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EXACT SOLUTIONS OF TIME FRACTIONAL KOLMOGOROV EQUATION BY USING LIE SYMMETRY ANALYSIS

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Abstract. In this work the Lie point symmetries admitted by α-time fractional Kolmogorov differential equation are constructed. It is shown that the obtained Lie point symmetries is used to transform α-time fractional Kolmogorov equation into a second order ordinary differential equation which is solved in terms of special function.This analysis will be used to construct some exact solutions.

1. Introduction

The first serious attempt to give a logical definition of a fractional derivative is due to Liouville [1, 2] . There are several approaches to define the fractional calculus, e.g. Riemann- Liouville, Grunwald-Letnikow, Caputo, and Generalized Functions approach[1, 2, 12]. Recently, differential equations of fractional order arise very naturally in solving a wide variety of phenomena in fluid mechanics, biology, physics and other areas of science [2, 12]. There is no well defined method to analyze them. It has been studied relatively little, therefore few methods are used like Laplace transform method, Fourier transform method, variation alteration method, Adomian decomposition method, finite difference method and so on [1, 2, 10, 11].

Lie's symmetry group method for differential equations provides a powerful and fundamental framework to the exploitation procedures leading to the integration and plays a significant role in studying differential equations [see e.g [3]]. Methods of constructing point symmetries admitted by differential equations were intensively developed by several researchers [3, 4, 5, 6, 7, 8]. Consequently, different applications were discussed as reduction of order or number of independent variables, also construction of integrator factor, exact solutions, conserved lows and other applications .

Methods of group analysis adapted to study symmetry properties of differential equations with fractional order derivatives is not yet studied widely except in

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special papers[7, 8, 9]. For example, in [7] the authors develops a framework to obtain the α-th extended infinitesimal generator which is used in invariance criterion. This allow them to establish an algorithm to determine the infinitesimal transformations admitted by the studied fractional differential equation. This Lie symmetry analysis is effectively used to find exact solutions of fractional differential equation as it was done for non fractional different linear and nonlinear partial differential equations[3, 4, 5].

The main objective of this paper is to compute point symmetries admitted by α-time fractional Kolmogorov differential equation given by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = x\frac{\partial^2 u}{\partial x^2} + \frac{\gamma x}{1+\frac{1}{2}\gamma x}\frac{\partial u}{\partial x}, \quad \gamma > 0,\ x > 0. \tag{1}$$

where $\frac{\partial^\alpha u}{\partial t^\alpha}$ is the fractional derivative of order α with $0 < \alpha < 1$. The case $\alpha = 1$ is extensively studied and fundamental solution is given[6].

2. Preliminaries and Basic Definitions

In this section, we set up notation, basic definitions and main results of fractional calculus. To begin with, recall that there are several approaches to define the fractional calculus, e.g. Riemann- Liouville, Grunwald-Letnikow, Caputo, and Generalized Functions approach.

Definition 1 The Riemann-Liouville fractional derivative operator D^α $(\alpha \geq 0)$ of a function $u(t,x)$, is defined as

$$D^\alpha u(t,x) = \frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \frac{\partial^n u}{\partial t^n},\ \alpha = n \in \mathbb{N} \\ \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial t^n}\int_0^t \frac{u(\upsilon,x)}{(t-\upsilon)^{\alpha+1-n}}d\upsilon, \quad n-1 < \alpha < n,\ n \in \mathbb{N}, \end{cases}$$

where Γ is the well-known gamma function, and some properties of the operator D^α are as follows

$$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}t^{\gamma-\alpha},\ \alpha > 0,\ \gamma > -1,\ t > 0.$$

$$D^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)},\ \alpha \geq 0,\ t > 0.$$

Definition 2 The Caputo fractional derivative D_c^α of a function $u(t,x)$ is defined as

$$D_c^\alpha u(t,x) = \frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \frac{\partial^n u}{\partial t^n},\ \alpha = n \in \mathbb{N} \\ \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u_\upsilon^{(n)}(\upsilon,x)}{(t-\upsilon)^{\alpha+1-n}}d\upsilon, \quad n-1 < \alpha < n,\ n \in \mathbb{N}, \end{cases}$$

where $u_\upsilon^{(n)}(\upsilon,x) = \frac{\partial^n u(\upsilon,x)}{\partial \upsilon^n}$ and some properties of this derivation are as follows

$$D_c^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}t^{\gamma-\alpha},\ \alpha > 0,\ \gamma > -1,\ t > 0.$$

$$D_c^\alpha 1 = 0,\ \alpha \geq 0,$$

$$D^\alpha\left(u(t,x) - \sum_{k=0}^{n-1}\frac{t^k}{k!}u_t^{(k)}(0^+,x)\right) = D_c^\alpha u(t,x),\ n-1 < \alpha < n,\ t > 0.$$

Some other properties of fractional derivative can be found in [1, 2]. In this paper, we adopt the Riemann-Liouville fractional derivative and we can do the same in terms of Caputo contribution. Since in this paper $0 < \alpha < 1$ the Riemann-Liouville fractional derivative definition becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(\upsilon, x)}{(t-\upsilon)^\alpha} d\upsilon.$$

3. Lie symmetry analysis method

Now consider the one parameter group of point transformations of the form

$$\begin{aligned} \widetilde{t} &= t + \varepsilon\tau(t,x,u) + o(\varepsilon), \\ \widetilde{x} &= x + \varepsilon\xi(t,x,u) + o(\varepsilon), \\ \widetilde{u} &= u + \varepsilon\varphi(t,x,u) + o(\varepsilon), \end{aligned} \quad (2)$$

with coresponding infinitesimal generator which is

$$X = \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \varphi(t,x,u)\frac{\partial}{\partial u}.$$

Definition 3 A solution $u = \theta(t,x)$ is said to be invariant solution of fractional partial differential equation (1) if and only if

a) $u = \theta(t,x)$ is an invariant surface, i.e. $X\theta = 0 \Rightarrow (\tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \varphi\frac{\partial}{\partial u})\theta = 0,$
b) $u = \theta(t,x)$ satisfies equation (1).

As equation (1) is a time fractional partial differential equation having the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = F(t,x,u,u_x,u_{xx},...),$$

where subscripts denote partial derivatives, we will need to extend the infinitesimal generator X to X^α of the form

$$X^\alpha = X + \varphi^x \frac{\partial}{\partial u_x} + \varphi^{xx}\frac{\partial}{\partial u_{xx}} + \varphi^\alpha \frac{\partial}{\partial u^\alpha},$$

such that $u^\alpha = \frac{\partial^\alpha u}{\partial t^\alpha}$ and $\varphi^x, \varphi^{xx}, \varphi^\alpha$ are extended infinitesimals of order 1,2 and α respectively. φ^x and φ^{xx} have the form[3, 4, 5]

$$\varphi^x = \varphi_x + (\varphi_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \quad (3)$$

$$\begin{aligned} \varphi^{xx} &= \varphi_{xx} + (2\varphi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\varphi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t \\ &- \xi_{uu}u_x^3 - \tau_{uu}u_x^2 u_t + (\varphi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} \\ &- \tau_u u_{xx}u_t - 2\tau_u u_x u_{xt}. \end{aligned} \quad (4)$$

Nevertheless, the αth extended infinitesimal φ^α related to Riemann-Liouville fractional time derivative is established and has the following form[7]

$$\varphi^\alpha = D_t^\alpha(\varphi) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u),$$

where D_t^α denotes the total time fractional derivative.

Let us recall some useful properties concerned the total fractional derivative operator D_t^ν. We start with the generalized Leibnitz's formula [1, 2, 12] given by

$$D_t^\nu(f(t)g(t)) = \sum_{n=0}^{+\infty} \binom{\nu}{n} D_t^{\nu-n} f(t) D_t^n g(t), \qquad \nu > 0,$$

where

$$\binom{\nu}{n} = \frac{(-1)^{n-1}\nu\Gamma(n-\nu)}{\Gamma(1-\nu)\Gamma(n+1)},$$

and the chain rule for composite function [1, 2, 12] which is

$$\frac{d^n g(y(t))}{dt^n} = \sum_{k=0}^{n}\sum_{r=0}^{k}\left(\begin{array}{c}k\\ r\end{array}\right)\frac{1}{k!}[-y(t)]^r \times \frac{d^n}{dt^n}[(y(t))^{k-r}]\frac{d^k g(y)}{dy^k}.$$

Since $u = u(t,x)$ and $\varphi = \varphi(t,x,u)$ an important formula gives $D_t^\alpha[\varphi(t,x,u)]$ in terms of partial derivatives found in[1, 7] which is given by:

$$\begin{aligned} D_t^\alpha[\varphi(t,x,u)] &= \sum_{n=0}^{+\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{r=0}^{k}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \times \\ &\times \quad [-u(t,x)]^r\frac{\partial^m}{\partial t^m}([u(t,x)]^{k-r})\frac{\partial^{n-m+k}\varphi(t,x,u)}{\partial t^{n-m}\partial u^k}. \end{aligned}$$

From the above formula and applying Leibnitz formula for $\frac{\partial^\alpha(u\varphi_u)}{\partial t^\alpha}$ the expression of $D_t^\alpha\varphi(t,x,u)$ will have the following form

$$D_t^\alpha(\varphi) = \frac{\partial^\alpha\varphi}{\partial t^\alpha} + \varphi_u\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha\varphi_u}{\partial t^\alpha} + \sum_{n=1}^{+\infty}\binom{\alpha}{n}\frac{\partial^n\varphi_u}{\partial t^n}D_t^{\alpha-n}(u) + \mu,$$

with

$$\begin{aligned} \mu &= \sum_{n=2}^{+\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!} \times \\ &\times \quad \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\varphi}{\partial t^{n-m}\partial u^k}. \end{aligned} \tag{5}$$

Consequently, the α-th extended infinitesimal φ^α can be rewritten as

$$\begin{aligned} \varphi^\alpha &= \frac{\partial^\alpha\varphi}{\partial t^\alpha} + (\varphi_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha\varphi_u}{\partial t^\alpha} + \mu \\ &+ \sum_{n=1}^{+\infty}\left[\binom{\alpha}{n}\frac{\partial^n\varphi_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u) \\ &- \sum_{n=1}^{+\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x). \end{aligned} \tag{6}$$

Here in our case, the term μ vanishes. This is a consequence of $\mu_{uu} = 0$ which is obtained in analyzing of invariance condition studied in next section.

Definition 4 By definition, transformations (2) form a symmetry group of equation (1) if they transform solution of equation (1) to an other solution of the same equation. Therefore, equation (1) is invariant under transformations (2) if and only if the invariance condition is satisfies,i.e

$$\varphi^\alpha - (u_{xx} + f'(x)u_x)\xi - f(x)\varphi^x - x\varphi^{xx} = 0, \quad \text{whenever} \quad \Delta = 0. \tag{7}$$

with $\Delta = u^\alpha - xu_x - f(x)u_{xx}$ and f is the drift function given by

$$f(x) = \frac{\gamma x}{1 + \frac{1}{2}\gamma x}.$$

Equation (7) defines all infinitesimal symmetries of equation (1)

Theorem 1 The basis of the symmetry Lie algebras admitted by equation (1) is spanned by vector fields

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = u\frac{\partial}{\partial u},$$
$$X_3 = \alpha x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{2\alpha}{2+\gamma x}u\frac{\partial}{\partial u}, \qquad X_h = h(t,x)\frac{\partial}{\partial u},$$

where $h(t,x)$ is an arbitrary solution of equation (1).

Proof. To obtain general form of infinitesimals ξ, τ and φ, we need to substitute expressions (3),(4) and (6) into invariance condition (7). The equation depends on variables $u_x, u_{xx}, u_t, u_{xt}, ...$ and $D_t^{\alpha-n}u, D_t^{\alpha-n}u_x$ for $n = 1, 2, ...$ These variables are considered to be independent variables[7].

Splitting the defining equation (7) with respect to independents variables mentioned above leads to the system of infinitely equations

$$\xi_t = \xi_u = \tau_x = \tau_u = \varphi_{uu} = 0, \tag{8}$$
$$x\varphi_u - \alpha x\tau_t - \xi - x(\varphi_u - 2\xi_x) = 0, \tag{9}$$
$$-\alpha f(x)\tau_t - f'(x)\xi + f(x)\xi_x - x(2\varphi_{xu} - \xi_{xx}) = 0, \tag{10}$$
$$\frac{\partial^\alpha \varphi}{\partial t^\alpha} - u\frac{\partial^\alpha \varphi_u}{\partial t^\alpha} - f(x)\varphi_x - x\varphi_{xx} = 0, \tag{11}$$
$$\binom{\alpha}{n}\frac{\partial^n \varphi_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}\tau, \qquad n = 0, 1, 2, ... \tag{12}$$

Examining equations (8) and (9) in the above system, we readily obtain

$$\xi(x) = a\alpha x + m\sqrt{x}, \qquad \text{and} \qquad \tau(t) = at + b,$$

with a, b and m are arbitrary constants. In virtue of $\varphi_{uu} = 0$, φ must be linear in u. Thus

$$\varphi = g(t,x)u + h(t,x),$$

for some functions $g(t,x)$ and $h(t,x)$. On the other hand, equation (12) requires that

$$\frac{\partial \varphi_u}{\partial t} = 0, \qquad \text{then} \qquad g = g(x).$$

We substitute these in equation (11) to derive that the function h is an arbitrary solution of the original fractional differential equation (1). Substituting ξ and τ by their expressions into equation (10) leads to the equation

$$g'(x) = \frac{1}{4x\sqrt{x}}mf(x) - \frac{1}{2}(a\alpha + \frac{\sqrt{x}}{x}m)f'(x) - \frac{m}{8x\sqrt{x}}. \tag{13}$$

Differentiation of the equation (11) over u leads to

$$xg''(x) + f(x)g'(x) = 0. \tag{14}$$

Differentiation of equation (13) with respect the independent variable x and substituting this into equation (14), leads

$$-\frac{a\alpha}{2}x\frac{d}{dx}\mathfrak{L}f + 16^{-1}mx^{-\frac{1}{2}}[3 + 8\mathfrak{L}f - 8x\frac{d}{dx}\mathfrak{L}f] = 0, \tag{15}$$

with $\mathfrak{L}f = xf' - f + \frac{f^2}{2}$. Finally, equation (15) determines m and a. Obviously, the drift function in this case satisfies

$$\mathfrak{L}f = xf' - f + \frac{f^2}{2} = 0.$$

Thus the constant a appeared in equation (15) is arbitrary and m vanishes. This implies that infinitesimals ξ, τ and φ becomes

$$\xi = a\alpha x, \qquad \tau = at + b, \qquad \varphi = (\frac{2a\alpha}{2+\gamma x} + c)u + h(t,x),$$

where a, b, c are arbitrary constants and $h(t,x)$ is an arbitrary solution of equation (1). Finally, the symmetry algebra admitted by equation (1) is spanned by infinitesimal generators

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= u\frac{\partial}{\partial u}, \\ X_3 &= \alpha x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \frac{2\alpha}{2+\gamma x}u\frac{\partial}{\partial u}, \\ X_h &= h(t,x)\frac{\partial}{\partial u}. \end{aligned}$$

Theorem 2 The similarity variable and similarity transformation corresponding to the infinitesimal generator X_3 reduce Kolmogorov time fractional equation to second order ordinary differential equation

$$z\psi''(z) + (2 - vz)\psi'(z) = 0, \qquad \text{with } v = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)},$$

which gives arise of an exact solution of (1) given by

$$u_1(t,x) = \frac{c_1 x}{2+\gamma x} + \frac{c_2 x}{2+\gamma x}\left[-\frac{\exp(vxt^{-\alpha})}{vxt^{-\alpha}} - Ei(1, -vxt^{-\alpha})\right], \tag{16}$$

with Ei is the exponential integral special function.

Proof. Similarity variables corresponding to infinitesimal generator X_3 can be obtained by solving the characteristic equation

$$\frac{dx}{\alpha x} = \frac{dt}{t} = \frac{(2+\gamma x)du}{2\alpha u}.$$

The invariants of this operator are of the form

$$z = xt^{-\alpha}, \quad \text{and} \quad I = \frac{2+\gamma x}{x}u.$$

Exact solution which can be constructed in this case will be of the form

$$u(t,x) = \frac{x}{2+\gamma x}\psi(z),$$

where the function ψ satisfies the reduced second order ordinary differential equation of time fractional Kolmogorov differential equation given by

$$z\psi''(z) + (2 - vz)\psi'(z) = 0, \qquad \text{with } v = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}.$$

From the out put of Maple, the solution of the above second order differential equation is given in terms of exponential integral special function Ei by

$$\psi(z) = c_1 + c_2(-\frac{\exp(vz)}{vz} - Ei(1, -vz)),$$

where c_1 and c_2 are arbitrary constants. Then, it immediately leads to an exact solution of equation (1) which has the expression

$$u_1(t,x) = \frac{c_1 x}{2+\gamma x} + \frac{c_2 x}{2+\gamma x}\left[-\frac{\exp(vxt^{-\alpha})}{vxt^{-\alpha}} - Ei(1, -vxt^{-\alpha})\right]. \tag{17}$$

Remark 1 To obtain the group transformation generated by infinitesimal generator X_3, we solve the system of first order ordinary differential equations,

$$\begin{aligned}\frac{d\widetilde{t}}{d\varepsilon} &= \xi(\widetilde{t}, \widetilde{x}, \widetilde{u}),\\ \frac{d\widetilde{x}}{d\varepsilon} &= \varphi(\widetilde{t}, \widetilde{x}, \widetilde{u}),\\ \frac{d\widetilde{u}}{d\varepsilon} &= \varphi(\widetilde{t}, \widetilde{x}, \widetilde{u}),\end{aligned}$$

subject to the initial conditions

$$\widetilde{t}(0) = t, \qquad \widetilde{x}(0) = x, \qquad \widetilde{u}(0) = u.$$

The one-parameter groups G generated by X_3 is given as follows. The entries gives the transformed point $\exp(\varepsilon X_3)(t, x, u) = (\widetilde{t}, \widetilde{x}, \widetilde{u})$:

$$G \quad : \quad (xe^{\alpha\varepsilon}, te^{\varepsilon}, e^{\frac{\alpha}{2}\varepsilon}\sqrt{\frac{2+\gamma x}{2+\gamma x e^{\alpha\varepsilon}}}u).$$

As G is a symmetry group, so if $u(t, x)$ is a solution of equation (1), then is the transformed functions $\varepsilon.u(t, x)$ which is given by

$$\varepsilon.u(t,x) \quad = \quad e^{\frac{\alpha}{2}\varepsilon}\sqrt{\frac{2+\gamma x e^{-\alpha\varepsilon}}{2+\gamma x}}u(e^{-\varepsilon}t, e^{-\alpha\varepsilon}x),$$

where ε is sufficiently small real number. Now we consider the applications of the above transformation. If we start from exact solution obtained in (16), we get a new exact solution of (1)as:

$$u_2(t,x) = e^{\frac{\alpha}{2}\varepsilon}\sqrt{\frac{2+\gamma x e^{-\alpha\varepsilon}}{2+\gamma x}}u_1(e^{-\varepsilon}t, e^{-\alpha\varepsilon}x). \tag{18}$$

Further more, we continue this iteration process, we can derive other exact solutions of equation (1).

4. Conclusion

In this paper, the fractional prolongation formulae of infinitesimal generator was successfully used to compute Lie point symmetries of α-time fractional Kolmogorov equation. It is shown that a set of defining equations for infinitesimals transformations is completely solved. Consequently, the equation (1) is invariant under translation in t generated by X_1, scaling in u generated by X_2, α-scaling generated by X_3 and one parameter group generated by X_h reflecting linearity of equation (1). Therefore, using Lie point symmetry analysis we see that fractional

Kolmogorov equation can be transformed into a second order ordinary differential equation which is solvable in terms of special function and then some exact solutions was found.
An important extension of this analysis is to study this equation with a large family of drift functions. This perspective will be tackled in coming paper.

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