

LINEAR SPACE-TIME FRACTIONAL REACTION-DIFFUSION EQUATION WITH COMPOSITE FRACTIONAL DERIVATIVE IN TIME

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ABSTRACT. In this paper, we consider linear space-time fractional reaction-diffusion equation with composite fractional derivative as time derivative and Riesz-Feller fractional derivative with skewness zero as space derivative. We apply Laplace and Fourier transforms to obtain its solution.

1. INTRODUCTION

Reaction-Diffusion equations have found numerous applications in pattern formation in biology, chemistry and physics. In recent works authors have demonstrated the depth of mathematics and related physical issues of reaction-diffusion equations such as nonlinear phenomena, stationary and spatio-temporal dissipative pattern formation, oscillations, waves etc. [4, 5]. Interest in fractional reaction-diffusion equations has increased because the equation exhibits self organization phenomena and introduces a new parameter, the fractional index, into the equation.

The classical reaction-diffusion equations are useful to model the spread of invasive species [12]. In this model the population density $u(x, t)$ at location x and time t is solution of the reaction-diffusion equation in its simplest form as given by

$$\frac{\partial u(x, t)}{\partial t} = d \frac{\partial^2 u(x, t)}{\partial x^2} + R(u) \quad (1)$$

where d is the diffusion coefficient and $R(u)$ is a function representing reaction kinetics which may be linear or nonlinear. If we set $R(u) = 0$, in (1) it reduces to diffusion equation, for $R(u) = u(x, t) - u^3(x, t)$, it reduces to Ginzburg-Landau equation, for $R(u) = u(x, t) - u^2(x, t)$, it reduces to Fisher equation and for $R(u) = \sigma u(x, t) [1 - u^2(x, t)]$, it reduces to Fisher-Kolmogorov equation.

The main shortcoming of this model in real applications is its unrealistically slow spreading, since typical invasive species have population densities that spread

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faster than t , with power law leading edges [13]. The ordinary reaction-diffusion equation is inadequate to model many real situations. Also, solution to fractional reaction-diffusion equation spread faster than the counterpart of ordinary reaction-diffusion equations. Fractional generalizations of reaction-diffusion equation, have been studied and solved by many researchers, namely Atabong & Oyesanya [2], Henry and Wearne [8], Seki *et al.*[21], Akil *et al.*[1], Saxena *et al.*[17,18, 19].

Linear space-time fractional reaction-diffusion equation on a finite domain $0 < x < L, t > 0$ with $0 < \alpha \leq 1$ and $1 < \beta \leq 2$ is given by [24, 25]

$${}_t^{\alpha} D_t^{\alpha} u(x, t) = b(x) {}_x^{\beta} D_x^{\beta} u(x, t) - c(x) u(x, t) + f(x, t) \tag{2}$$

where the coefficient of diffusion $b(x) > 0$, reaction term $c(x) > 0$ the function $f(x, t)$ represents source or sink and ${}_t^{\alpha} D_t^{\alpha}$ and ${}_x^{\beta} D_x^{\beta}$ are fractional derivatives considered in Caputo sense. Yu *et al.* [25] and Yildirim and Sezer [24] have used Adomian decomposition method and homotopy perturbation method respectively to obtain numerical solutions of these equations and Garg and Manohar [6] obtained analytical solutions of linear space-time fractional reaction-diffusion equations of the form (2) using generalized differential transform method.

2. PRELIMINARIES

The **Riemann-Liouville fractional integral** of order α is defined as [12]

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad 0 < \alpha \leq 1 \tag{3}$$

with $I_t^0 f(t) = f(t)$.

The **Riemann-Liouville fractional derivative** of order $\alpha, m - 1 < \alpha \leq m, m \in \mathbb{N}$ is defined as the left inverse of the corresponding Riemann-Liouville fractional integral, [12] i.e.

$$D_t^{\alpha} f(t) = D^m I_t^{m-\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} D^m \int_0^t (t - \tau)^{m-\alpha-1} f(\tau) d\tau, & \text{For } m - 1 < \alpha < m \\ D^m f(t), & \text{For } \alpha = m, D^m \equiv \frac{d^m}{dt^m}. \end{cases} \tag{4}$$

The **Caputo fractional derivative** of order $\alpha, m - 1 < \alpha \leq m, m \in \mathbb{N}$, is defined as [3]

$${}_t^{\alpha} D_t^{\alpha} f(t) = I_t^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t - \tau)^{m-\alpha-1} D^m f(\tau) d\tau. \tag{5}$$

The **Hilfer fractional derivative** of order $0 < \alpha \leq 1$ and type $0 \leq \beta \leq 1$ is defined by Hilfer [9] as follows

$$\left(D_t^{\alpha, \beta} f \right) (t) = \left(I_t^{\beta(1-\alpha)} D \left(I_t^{(1-\beta)(1-\alpha)} f \right) \right) (t), \tag{6}$$

The above definition, in the case $\beta = 0$, reduces to the classical Riemann-Liouville fractional derivative as given below [22]

$$\left(D_t^{\alpha, 0} f \right) (t) = D^n \left(I_t^{(1-\alpha)} f \right) (t) = (D_t^{\alpha} f) (t) \tag{7}$$

and in the case $\beta = 1$, it reduces to the Caputo fractional derivative as

$$\left(D_t^{\alpha, 1} f \right) (t) = \left(I_t^{(1-\alpha)} D^n f \right) (t) = ({}_t^{\alpha} D_t^{\alpha} f) (t). \tag{8}$$

For $0 < \beta < 1$, it interpolates continuously between these two derivatives.

Recently this definition is extended for $n - 1 < \alpha \leq n, n \in \mathbb{N}, 0 \leq \beta \leq 1$, and is termed as **composite fractional derivative** or **generalized Riemann-Liouville fractional derivative** by Hilfer *et al.* [10]. It is given by

$$\left(D_t^{\alpha, \beta} f \right) (t) = \left(I_t^{\beta(n-\alpha)} D^n \left(I_t^{(1-\beta)(n-\alpha)} f \right) \right) (t). \quad (9)$$

The **Laplace transform** $L[g(t); s] = \int_0^\infty g(t) e^{-st} dt$ of the composite fractional derivative (9) is given by [22]

$$L \left[D_t^{\alpha, \beta} f(t) \right] = s^\alpha L[f(t)] - \sum_{k=0}^{n-1} s^{n-k-1-\beta(n-\alpha)} D^k \left(I_t^{(1-\alpha)(1-\beta)} f \right) (0). \quad (10)$$

The **Riesz-Feller fractional derivative** of order $\gamma, 0 < \gamma \leq 2$ and skewness θ is defined as [11]

$$F \{ {}_x D_\theta^\gamma f(x); k \} = -\psi_\gamma^\theta(k) \bar{f}(k), \quad (11)$$

where $\psi_\gamma^\theta(k) = |k|^\gamma e^{i(\text{sign}k)\theta\pi/2}$, $|\theta| \leq \min\{\gamma, 2-\gamma\}$, $\bar{f}(k)$ is the Fourier transform of the function $g(x)$, defined as

$$\bar{f}(k) = F \{ f(x); k \} = \int_{-\infty}^\infty f(x) e^{ikx} dx. \quad (12)$$

with Fourier inverse given by

$$f(x) = F^{-1} \{ \bar{f}(k); x \} = \frac{1}{2\pi} \int_{-\infty}^\infty \bar{f}(k) e^{-ikx} dk. \quad (13)$$

The Riesz-Feller fractional derivative with skewness $\theta = 0$ and $\gamma \neq 1$ may also be termed as Riesz fractional derivative, since it is left inverse of a fractional integral, introduced by Marcel Riesz in late 1940's, known as Riesz potential. In this case we write ${}_x D_0^\gamma \equiv \frac{d^\gamma}{d|x|^\gamma}$, and thus Riesz-Feller fractional derivative of order $\gamma, 0 < \gamma \leq 2, \gamma \neq 1$ and skewness zero is defined by

$$\frac{d^\gamma}{d|x|^\gamma} f(x) = F^{-1} \{ -|k|^\gamma \bar{f}(k); x \}, \quad (14)$$

The **Mittag-Leffler function** which is a generalization of exponential function is defined as [13]

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + 1)}, \text{Re}(\alpha) > 0. \quad (15)$$

A generalization of (15) is given in the form [23]

$$E_{\alpha, \beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + \beta)}, \alpha, \beta \in C, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0. \quad (16)$$

3. SOLUTION OF LINEAR SPACE-TIME FRACTIONAL REACTION-DIFFUSION EQUATION

Theorem 1. We consider the linear space-time fractional reaction-diffusion equation for the field variable $u(x, t)$ as

$$D_t^{\alpha, \beta} u(x, t) = \eta \frac{\partial^\gamma}{\partial |x|^\gamma} u(x, t) + cu(x, t) + \phi(x, t), \quad t > 0, \quad -\infty < x < \infty, \quad (17)$$

with boundary conditions

$$u(\pm\infty, t) = 0, \quad t > 0 \quad (18)$$

and initial condition

$$\left\{ I_t^{(1-\beta)(1-\alpha)} u(x, t) \right\}_{t \rightarrow 0^+} = g(x), \quad -\infty < x < \infty, \quad (19)$$

where $D_t^{\alpha, \beta}$ is the composite fractional derivative operator, defined by (6) with $0 < \alpha \leq 1, 0 \leq \beta \leq 1, \frac{\partial^\gamma}{\partial |x|^\gamma}$, is the Riesz-Feller fractional derivative of order $\gamma, 0 < \gamma \leq 2, \gamma \neq 1$ and skewness zero, $\eta > 0$ is diffusion coefficient, $c > 0$ is a constant with reaction term and $\phi(x, t)$ represents source or sink.

The solution of the problem (18)-(19) is given by

$$u(x, t) = t^{-(1-\alpha)(1-\beta)} \int_{-\infty}^{\infty} \bar{g}(k) E_{\alpha, \alpha+\beta(1-\alpha)} \{-(\eta |k|^\gamma + c) t^\alpha\} e^{-ikx} dk + \int_{-\infty}^{\infty} e^{-ikx} \int_0^t \tau^{\alpha-1} E_{\alpha, \alpha} \{(\eta |k|^\gamma + 1) \tau^\alpha\} \bar{\phi}(k, t - \tau) dk d\tau, \quad (20)$$

where $\bar{g}(k)$ and $\bar{\phi}(k, t)$ are Fourier transforms of the functions $g(x)$ and $\phi(x, t)$ respectively.

Proof. Taking Laplace transform of equation (17) with respect to ‘ t ’, using condition (19) and result (10), we get

$$s^\alpha U(x, s) - s^{\beta(\alpha-1)} g(x) = \eta \frac{d^\gamma}{d|x|^\gamma} U(x, s) + cU(x, s) + \Phi(x, s), \quad (21)$$

where $U(x, s)$ and $\Phi(x, s)$ are Laplace transforms of functions $u(x, t)$ and $\phi(x, t)$ respectively.

Taking Fourier transform of the equation(21) with respect to ‘ x ’ and using(14), we get

$$s^\alpha \bar{U}(k, s) - s^{-\beta(1-\alpha)} \bar{g}(k) = \eta \{ -|k|^\gamma \bar{U}(k, s) \} + c\bar{U}(k, s) + \bar{\Phi}(k, s), \quad (22)$$

where $\bar{U}(k, s)$ and $\bar{\Phi}(k, s)$ are Fourier transforms of functions $u(x, t)$ and $\phi(x, t)$ respectively.

On simplification,(22) gives

$$\bar{U}(k, s) = \frac{s^{-\beta(1-\alpha)} \bar{g}(k) + \bar{\Phi}(k, s)}{s^\alpha + \eta |k|^\gamma - c}. \quad (23)$$

Taking inverse Laplace transform of the above equation, using convolution theorem for Laplace transform and the following known result [16]

$$L \left[t^{-(1-\alpha)(1-\beta)} E_{\alpha, 1-(1-\alpha)(1-\beta)} (-\lambda t^\alpha) \right] = \frac{s^{-\beta(1-\alpha)}}{s^\alpha + \lambda}, \quad s, \lambda \in \mathbb{R}^+, \quad (24)$$

we obtain

$$\bar{u}(k, t) = t^{-(1-\alpha)(1-\beta)} \bar{g}(k) E_{\alpha, \alpha+\beta(1-\alpha)} \{-(\eta |k|^\gamma + c) t^\alpha\} + \int_0^t \tau^{\alpha-1} E_{\alpha, \alpha} \{(\eta |k|^\gamma + 1) \tau^\alpha\} \bar{\phi}(k, t - \tau) d\tau, \quad (25)$$

where $\bar{u}(k, t)$ is Fourier transform of $u(x, t)$.

Taking inverse Fourier transform of (25), we obtain the desired result (20).

Special Cases

(1) On taking $\beta = 1$ in Theorem 1, we get the following result

Corollary 1. Consider the linear space-time fractional reaction-diffusion equation with Caputo fractional derivative in time for the field variable $u(x, t)$

$$*_t D_t^\alpha u(x, t) = \eta \frac{\partial^\gamma}{\partial |x|^\gamma} u(x, t) + cu(x, t) + \phi(x, t), \quad t > 0, \quad -\infty < x < \infty, \quad (26)$$

with boundary conditions

$$u(\pm\infty, t) = 0, \quad t > 0 \quad (27)$$

and initial condition

$$\{u(x, t)\}_{t \rightarrow 0^+} = g(x), \quad -\infty < x < \infty, \quad (28)$$

where ${}_t D_t^\alpha$ is the Caputo fractional derivative operator, defined by (5) with $0 < \alpha \leq 1$ and all other symbols are as explained in Theorem 1.

The solution of problem (26)-(28) is given by

$$u(x, t) = \int_{-\infty}^{\infty} \bar{g}(k) E_\alpha \{-(\eta |k|^\gamma + c) t^\alpha\} e^{-ikx} dk + \int_{-\infty}^{\infty} e^{-ikx} \int_0^t \tau^{\alpha-1} E_{\alpha, \alpha} \{(\eta |k|^\gamma + 1) \tau^\alpha\} \bar{\phi}(k, t - \tau) dk d\tau, \quad (29)$$

where $\bar{g}(k)$ and $\bar{\phi}(k, t)$ are Fourier transforms of the functions $g(x)$ and $\phi(x, t)$ respectively.

Further setting $\gamma = 2$, we obtain solution of linear time fractional reaction-diffusion equation with Caputo fractional derivative in time.

- (2) On taking $\beta = 0$ in Theorem 1, we get the following result

Corollary 2. Consider the linear space-time fractional reaction-diffusion equation with Riemann-Liouville fractional derivative in time for the field variable $u(x, t)$

$$D_t^\alpha u(x, t) = \eta \frac{\partial^\gamma}{\partial |x|^\gamma} u(x, t) + cu(x, t) + \phi(x, t), \quad t > 0, \quad -\infty < x < \infty, \quad (30)$$

with boundary conditions

$$u(\pm\infty, t) = 0, \quad t > 0 \quad (31)$$

and initial condition

$$\left\{ I_t^{(1-\alpha)} u(x, t) \right\}_{t \rightarrow 0^+} = g(x), \quad -\infty < x < \infty, \quad (32)$$

where D_t^α is the Riemann-Liouville fractional derivative operator, defined by (4) with $0 < \alpha \leq 1$ and all other symbols are as explained in Theorem 1.

The solution of problem (30)-(32) is given by

$$u(x, t) = t^{-(1-\alpha)} \int_{-\infty}^{\infty} \bar{g}(k) E_{\alpha, \alpha} \{-(\eta |k|^\gamma + c) t^\alpha\} e^{-ikx} dk + \int_{-\infty}^{\infty} e^{-ikx} \int_0^t \tau^{\alpha-1} E_{\alpha, \alpha} \{(\eta |k|^\gamma + 1) \tau^\alpha\} \bar{\phi}(k, t - \tau) dk d\tau, \quad (33)$$

where $\bar{g}(k)$ and $\bar{\phi}(k, t)$ are Fourier transforms of the functions $g(x)$ and $\phi(x, t)$ respectively.

Further setting $\gamma = 2$, we obtain solution of the linear time fractional reaction-diffusion equation with Riemann-Liouville fractional derivative in time.

- (3) If we take $\alpha = 1$ in Theorem 1, we get the following result

Corollary 3. Consider the linear space fractional reaction-diffusion equation for the field variable $u(x, t)$

$$D_t u(x, t) = \eta \frac{\partial^\gamma}{\partial |x|^\gamma} u(x, t) + cu(x, t) + \phi(x, t), \quad t > 0, \quad -\infty < x < \infty, \quad (34)$$

with boundary conditions

$$u(\pm\infty, t) = 0, \quad t > 0 \quad (35)$$

and initial condition

$$\{u(x, t)\}_{t \rightarrow 0^+} = g(x), \quad -\infty < x < \infty, \quad (36)$$

where the symbols are as explained in Theorem 1.

The solution of problem (34)-(36) is given by

$$u(x, t) = \int_{-\infty}^{\infty} \bar{g}(k) e^{\{-(\eta|k|^\gamma + c)t\} - ikx} dk + \int_{-\infty}^{\infty} e^{-ikx} \int_0^t e^{\{(\eta|k|^\gamma + 1)\tau\}} \bar{\phi}(k, t - \tau) dk d\tau, \quad (37)$$

where $\bar{g}(k)$ and $\bar{\phi}(k, t)$ are Fourier transforms of the functions $g(x)$ and $\phi(x, t)$ respectively.

Further setting $\gamma = 2$, we obtain solution of the linear reaction-diffusion equation.

- (4) Making $c \rightarrow 0$ in Theorem 1, we obtain solution of inhomogeneous fractional diffusion equation with composite fractional time derivative and Riesz Feller fractional space derivative. The problem is same as considered recently by Saxena *et al.*[20]. (They have termed it as fractional reaction-diffusion equation)
- (5) Making $c \rightarrow 0$ and $\phi(x, t) = 0$ in Theorem 1, we obtain solution of the generalized space-time fractional diffusion equation with composite fractional time derivative as studied by Tomovski [22].
- (6) Making $c \rightarrow 0$ and $\gamma = 2$ in Theorem 1, we obtain solution of time fractional inhomogeneous diffusion equation with composite fractional time derivative as studied by Sandev *et al.* [16].
- (7) Making $c \rightarrow 0$ and $\beta = 1$ in Theorem 1, we obtain solution of space-time fractional diffusion equation with Caputo fractional time derivative as studied by Houbold *et al.* [7].

Conclusion

In this research work, we consider a linear space-time fractional reaction-diffusion equation with composite fractional derivative for time and Riesz-Feller fractional derivative with skewness zero for space. Since fractional reaction-diffusion equation models many real world problems more realistically and composite fractional derivative leads to a very flexible framework for the description of complex processes, the problem considered here addresses to more realistic and complex class of problems. We apply Laplace and Fourier transforms to obtain its solution and also provide solutions of some new or known reaction-diffusion equations with other well known fractional derivatives.

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