{6}$$

Lemma 3 [2] For $g(t) \in C(0,1)$, the homogenous fractional order differential equation $D^{\alpha}_{0+}g(t) \ = \ 0$ has a solution

$$g(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1},\ c_i \in R, i = 1,2,3,\cdots,n. \tag{7}$$

We use the notations $\Delta = \frac{1-\delta\eta^{1-p}}{\Gamma(2-p)}$, $G_1(t,s) = \frac{-1}{\Gamma(\alpha)}(t-s)^{\alpha-1}$, $G_2(t,s) = \frac{-\delta t}{\Delta\Gamma(\alpha-p)}(\eta-s)^{\alpha-p-1}$ and $G_3(t,s) = \frac{t}{\Delta\Gamma(\alpha-p)}(1-s)^{(\alpha-p-1)}$.

Lemma 4 For $y \in C[0,1]$, $0 < \delta < p < 1$ and $0 < \eta \leq 1$, the boundary value problem for fractional differential equation

$$\begin{cases} D^\alpha u(t) + y(t) = 0 & 1 < \alpha \leq 2, \\ u(0) = 0,\ D^p u(1) = \delta D^p u(\eta), & 0 < p < 1 \end{cases} \tag{8}$$

has a solution of the form

$$u(t) = \int_0^1 G(t,s)\, y(s)\, ds, \tag{9}$$

where

$$G(t,s) = \begin{cases} G_1(t,s) + G_2(t,s) + G_3(t,s) & 0 \leq s \leq t \leq \eta \leq 1\ , \\ G_2(t,s) + G_3(t,s) & 0 \leq t \leq s \leq \eta \leq 1\ , \\ G_3(t,s) & 0 \leq t \leq \eta \leq s \leq 1, \\ G_1(t,s) + G_2(t,s) + G_3(t,s) & 0 \leq s \leq \eta \leq t \leq 1\ , \\ G_1(t,s) + G_3(t,s) & 0 \leq \eta \leq s \leq t \leq 1\ , \\ G_3(t,s) & 0 \leq \eta \leq t \leq s \leq 1\ . \end{cases}$$

Proof. Applying the operator I^α on (8) and using lemma 1, we obtain

$$u(t) = -I_0^\alpha y(t) + c_1 + c_2 t. \tag{10}$$

The boundary condition $u(0) = 0$ implies $c_1 = 0$, thus we have

$$u(t) = -I_0^\alpha\ y(t) + c_2 t \text{ which implies } D_{0^+}^p\ u(t) = -I^{\alpha-p} y(t) + c_2 \frac{t^{1-p}}{\Gamma(2-p)}.$$

The boundary condition $D^p u(1) = \delta D^p\ u(\eta)$ yields $c_2 = \frac{1}{\Delta}(-\delta I^{\alpha-p}\ y(\eta) + I^{\alpha-p}\ y(1))$ where $\Delta = \frac{1-\delta\eta^{1-p}}{\Gamma(2-p)}$. Hence, (10) takes the form

$$u(t) = -I^\alpha\ y(t) + \frac{t}{\Delta}(-\delta I^{\alpha-p}\ y(\eta) + I^{\alpha-p}\ y(1)) \tag{11}$$

which can be rewritten as

$$\begin{aligned} u(t) &= \frac{-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ y(s)\ ds + \frac{t}{\Delta\Gamma(\alpha-p)}(-\delta \int_0^\eta (\eta-s)^{\alpha-p-1}\ y(s)\ ds \\ &\quad + \int_0^1 (1-s)^{\alpha-p-1}\ y(s)\ ds) = \int_0^1 G(t,s)\ y(s)\ ds. \end{aligned}$$

2. MAIN RESULTS

We consider the space $E = \{u(t) \in C[0,1] : u'(t) \in C[0,1]\}$ with the norm defined by $\|u\|_1 = \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} |u'(t)|$. E is a Banach space [20]. In view of Lemma (1), the integral form of the BVP (4) is given by

$$\begin{aligned} u(t) &= -I^\alpha f(t,u(t),u'(t)) + \frac{t}{\Delta}\big(-\delta I^{\alpha-p} f(\eta,u(\eta),u'(\eta)) + I^{\alpha-p} f(1,u(1),u'(1))\big) \\ &= \int_0^1 G(t,s) f(s,u(s),u'(s))\ ds,\ t \in [0,1]. \end{aligned} \tag{12}$$

By a solution of the the BVP (8), we mean solution of the integral equation (12). Define an operator $T : E \to E$ by

$$Tu(t) = \int_0^1 G(t,s)\ f(s,u(s),u'(s))\ ds,\ t \in [0,1]. \tag{13}$$

By continuity of $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and G on $[0,1] \times [0,1]$, the operator T is continuous and solutions of the integral equation (12) are fixed points of the operator T. In view of (12), we have

$$\begin{aligned}(Tu)'(t) &= -I^{\alpha-1}f(t,u(t),u'(t)) + \frac{1}{\Delta}\big(-\delta\ I^{\alpha-p}f(\eta,u(\eta),u'(\eta))\\ &\quad + I^{\alpha-p}\ f(1,u(1),u'(1))\big).\end{aligned} \tag{14}$$

Assume that the following growth conditions hold:

(B1) There exists a nonnegative function $m(t) \in L_1([0,1])$ such that

$$|f(t,u(t),u'(t))| \leq m(t) + a_1\ |u(t)|^{\lambda_1} + a_2\ |u'(t)|^{\lambda_2},$$

where $a_1, a_2 \in \mathbb{R}$ are nonnegative constants and $0 < \lambda_1, \lambda_1 < 1$.

(B2) There exists a nonnegative function $m(t) \in L_1([0,1])$ such that

$$|f(t,u(t),u'(t)))| \leq m(t) + a_1\ |u(t)|^{\lambda_1} + a_2\ |u'(t)|^{\lambda_2},$$

where $a_1, a_2 \in \mathbb{R}$ are nonnegative constants and $\lambda_1,\ \lambda_1 > 1$

(B3) There exist a constant $k > 0$ such that

$$|f(t,u(t),u'(t))) - f(t,v(t),v'(t)))| \leq k\ (|u(t) - v(t)| + |u'(t) - v'(t)|)$$

for each $t \in [0,1]$ and all $u,\ v,\ u',\ v'$ real valued functions of t.

For convenience, use the following notations

$$\begin{aligned}h_1(t) &= \int_0^1 G(t,s)\ m(s)\ ds + \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\ m(s)\ ds +\\ &\frac{1}{\Delta}\Big(\frac{\delta}{\Gamma(\alpha-p)}\int_0^\eta (\eta-s)^{\alpha-p-1}\ m(s)\ ds + \frac{1}{\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1}\ m(s)\ ds\Big)\end{aligned} \tag{15}$$

$$h_2(t) = \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{(t+1)(\delta\eta^{\alpha-p}+1)}{\Delta\Gamma(\alpha-p+1)} + \frac{1}{\Gamma(\alpha)}t^{\alpha-1} \tag{16}$$

$$\omega_1(t) = \frac{1}{\Gamma\alpha+1}t^\alpha + \frac{t}{\Delta\Gamma(\alpha-p+1)}(\delta\eta^{\alpha-p-1}+1),\ \omega_2(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\delta\eta^{\alpha-p}+1}{\Delta\Gamma(\alpha-p+1)}$$

and $\varpi = \omega_1 + \omega_2$. Choose $R \geq max\{3h_1,\ (3k_1h_2)^{\frac{1}{1-\lambda_1}},\ (3k_2h_2)^{\frac{1}{1-\lambda_2}}\}$, where $k_1 = max_{t\in[0,1]}\ a_1(t)$, $k_2 = max_{t\in[0,1]}\ a_2(t)$ and consider a closed bounded subset $U = \{u(t) \in E : \|u\|_1 \leq R, t \in [0,1]\}$ of E.

Theorem Under the assumption $(B1)$ the BVP (4) has at least one solution.

Proof. The proof is based on Schauder's fixed point theorem. For $u \in U$, using

$B1$, we have

$$\begin{aligned}
|Tu(t)| \;\le\; & |\int_0^1 |G(t,s)|\ m(s)\ ds + (a_1(t)|u|^{\lambda_1} + a_2(t)|u'|^{\lambda_2}) \int_0^1 |G(t,s)|\ ds \\
& \le \int_0^1 |G(t,s)|\ m(s)\ ds + (a_1(t)|u|^{\lambda_1} + a_2(t)|u'|^{\lambda_2}) (\int_0^t (|G_1(t,s)) + G_2(t,s) \\
& + G_3(t,s)|) ds + \int_t^\eta |G_2(t,s) + G_3(t,s)| ds + \int_\eta^1 |G_3(t,s)|\ ds) \\
& = \int_0^1 G(t,s)\ m(s)\ ds + (a_1(t)|u|^{\lambda_1} + a_2(t)|u'|^{\lambda_2}) (\frac{t^\alpha}{\Gamma(\alpha+1)} \\
& + \frac{\delta t \eta^{\alpha-p}}{\Delta\Gamma(\alpha-p+1)} + \frac{t}{\Delta\Gamma(\alpha-p+1)}) \\
& \le \int_0^1 G(t,s)\ m(s)\ ds + (k_1|u|^{\lambda_1} + k_2\ |u'|^{\lambda_2}) (\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{\delta t \eta^{\alpha-p}}{\Delta\Gamma(\alpha-p+1)} \\
& + \frac{t}{\Delta\Gamma(\alpha-p+1)})
\end{aligned} \tag{17}$$

and in view of (14), we have

$$\begin{aligned}
|T'u(t)| \;\le\; & \frac{1}{\Gamma(\alpha-1)} (\int_0^t (t-s)^{\alpha-2}\ m(s)\ ds + (k_1|u|^{\lambda_1} + k_2|u'|^{\lambda_2}) \\
& \int_0^t (t-s)^{\alpha-2}\ ds) + \frac{1}{\Delta} (\frac{\delta}{\Gamma(\alpha-p)} (\int_0^\eta (\eta-s)^{\alpha-p-1}\ m(s)\ ds \\
& + (k_1\ |u|^{\lambda_1} + k_2\ |u'|^{\lambda_2}) \int_0^\eta (\eta-s)^{\alpha-p-1}\ ds) + \frac{1}{\Gamma(\alpha-p)} \\
& (\int_0^1 (1-s)^{\alpha-p-1}\ m(s)\ ds + (k_1|u|^{\lambda_1} + k_2|u'|^{\lambda_2}) \int_0^1 (1-s)^{\alpha-p-1}\ ds)) \\
& = \big(\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} + \frac{1}{\Delta}\ (\frac{\delta}{\Gamma(\alpha-p)} \int_0^\eta (\eta-s)^{\alpha-p-1} \\
& + \frac{1}{\Gamma(\alpha-p)} \int_0^1 (1-s)^{\alpha-p-1}) \big)\ m(s)\ ds + (\frac{1}{\Gamma(\alpha)} t^{\alpha-1} + \frac{1}{\Delta} \frac{\delta\eta^{\alpha-p}+1}{\Gamma(\alpha-p+1)}) \\
& (k_1\ |u|^{\lambda_1} + k_2\ |u'|^{\lambda_2}).
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
\|Tu(t)\|_1 \;\le\; & \int_0^1 G(t,s)\ m(s)\ ds + (k_1\ |u|^{\lambda_1} + k_2\ |u'|^{\lambda_2}) (\frac{t^\alpha}{\Gamma(\alpha+1)} \\
& + \frac{\delta t \eta^{\alpha-p}}{\Delta\Gamma(\alpha-p+1)} + \frac{t}{\Delta\Gamma(\alpha-p+1)}) + (\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \\
& + \frac{1}{\Delta} (\frac{\delta}{\Gamma(\alpha-p)} \int_0^\eta (\eta-s)^{\alpha-p-1} + \frac{1}{\Gamma(\alpha-p)} \int_0^1 (1-s)^{\alpha-p-1})) \ m(s)\ ds
\end{aligned} \tag{18}$$

$$
\begin{aligned}
&+(\frac{1}{\Gamma(\alpha)}t^{\alpha-1}+\frac{1}{\Delta}\frac{\delta\eta^{\alpha-p}+1}{\Gamma(\alpha-p+1)})(k_1\ |u|^{\lambda_1}+k_2\ |u'|^{\lambda_2})\\
&\leq (\int_0^1 G(t,s)+\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}+\frac{1}{\Delta}(\frac{\delta}{\Gamma(\alpha-p)}\int_0^\eta (\eta-s)^{\alpha-p-1}\\
&+\frac{1}{\Gamma(\alpha-p)}\int_0^1 (1-s)^{\alpha-p-1}))\ m(s)\ ds+(\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{(t+1)(\delta\eta^{\alpha-p}+1)}{\Delta\Gamma(\alpha-p+1)}\\
&+\frac{1}{\Gamma(\alpha)}t^{\alpha-1})(k_1\ |R|^{\lambda_1}+k_2\ |R|^{\lambda_2}).
\end{aligned}
$$

Using (15), (16) and (18), we have

$$
\|Tu(t)\|_1 \ \leq\ h_1+(k_1\ |R|^{\lambda_1}+k_2\ |R|^{\lambda_2})\ h_2\ \leq\ \frac{R}{3}+\frac{R}{3}+\frac{R}{3}\ =\ R
$$

which implies that $TU\ \subset\ U$.

Now we show that T is completely continuous operator. Let

$$
M\ =\ \max\{|f(t,u(t),u'(t))|\ :\ t\in[0,1],\ u\in U\},
$$

then

$$
\begin{aligned}
|Tu(t)-Tu(\tau)|\ &=\ |\int_0^1 G(t,s)f(s,u(s),u'(s))\ ds-\int_0^1 G(\tau,s)f(s,u(s),u'(s))\ ds|\\
&\leq\ M\ |\int_0^1 (G(t,s)-G(\tau,s))\ ds|\\
&=\ M|\int_0^t (t-s)^{\alpha-1}ds-\int_0^\tau (\tau-s)^{\alpha-1}ds+\frac{t-\tau}{\Delta\Gamma(\alpha-p)}\\
&(-\delta\int_0^\eta (\eta-s)^{\alpha-p-1}\ ds+\int_0^1 (1-s)^{\alpha-p-1}\ ds)|\\
&\leq\ M(\frac{1}{\Gamma(\alpha+1)}(t^\alpha-\tau^\alpha)+\frac{t-\tau}{\Delta\Gamma(\alpha-p+1)}(\delta\eta^{\alpha-p}+1)),
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
|T'u(t)-T'u(\tau)|\ &=|-I^{\alpha-1}(f(t,u(t),u'(t))-f(\tau,u(\tau),u'(\tau))|\\
&\leq\ \frac{M}{\Gamma(\alpha-1)}(\int_0^t (t-s)^{\alpha-2}\ ds-\int_0^\tau (\tau-s)^{\alpha-2}ds)\\
&=\ \frac{M}{\Gamma(\alpha)}(t^{\alpha-1}-\tau^{\alpha-1}).
\end{aligned}
$$

thus

$$
\begin{aligned}
\|Tu(t)-Tu(\tau)\|_1\ &\leq\ M(\frac{1}{\Gamma(\alpha+1)}(t^\alpha-\tau^\alpha)+\frac{t-\tau}{\Delta\Gamma(\alpha-p+1)}(\delta\eta^{\alpha-p}+1))\\
&+\frac{M}{\Gamma(\alpha)}(t^{\alpha-1}-\tau^{\alpha-1}).
\end{aligned}
$$

Since the functions $t^{\alpha-1},\tau^{\alpha-1},t^\alpha,\tau^\alpha$, are uniformly continuous on the interval [0,1], it follows that T is equicontinuous and by Arzela-Ascoli theorem T is completely continuous. By Schauder fixed point theorem T has a fixed point.

Lemma Under the assumption $(B2)$, the boundary value problem (4) has a solution.

Proof. The proof is similar like theorem 2, so we exclude the proof.

Theorem Assume that (B3) hold. If $k\varpi < 1$ then the problem (4) has a unique solution.

Proof. By the help of our supposition (B3), we have the following estimates

$$\begin{aligned}
|Tu(t) - Tv(t)| &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,u(s),u'(s)) - f(s,v(s),v'(s))|\, ds \\
&+ \frac{t}{\Delta\Gamma(\alpha-p)} (\delta \int_0^\eta (\eta-s)^{\alpha-p-1} |f(s,u(s),u'(s)) \\
&- f(s,v(s),v'(s))|\, ds + \int_0^1 (1-s)^{\alpha-p-1} |f(s,u(s),u'(s)) \\
&- f(s,v(s),v'(s))|\, ds) \\
&\leq \frac{1}{\Gamma(\alpha+1)} t^\alpha (k\{|u-v| + |u'-v'|\})) + \frac{t}{\Delta\Gamma(\alpha-p+1)} \\
&(\delta\eta^{\alpha-p-1}(k\{|u-v| + |u'-v'|\}) + (k\{|u-v| + |u'-v'|\})) \\
&\leq k\{|u-v| + |u'-v'|\}(\frac{1}{\Gamma(\alpha+1)} t^\alpha + \frac{t}{\Delta\Gamma(\alpha-p+1)} \\
&(\delta\eta^{\alpha-p-1} + 1)) = k\omega_1 \|u-v\|_1
\end{aligned} \tag{20}$$

By the help of (14) we have the following estimates

$$\begin{aligned}
|T'u(t) - T'v(t)| &= |\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} |f(s,u(s),u'(s)) \\
&- f(s,v(s),v'(s))|\, ds + \frac{1}{\Delta\Gamma(\alpha-p)} (-\delta \int_0^\eta (\eta-s)^{\alpha-p-1} \\
&|f(s,u(s),u'(s)) - f(s,v(s),v'(s))|\, ds + \int_0^1 (1-s)^{\alpha-p-1} \\
&|f(s,u(s),u'(s)) - f(s,v(s),v'(s))|)|\, ds \\
&\leq k\ \{|u-v| + |u'-v'|\}(\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{\delta\eta^{\alpha-p}+1}{\Delta\Gamma(\alpha-p+1)}) \\
&= k\omega_2 \|u-v\|_1
\end{aligned} \tag{21}$$

Thus by the help of (20) and (21) we have

$$\begin{aligned}
\|Tu(t) - Tv(t)\|_1 &\leq k\ \omega_1 \|u-v\|_1 + k\ \omega_2 \|u-v\|_1 \\
&= k(\omega_1 + \omega_2)\|u-v\|_1 = k\varpi \|u-v\|_1
\end{aligned} \tag{22}$$

thus by contraction mapping principle the boundary value problem (4) has a unique solution.

Example 1

$$\begin{aligned}
&D_t^{\frac{3}{2}} u(t) = \tfrac{\cos(t)}{35(5+7|u(t)|+5|Du(t)|)} \\
&u(0) = 0, \quad D^{\frac{1}{3}} u(1) = \tfrac{1}{10} D^{\frac{1}{3}} u(\tfrac{1}{3})
\end{aligned} \tag{23}$$

For the unique solution of problem (23), we apply theorem (2) with $f(t,u(t),u'(t)) = \frac{\cos(t)}{35(5+7|u(t)|+5|Du(t)|)}$, $t \in [0,1]$, $u(t), u'(t) \in [0,\infty)$, $\alpha = \frac{3}{2}$, $p = \frac{1}{3}$, $\delta = \frac{1}{10}$, $\eta = \frac{1}{3}$ and for $u(t), u'(t), v(t), v'(t)$ we have that $|f(t,u(t),u'(t)) - f(t,v(t),v'(t))| \leq$

$\frac{1}{5}\{|u-v|+|u'(t)-v'(t)|\}$ and thus condition (A4) is satisfied. By computation we have $\varpi = 2.4389$ and $k\varpi = .4878 < 1$. Thus by theorem (2) the problem (23) has a unique solution.

References

[1] R. Hilfer(Ed), Application of fractional calculus in physics, World scientific publishing Co. Singapore, 2000.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and applications of fractional differential equations, Volume 24, North-Holland Mathematics Studies,Amsterdame, 2006.

[3] K. S. Miller and B. Ross. An introduction to the fractional calculus and fractional differential equations, Wiley, 1993.

[4] K. B. Oldhalm and J. Spainer, The fractional calculus, Academic Press, New York, 1974.

[5] I. Podlubny, Fractional differential equations, Academic Press, New York,1999.

[6] J. Sabatier, O. P. Agrawal, J. A. Tenreiro and Machado, Advances in Fractional Calculus, Springer, 2007.

[7] M. El-Shahed, On the existence of positive solutions for a boundary value problem of fractional order, Thai J. Math.,5 (2007), 143-150.

[8] M. Benchohra, J. R. Graef and S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, Appl. Anal., 87 (2008), 851–863.

[9] G. A. Anastassiou, On right fractional calculus, Chaos, Solitons and fractals, vol. 42, no. 1, pp. 365–376, 2009.

[10] B. Ahmad, J. J. Nieto, Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations, Abstr. Appl. anal., (2009) art. ID 494720, 9pp.

[11] B. Ahmad, J. J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three point boundary conditions, Comput. Math. Appl., 58 (2009), 1838–1843.

[12] C. Yuan, D. Jiang and X. Xu,Singular positone and semipositone boundary value problems of nonlinear fractional differential equations, Math. Probl. Engineering, 2009(2009), Article ID 535209, 17 pages.

[13] M. Rehman and R. A. Khan, Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations, Appl. Math. Letters, 23(2010), 1038–1044.

[14] X. Dou, Y. Li, P. Liu, Existence of solutions for a four point boundary value problem of a nonlinear fractional differential equation, Op. Math., 31 (2011), 359–372.

[15] C. S. Goodrich, Existence and uniqueness of solutions to a fractional difference equation with non-local conditions, Comput. Math. Appl., 61 (2011), 191–202.

[16] R. A. Khan and M. Rehman, Existence of Multiple Positive Solutions for a General System of Fractional Differential Equations, Commun. Appl. Nonlinear Anal, 18(2011), 25–35.

[17] R. A. Khan, M. Rehman and N. Asif, Three point boundary value problems for nonlinear fractional differential equations, Acta. Mathematica. Scientia, 31B4(2011) 1–10.

[18] M. Rehman and R. A. Khan, A note on boundary value problems for a coupled system of fractional differential equations, Comput. Math. Appl, 61(2011),2630–2637.

[19] M. Rehman, R. A. Khan and J. Henderson, Existence and uniqueness of solutions for nonlinear fractional differential equations with integral boundary conditions, *Fract. Dyn.Syst*, 1 (2011), 29–43.

[20] X. Su, Boundary value problem for a coupled system of nonlinear fractional differential equations, Appl. Math. Lett., 22 (2009) 64–69.

[21] R. A. Khan, Three-point boundary value problems for higher order nonlinear fractional differential equations, J. Appl. Math. Informatics, 31(2013), 221–228.

[22] Q. Zhang, S. Chen and J. Lu,Upper and lower solution method for fourth-order four point boundary value problems, Journal of Computational and Applied Mathematics, 196(2006)387–393.

[23] M. Benchohra, N. Hamidi and J. Henderson, Fractional differential equations with anti-periodic boundary conditions, Numerical Funct. Anal. and Opti., 34(2013), 404–414.

[24] Jiang, D, Yuan, C: The positive properties of the Green function for Direchlet-type boundary value problems of nonlinear fractional differential equations and its application, Nonlinear Anal. 15, 710–719 (2010).

[25] Z. Bai and H. Lu, Positive solutions for boundary value problem of nonlinear fractional differential equation, Journal of Mathematical Analysis and Applications, vol. 311, no. 2, pp. 495–505, 2005.

R. A. Khan
Mathematics Department, University of Malakand, Dir Lower, KPK, Pakistan
E-mail address: rahmat_ alipk@yahoo.com,

H. Khan
Shaheed Benazir Bhutto University, Dir Upper, KPK, Pakistan
E-mail address: hasibkhan13@yahoo.com