Journal of Fractional Calculus and Applications,
Vol. 5(1) Jan. 2014, pp. 73-84.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

SUFFICIENT CONDITIONS FOR THE EXISTENCE AND UNIQUENESS OF SOLUTIONS TO IMPULSIVE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

RAJIB HALOI, PRADEEP KUMAR AND DWIJENDRA N. PANDEY

Abstract. In this article we prove the sufficient conditions for the existence and uniqueness of piecewise continuous ($\mathcal{PC}$) mild solutions to impulsive fractional integro-differential equations with deviating arguments in a Banach space. The results are obtained by using the theory of analytic semigroup and the Banach fixed point theorem.

1. Introduction

The objective of this article is to study the existence and uniqueness of the solutions to the following problem in a complex Banach space $(X, \|\cdot\|)$:

$$\left.\begin{array}{rcl} {}^C D_t^{\eta} u(t) & = & Au(t) + f(t, u(t), u(\psi(t, u(t)))) \\ & & \quad + \displaystyle\int_0^t a(t,\tau) g(\tau, u(\tau)) d\tau, \quad t \in J = [0, b]; \\ \Delta u|_{t=t_k} & \equiv & I_k(u(t_k^-)) = u(t_k^+) - u(t_k^-), \\ u(0) & = & u_0, \end{array}\right\} \tag{1}$$

where $u : \mathbb{R}_+ \to X$ and $u_0 \in X$. The functions $f : \mathbb{R}_+ \times X \times X \to X$, $g : \mathbb{R}_+ \times X \to X$ and $\psi : \mathbb{R}_+ \times X \to \mathbb{R}_+$ are three non-linear functions and satisfy some appropriate conditions, the function $a : [0, T] \times [0, T] \to \mathbb{C}$ is a continuous function on $[0, T]$ for a fixed $T \in J$. Through the article we denote $u(t_k^+) - u(t_k^-) = x_k$.

Many processes in various fields of engineering and science such as physics, electrochemistry, electro-magnetics, control theory, visco-elasticity, porous media, etc. can be modeled as differential equation with fractional order. The fractional integro-differential equations has played an important role in exploring various characteristics of different branch of science and engineering. The fractional differential equations also describe the memory and hereditary properties of various materials and processes. The plentiful occurrence and applications of fractional differential

---

2000 *Mathematics Subject Classification.* 34G20, 34K37, 34K45, 35R12, 45J05.

Key words and phrases. Impulsive differential equation, Deviating arguments, Analytic semigroup, Banach fixed point theorem.

Submitted Aug. 10, 2013 Accepted Sep. 29, 2013.

equations motivate the rapid developments and gained much attention in the recent years. The details on the theory and its applications can be found in Agrawal et al. [1], Hilfer [14], Kilbas et al.[15, 23], Lakshmikantham et al. [17], Miller and Ross [18], Oldham and Spanier [20], Podlubny [22].

Impulsive effects are common in the process where the short-term perturbations are to be considered. The differential equations with memory effects and impulse effects are medelled as impulsive integro-differential equations. In recent years, there has been a growing interest in the study of fractional differential equations as these equations approach the simulation processes in the control theory, physics, chemistry, population dynamics, biotechnology, economics and so on. The investigation of existence and uniqueness of mild solutions for differential and integro-differential equations with impulse effects have been discussed by many authors [6, 12, 3, 8, 19, 24, 25, 28].

The theory of differential equations with deviating arguments is one of the important and significant branch of nonlinear analysis with numerous applications to physics, mechanics, control theory, biology, ecology, economics, theory of nuclear reactors, engineering, natural sciences, and many other areas of science and technology [7]. Recently, the study of differential equations with impulsive and deviating arguments has studied by some authors [2, 3, 5, 8, 9, 11, 10, 13, 16, 19, 24, 25, 26, 28]. In [26], Wang et al. have discussed the existence and uniqueness of the following fractional differential equation with impulse in a Banach space X,

$$\left.\begin{array}{rcl} {}^CD_t^\eta u(t) &=& Au(t)+f(t,u(t)), \quad t\in J=[0,b], t\neq t_k, 0<\eta\leq 1,\\ \Delta u|_{t=t_k} &\equiv& I_k(u(t_k^-))=u(t_k^+)-u(t_k^-), \quad k=1,2,3,...n,\\ u(0) &=& u_0, \end{array}\right\} \tag{2}$$

where ${}^CD_t^\eta$ denotes the Caputo fractional derivative of order η and $A: D(A)\subset X\to X$ generates a C_0- semigroup on X. The results are established by the fixed point theorem with appropriate f.

Using the theory of analytic semigroup and the Banach fixed point, Borai and Debbouche [5] have studied the existence and uniqueness of solution to the following equation

$$\left.\begin{array}{rcl} {}^CD_t^\eta u(t) &=& Au(t)+f(t,u(t))\\ && +\displaystyle\int_0^t a(t-\tau)g(\tau,u(\tau))d\tau, t\in J=[0,T],\\ u(0) &=& u_0, \end{array}\right\} \tag{3}$$

where $u:\mathbb{R}_+\to X$ and $u_0\in X$. The functions $f:\mathbb{R}_+\times X\to X$, $g:\mathbb{R}_+\times X\to X$ and $\psi:\mathbb{R}_+\times X\to\mathbb{R}_+$ satisfy some appropriate conditions, the function $a:[0,T]\to\mathbb{C}$ is a complex valued continuous function.

However, the study of solutions to impulsive fractional differential equations with deviating arguments need to pay much of attention. The article is devoted to establish the existence and uniqueness of (1) which are new and complement to the existing ones that generalizes some results in [9, 26, 5, 28]. The paper is organized as follows. In Section 2, we recall the definition of the Caputo fractional derivative, Riemann-Liouville integral, the theory of semigroup of bounded linear operators and some lemmas that are used in the remaining part of the article. In Section 3, we study the existence and the uniqueness of $\mathcal{PC}$ mild solutions equation (1). Finally, an example is provided to illustrate the main results in Section 4.

2. Preliminaries and assumptions

In this section, we will introduce some basic definitions, notations and lemmas which are used throughout this paper.

This section is aimed to collect assumptions, preliminaries and lemma required to prove our main results. We briefly outline the facts concerning analytic semigroups, fractional powers of operators, and fractional derivatives. For more details, we refer to [21, 4, 14, 15, 16, 17, 18, 20, 22, 23].

Let $(X, \|\cdot\|)$ be a complex Banach space. Let $\{A|A : D(A) \subset X \to X\}$ be a family of closed linear operators on the Banach space X such that

- **(B1)** The domain $D(A)$ of A is dense in X.
- **(B2)** The resolvent $R(\lambda; A)$ exists for all Re $\lambda \leq 0$ and there is a constant $C > 0$ such that
$$\|R(\lambda; A)\| \leq \frac{C}{|\lambda|+1}, \text{ Re } \lambda \leq 0,$$

Assumptions **(B1)** and **(B2)** implies that $-A$ generates an analytic semigroup of bounded operators, denoted by $S(t)$, $t \geq 0$. Then there exist constants $\tilde{M} \geq 1$ and $\omega \geq 0$ such that
$$\|S(t)\| \leq \tilde{M}e^{\omega t}, \quad t \geq 0.$$

We may assume without loss of generality that $\|S(t)\|$ is uniformly bounded by M, i.e., $\|S(t)\| \leq M$ for $t \geq 0$. We also note that [21, Lemma 4.2,pp. 52]
$$\left\|\frac{d^i}{dt^i}S(t)\right\| \leq M_i, \ t > t_0$$
for some positive constant M_i. It follows from the assumption **(B2)** that the negative fractional powers of the operator A is well defined. For $\alpha > 0$, define the negative fractional powers $A^{-\alpha}$ by
$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty \tau^{\alpha-1}S(\tau)d\tau.$$
It can be seen that $A^{-\alpha}$ is one-to-one and bounded linear operator on X. Define the positive fractional powers of A by $A^\alpha \equiv [A^{-\alpha}]^{-1}$. Then A^α is closed linear operator with dense domain $D(A^\alpha)$ in X, and $D(A^\alpha) \subset D(A^\gamma)$ if $\alpha > \gamma > 0$. For $0 < \alpha \leq 1$, let $X_\alpha = D(A^\alpha)$ and equip this space with the graph norm
$$\|x\|_\alpha = \|A^\alpha x\|.$$
Then X_α is a Banach space endowed with this norm. If $0 < \alpha \leq 1$, the embedding $X_1 \hookrightarrow X_\alpha \hookrightarrow X$ are dense and continuous. For each $\alpha > 0$, define $X_{-\alpha} = (X_\alpha)^*$, the dual space of X_α, and endow with the natural norm
$$\|x\|_{-\alpha} = \|A^{-\alpha}x\|.$$
Then $X_{-\alpha}$ is a Banach space endowed with this norm. The following lemma hold.

Lemma 1 [21] Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$ with $\|S(t)\| \leq M$ for $t \geq 0$ and $0 \in \rho(-A)$. Then we have the following:

- (i) X_α is a Hilbert space for $0 \leq \alpha \leq 1$;
- (ii) For any $0 < \delta \leq \alpha$ implies $D(A^\alpha) \subset D(A^\delta)$, the embedding $X_\alpha \hookrightarrow X_\delta$ is continuous;

(iii) The operator $A^\alpha S(t)$ is bounded for every $t > 0$ and

$$\|A^\alpha S(t)\| \leq C_\alpha t^{-\alpha}.$$

The following assumptions are necessary for proving the main results. Let f, g and ψ be three continuous functions. For $0 < \alpha \leq 1$, let V_α and $V_{\alpha-1}$ be open sets in X_α and $X_{\alpha-1}$ respectively. For each $v_1 \in V_\alpha$ and $v_2 \in V_{\alpha-1}$, there are balls such that $B_1(v_1, r_1) \subset V_\alpha$, and $B_{\alpha-1}(v_2, r_2) \subset V_{\alpha-1}$ for $r_1, r_2 > 0$. We will assume the following conditions.

(B3) There exist constants $C_f = C_f(t, v_1, v_2, r_1, r_2) > 0$ such that the nonlinear map $f : [0, T] \times V_\alpha \times V_{\alpha-1} \to X$ satisfies

$$\|f(t_1, u_1, w_1) - f(t_2, u_2, w_2)\| \leq C_f(\|u_1 - u_2\|_\alpha + \|w_1 - w_2\|_{\alpha-1}) \tag{4}$$

for all $u_1, u_2 \in B_\alpha$, $w_1, w_2 \in B_{\alpha-1}$ and $t_1, t_2 \in [0, T]$.

(B4) There exist constants $C_\psi = C_\psi(v_1, t, r_1) > 0$ such that $\psi(0, \cdot) = 0$, $\psi : [0, T] \times V_\alpha \to [0, T]$ satisfies

$$|\psi(t_1, u_1) - \psi(t_2, u_2)| \leq C_\psi(\|u_1 - u_2\|_\alpha) \tag{5}$$

for all $u_1, u_2 \in B_\alpha$ and $t_1, t_2 \in [0, T]$.

(B5) There exists a positive constant $C_g = C_g(v_1, t, r_1)$ such that the continuous map $g : [0, T] \times V_\alpha \to X$ satisfies

$$\|g(t, x) - g(t, x')\| \leq C_g\|x - x'\|_\alpha \tag{6}$$

for all $x, x' \in B_\alpha$ and $t \in [0, T]$.

(B6) The functions $I_k : X_\alpha \to X_\alpha$ are continuous and there exist constants C_k such that

$$\|I_k(u)\|_\alpha \leq C_k$$

for $k = 1, 2, 3, ..., n$.

(B7) There exist positive constants D_k such that

$$\|I_k(u) - I_k(v)\|_\alpha \leq D_k\|u - v\|_\alpha \tag{7}$$

for $k = 1, 2, 3, ..., n$.

We recall the definition of fractional integral and derivative of a function.

Definition 1 The fractional integral of order η of a real valued absolutely continous function h on $[0, \infty)$ with the lower limit zero is defined as

$$I^\eta h(t) = \frac{1}{\Gamma(\eta)} \int_0^t \frac{h(s)}{(t-s)^{1-\eta}} ds, \ t > 0, \ \eta > 0$$

provided that the right hand side is defined pointwise on $[0, \infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 The Riemann-Liouville derivative of order η of a real valued absolutely continuous function h on $[0, \infty)$ with the lower limit zero is

$$^L D_t^\eta h(t) = \frac{1}{\Gamma(m-\eta)} \frac{d^m}{dt^m} \int_0^t \frac{h(s)}{(t-s)^{\eta+1-m}} ds, \ t > 0, \ m-1 \leq \eta \leq m.$$

Definition 3 The Caputo derivative of order η of a real valued absolutely continous function h on $[0, \infty)$ is defined as

$$^C D_t^\eta h(t) = ^L D_t^\eta \bigg(h(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} h^{(k)}(0) \bigg), \ t > 0, \ m-1 \leq \eta \leq m.$$

We consider the following fractional Cauchy problem:

$$\left.\begin{array}{rcl} {}^C D_t^\eta u(t) &=& Au(t) + f(t) \ t \in J, \\ u(0) &=& u_0. \end{array}\right\} \tag{8}$$

Definition 4 [27] A continuous function $u : J \to X$ is said to be a mild solution of problem (8) if u satisfies the following integral equation

$$u(t) = \mathcal{T}(t)u_0 + \int_0^t (t-s)^{\eta-1}\mathcal{S}(t-s)f(s)ds,$$

where

$$\mathcal{T}(t) = \int_0^\infty \zeta_\eta(\theta)S(t^\eta\theta)d\theta \text{ and } \mathcal{S}(t) = \eta\int_0^\infty \theta\zeta_\eta(\theta)S(t^\eta\theta)d\theta,$$

$$\zeta_\eta(\theta) = \frac{1}{\eta}\theta^{-1-\frac{1}{\eta}} \times \rho_\eta(\theta^{-\frac{1}{\eta}}),$$

$$\rho_\eta(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-n\eta-1}\frac{\Gamma(n\eta+1)}{n!}\sin(n\pi\eta), \ \theta \in (0,\infty),$$

ζ_η is a probability density function defined on $(0,\infty)$, that is

$$\zeta_\eta(\theta) \geq 0, \ \int_0^\infty \zeta_\eta(\theta)d\theta = 1.$$

The following Lemma is useful which is due to Zhou and Jiao [27].
Lemma 2

(i) For any $t \geq 0$, the operators $\mathcal{T}$ and $\mathcal{S}$ are bounded and statisfy

$$\|\mathcal{T}(t)v\| \leq M\|v\| \text{ and } \|\mathcal{S}(t)v\| \leq \frac{\eta M}{\Gamma(1+\eta)}\|v\|$$

for any $v \in X$, respectively.

(ii) The families $\{\mathcal{T}(t) : t \geq 0\}$ and $\{\mathcal{S}(t) : t \geq 0\}$ are strongly continuous.

(iii) For every $t > 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are compact operators if $S(t)$ is compact.

Using Lemma 2 and Lemma 2, the following lemma can be proved.
Definition 5 If $u_0 \in X_\alpha$ and f is a piecewise continuous function on J to X, then $Pu \in X_\alpha$, where the map P is defiend as

$$Pu(t) = \mathcal{T}(t)u_0 + \int_0^t (t-s)^{\eta-1}\mathcal{S}(t-s)f(s)ds.$$

3. Existence of Solutions

In this section we prove the main result for the existence of the solution to the equation (1). We define the following space

$$Y = \mathcal{PC}(X_\alpha) = \{u : J \to X_\alpha : u \in C((t_k, t_{k+1}], X_\alpha), k = 0, 1, \cdots, n, \\ u(t_k^-), u(t_k^+) \text{ exist}\},$$

where $J = [0, T_0]$ for some $0 < T_0 \leq T$. We put $b = T_0$. Then Y is a Banach space endowed with the supremum norm

$$\|u\|_{\mathcal{PC},\alpha} := \max\{\sup_{t\in J}\|u(t+0)\|_\alpha, \sup_{t\in J}\|u(t-0)\|_\alpha\}.$$

For $0 \leq \alpha < 1$, we define

$$Y_1 = \{u \in Y : \|u(t) - u(s)\|_{\alpha-1} \leq L|t-s|, \forall \ t, s \in (t_k, t_{k+1}], k = 0, 1, \cdots, n\},$$

where L is a suitable positive constant to be specified later.

Definition 6 By a $\mathcal{PC}$-mild solution to problem (1), we mean that a function $u \in Y \cap Y_1$ which satisfies the following integral equation

$$u(t) = \begin{cases} \mathcal{T}(t)u_0 \\ \quad + \int_0^t (t-s)^{\eta-1}\mathcal{S}(t-s)\big[f(s,u(s),u(\psi(s,u(s)))) \\ \qquad + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds, \quad t \in [0,t_1], \\ \\ \mathcal{T}(t)u_0 + \mathcal{T}(t-t_1)x_1 \\ \quad + \int_0^t (t-s)^{\eta-1}\mathcal{S}(t-s)\big[f(s,u(s),u(\psi(s,u(s)))) \\ \qquad + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds, \quad t \in (t_1,t_2], \\ \\ \vdots \\ \mathcal{T}(t)u_0 + \sum_{i=1}^k \mathcal{T}(t-t_i)x_i \\ \quad + \int_0^t (t-s)^{\eta-1}\mathcal{S}(t-s)\big[f(s,u(s),u(\psi(s,u(s)))) \\ \qquad + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds, \quad t \in (t_k,b]. \end{cases} \tag{9}$$

For a fixed $R > 0$, we define

$$\mathcal{W} = \{u \in Y \cap Y_1 : u(0) = u_0, \ \ \|u - u_0\|_{\mathcal{PC},\alpha} \le R\}.$$

Then $\mathcal{W}$ is a closed and bounded subset of Y_1 and is a Banach space. We choose T_0, $0 < T_0 \le T$ sufficiently small such that

$$\|(S(t^\eta\theta) - I)A^\alpha u_0\| \le \frac{R}{2} \quad \text{for} \quad t \in [0,T_0], \tag{10}$$

$$C_\alpha(N + \widetilde{N}a_{T_0})\frac{T_0^{\eta(1-\alpha)}}{1-\alpha} + M\sum_{k=1}^n C_k \le \frac{R}{2}, \tag{11}$$

$$[C_f(2 + LC_\psi) + a_{T_0}C_g]C_\alpha\frac{T_0^{\eta(1-\alpha)}}{1-\alpha} + M\sum_{k=1}^n D_k < 1, \tag{12}$$

where $a_{T_0} = \sup_{s\in[0,T_0]}\int_0^{T_0}|a(s,\tau)|d\tau$.

Assumptions **(B3)-(B4)** and $u \in \mathcal{W}$ imply that $f(t,u(t),u(\psi(t,u(t))))$ is continuous on $[0,T_0]$. Hence, there exist positive constants

$$N = C_fR(1 + LC_\psi) + N_0 \qquad \text{and} \quad N_0 = \|f(0,u_0,u_0)\|$$

such that

$$\|f(t,u(t),u(\psi(t,u(t))))\| \le N, \quad \text{for} \quad t \in [0,T_0]. \tag{13}$$

Also assumption **(B5)** implies that there exists a constant $\widetilde{N}$ such that

$$\|g(t,u(t))\| \le \widetilde{N},$$

where $\widetilde{N} = C_gR + \|g(0,u_0)\|$.

Theorem Let the assumptions **(B1)-(B7)** hold. Then Problem (1) has a unique mild solution on $[0,T_0]$.

Proof. For fixed $u_0 \in X_\alpha$, we define a map $\mathcal{P}$ on $\mathcal{W}$ as

$$(\mathcal{P}u)(t) = \begin{cases} \mathcal{T}(t)u_0 \\ \quad + \int_0^t (t-s)^{\eta-1}\mathcal{S}(t-s)\big[f(s,u(s),u(\psi(s,u(s)))) \\ \qquad + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds, \quad t \in [0,t_1], \\ \\ \mathcal{T}(t)u_0 + \mathcal{T}(t-t_1)x_1 \\ + \int_0^t (t-s)^{\eta-1}\mathcal{S}(t-s)\big[f(s,u(s),u(\psi(s,u(s)))) \\ \qquad + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds, \quad t \in (t_1,t_2], \\ \\ \vdots \\ \mathcal{T}(t)u_0 + \sum_{i=1}^k \mathcal{T}(t-t_i)x_i \\ \quad + \int_0^t (t-s)^{\eta-1}\mathcal{S}(t-s)\big[f(s,u(s),u(\psi(s,u(s)))) \\ \qquad + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds, \quad t \in (t_k,T_0] \end{cases} \tag{14}$$

for $u \in \mathcal{W}$. We will show that $\mathcal{P} : \mathcal{W} \to \mathcal{W}$. From Lemma 2, it is clear that $\mathcal{P}u \in Y$. That is $\mathcal{P}$ maps Y into Y itself. We begin with by showing that $\mathcal{P} : Y_1 \to Y_1$. If $u \in Y_1$ and $\varsigma_1, \varsigma_2 \in [0,t_1]$ with $\varsigma_2 > \varsigma_1 > 0$, then for $0 \le \alpha < 1$, we have

$$\begin{aligned} &\|(\mathcal{P}u)(\varsigma_2) - (\mathcal{P}u)(\varsigma_1)\|_{\alpha-1} \\ &\le \|\mathcal{T}(\varsigma_2)u_0 - \mathcal{T}(\varsigma_1)u_0\|_{\alpha-1} \\ &+ \|\int_0^{\varsigma_2} (\varsigma_2-s)^{\eta-1}\mathcal{S}(\varsigma_2-s)\big[f(s,u(s),u(\psi(s,u(s)))) + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds \\ &- \int_0^{\varsigma_1} (\varsigma_1-s)^{\eta-1}\mathcal{S}(\varsigma_1-s)\big[f(s,u(s),u(\psi(s,u(s)))) + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds\|_{\alpha-1} \\ &\le \|\mathcal{T}(\varsigma_2)u_0 - \mathcal{T}(\varsigma_1)u_0\|_{\alpha-1} + I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 = \| &\int_0^{\varsigma_1} \Big[(\varsigma_2-s)^{\eta-1}\mathcal{S}(\varsigma_2-s) - (\varsigma_1-s)^{\eta-1}\mathcal{S}(\varsigma_1-s)\Big]\big[f(s,u(s),u(\psi(s,u(s)))) \\ &+ \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds\|_{\alpha-1}, \end{aligned}$$

$$I_2 = \|\int_{\varsigma_1}^{\varsigma_2} (\varsigma_2-s)^{\eta-1}\mathcal{S}(\varsigma_2-s)\big[f(s,u(s),u(\psi(s,u(s)))) + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds\|_{\alpha-1}$$

Using the definition of $\mathcal{T}$, we get

$$\begin{aligned} \|\mathcal{T}(\varsigma_2)u_0 - \mathcal{T}(\varsigma_1)u_0\|_{\alpha-1} &= \|\int_0^\infty \zeta_\eta(\theta)S(\varsigma_2^\eta\theta)u_0 d\theta - \int_0^\infty \zeta_\eta(\theta)S(\varsigma_2^\eta\theta)u_0 d\theta\|_{\alpha-1} \\ &= \|\int_0^\infty \zeta_\eta(\theta)\int_{\varsigma_1}^{\varsigma_2} \frac{d}{d\varsigma}S(\varsigma^\eta\theta)u_0 d\varsigma d\theta\|_{\alpha-1} \\ &\le M_1\|u_0\|_{\alpha-1}(\varsigma_2-\varsigma_1). \end{aligned}$$

Again using the definition of $\mathcal{S}$, we obtain

$$
\begin{aligned}
I_1 = \| \int_0^{\varsigma_1} & \Big[(\varsigma_2 - s)^{\eta-1}\mathcal{S}(\varsigma_2 - s) - (\varsigma_1 - s)^{\eta-1}\mathcal{S}(\varsigma_1 - s)\Big] \\
& \big[f(s,u(s),u(\psi(s,u(s)))) + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds\|_{\alpha-1} \\
& = \| \int_0^{\varsigma_1} \Big[(\varsigma_2 - s)^{\eta-1}\eta\int_0^\infty \theta\zeta_\eta(\theta)S((\varsigma_2 - s)^\eta\theta)d\theta \\
& \qquad - (\varsigma_1 - s)^{\eta-1}\eta\int_0^\infty \theta\zeta_\eta(\theta)S((\varsigma_1 - s)^\eta\theta)d\theta\Big] \\
& \times \big[f(s,u(s),u(\psi(s,u(s)))) + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds\|_{\alpha-1} \\
& = \| \int_0^{\varsigma_1}\int_0^\infty \zeta_\eta(\theta)\Big[\frac{d}{d\varsigma}S((\varsigma - s)^\eta\theta)\Big]_{\varsigma=\varsigma_2} - \Big[\frac{d}{d\varsigma}S((\varsigma - s)^\eta\theta)\Big]_{\varsigma=\varsigma_1} d\theta \\
& \big[f(s,u(s),u(\psi(s,u(s)))) + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds\|_{\alpha-2} \\
& = \| \int_0^{\varsigma_1}\int_0^\infty \zeta_\eta(\theta)\int_{\varsigma_1}^{\varsigma_2}\frac{d^2}{d\varsigma^2}S((\varsigma - s)^\eta\theta)d\varsigma d\theta\big[f(s,u(s),u(\psi(s,u(s)))) \\
& + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds\|_{\alpha-2} \\
& \le T_0(N + \widetilde{N}a_{T_0})M_2\|A^{\alpha-2}\|(\varsigma_2 - \varsigma_1).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
I_2 & = \| \int_{\varsigma_1}^{\varsigma_2}(\varsigma_2 - s)^{\eta-1}\mathcal{S}(\varsigma_2 - s)\big[f(s,u(s),u(\psi(s,u(s)))) \\
& \qquad + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]ds\|_{\alpha-1} \\
& = \| \int_{\varsigma_1}^{\varsigma_2}\int_0^\infty \zeta_\eta(\theta)\frac{d}{d\varsigma}S((\varsigma - s)^\eta\theta)\Big|_{\varsigma=\varsigma_2} \\
& \qquad \times \big[f(s,u(s),u(\psi(s,u(s)))) + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau\big]d\theta ds\|_{\alpha-2} \\
& \le (N + \widetilde{N}a_{T_0})M_1\|A^{\alpha-2}\|(\varsigma_2 - \varsigma_1).
\end{aligned}
$$

If $u \in Y_1$ and $\varsigma_1, \varsigma_2 \in (t_1, t_2]$ with $\varsigma_2 > \varsigma_1 > 0$, then for $0 \le \alpha < 1$, we have

$$
\begin{aligned}
& \|(\mathcal{P}u)(\varsigma_2) - (\mathcal{P}u)(\varsigma_1)\|_{\alpha-1} \\
& \le \|\mathcal{T}(\varsigma_2)u_0 - \mathcal{T}(\varsigma_1)u_0\|_{\alpha-1} + \|\mathcal{T}(\varsigma_2 - t_1)x_1 - \mathcal{T}(\varsigma_1 - t_1)x_1\|_{\alpha-1} \\
& + \| \int_0^{\varsigma_2}(\varsigma_2 - s)^{\eta-1}\mathcal{S}(\varsigma_2 - s)\big[f(s,u(s),u(\psi(s,u(s)))) + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau,\big]ds \\
& - \int_0^{\varsigma_1}(\varsigma_1 - s)^{\eta-1}\mathcal{S}(\varsigma_1 - s)\big[f(s,u(s),u(\psi(s,u(s)))) + \int_0^s a(s,\tau)g(\tau,u(\tau))d\tau,\big]ds\|_{\alpha-1} \\
& \le \|\mathcal{T}(\varsigma_2)u_0 - \mathcal{T}(\varsigma_1)u_0\|_{\alpha-1} + \|\mathcal{T}(\varsigma_2 - t_1)x_1 - \mathcal{T}(\varsigma_1 - t_1)x_1\|_{\alpha-1} + I_1 + I_2,
\end{aligned}
$$

where

$$I_1 = \| \int_0^{\varsigma_1} \Big[(\varsigma_2 - s)^{\eta-1}\mathcal{S}(\varsigma_2 - s) - (\varsigma_1 - s)^{\eta-1}\mathcal{S}(\varsigma_1 - s) \Big] \big[f(s, u(s), u(\psi(s, u(s)))) + \int_0^s a(s,\tau) g(\tau, u(\tau)) d\tau \big] ds \|_{\alpha-1},$$

$$I_2 = \| \int_{\varsigma_1}^{\varsigma_2} (\varsigma_2 - s)^{\eta-1}\mathcal{S}(\varsigma_2 - s) \big[f(s, u(s), u(\psi(s, u(s)))) + \int_0^s a(s,\tau) g(\tau, u(\tau)) d\tau \big] ds \|_{\alpha-1}.$$

We have the following estimate

$$\begin{aligned} \|\mathcal{T}(\varsigma_2 - t_1)x_1 - \mathcal{T}(\varsigma_1 - t_1)x_1)\|_{\alpha-1} &= \| \int_0^\infty \zeta_\eta(\theta) \frac{d}{d\varsigma} S((\varsigma - t_1)^\eta \theta) x_1 d\varsigma d\theta \|_{\alpha-1} \\ &\leq M_1 \|x_1\|_{\alpha-1} (\varsigma_2 - \varsigma_1). \end{aligned}$$

We note that the estimates for I_1 and I_2 are same as in the previous case. In a similar way, we have the similar estimates for $t \in (t_2, t_3], t \in (t_3, t_4],..., t \in (t_k, T_0]$.

Thus we have $\mathcal{P}u \in Y_1$ for a suitable positive constant

$$L = \max\{M_1 \|u_0\|_{\alpha-1}, M_1\|x_1\|_{\alpha-1}, T_0(N + \widetilde{N} a_{T_0}) M_2 \|A^{\alpha-1}\|, (N + \widetilde{N} a_{T_0}) M_1 \|A^{\alpha-2}\|\}.$$

Next we will show that $\mathcal{P} : \mathcal{W} \to \mathcal{W}$. For $t \in (0, t_1]$ and $u \in \mathcal{W}$, we have

$$\begin{aligned} &\|\mathcal{P}u(t) - u_0\|_\alpha \\ &\leq \int_0^\infty \zeta_\eta(\theta) \|(S(t^\eta\theta) - I)A^\alpha u_0\| d\theta + \eta \int_0^t \int_0^\infty \theta \zeta_\eta(\theta)(t-s)^{\eta-1} \|S((t-s)^\eta\theta)A^\alpha\| \\ &\qquad \big[f(s, u(s), u(\psi(s, u(s)))) + \int_0^s a(s,\tau) g(\tau, u(\tau)) d\tau \big] ds \| \| d\theta ds \\ &\leq \frac{R}{2} + C_\alpha (N + \widetilde{N} a_{T_0}) \eta \int_0^t \int_0^\infty \theta^{1-\alpha} \zeta_\eta(\theta)(t-s)^{-\eta\alpha+\eta-1} d\theta ds \\ &\leq \frac{R}{2} + C_\alpha (N + \widetilde{N} a_{T_0}) \frac{T_0^{\eta(1-\alpha)}}{1-\alpha} \\ &\leq R. \end{aligned}$$

For $t \in (t_1, t_2]$ and $u \in \mathcal{W}$, we use then we have

$$\begin{aligned} &\|\mathcal{P}u(t) - u_0\|_\alpha \\ &\leq \int_0^\infty \zeta_\eta(\theta) \|(S(t^\eta\theta) - I)A^\alpha u_0\| d\theta + \|\mathcal{T}(t - t_1)x_1\|_\alpha \\ &\quad + \eta \int_0^t \int_0^\infty \theta \zeta_\eta(\theta)(t-s)^{\eta-1} \|S((t-s)^\eta\theta)A^\alpha\| \\ &\qquad \big[f(s, u(s), u(\psi(s, u(s)))) + \int_0^s a(s,\tau) g(\tau, u(\tau)) d\tau \big] ds \| \| d\theta ds \\ &\leq \frac{R}{2} + MC_1 + C_\alpha (N + \widetilde{N} a_{T_0}) \eta \int_0^t \int_0^\infty \theta^{1-\alpha} \zeta_\eta(\theta)(t-s)^{-\eta\alpha+\eta-1} d\theta ds \\ &\leq \frac{R}{2} + MC_1 + C_\alpha (N + \widetilde{N} a_{T_0}) \frac{T_0^{\eta(1-\alpha)}}{1-\alpha} \\ &\leq R. \end{aligned}$$

Thus we can also prove that $\|\mathcal{P}u(t) - u_0\|_\alpha \leq R$ for any $t \in [0, T_0]$. Hence, $\mathcal{P} : \mathcal{W} \to \mathcal{W}$.

Finally, we will claim that $\mathcal{P}$ is a contraction map. For $t \in [0, T_0]$, we have

$$
\begin{aligned}
&\|\mathcal{P}u(t) - \mathcal{P}v(t)\|_\alpha \\
&\leq M \sum_{k=1}^{n} D_k \|u-v\|_\alpha + \eta \int_0^t \int_0^\infty \theta \zeta_\eta(\theta)(t-s)^{\eta-1} \|S((t-s)^\eta \theta) A^\alpha\| \\
&\quad \|[f(s,u(s),u(\psi(u(s),s))) - f(s,v(s),v(\psi(v(s),s))))]\| \\
&+ \int_0^s |a(s,\tau)| \|g(\tau,u(\tau)) - g(\tau,v(\tau))\| d\tau d\theta ds \\
&\leq M \sum_{k=1}^{n} D_k \|u-v\|_\alpha + \eta [C_f(2+LC_\psi) + a_{T_0} C_g] C_\alpha \|u-v\|_\alpha \\
&\qquad \times \int_0^t \int_0^\infty \theta \zeta_\eta(\theta)(t-s)^{\eta-1} \|S((t-s)^\eta \theta) A^\alpha\| ds \\
&\leq \left[M \sum_{k=1}^{n} D_k + [C_f(2+LC_\psi) + a_{T_0} C_g] C_\alpha \frac{T_0^{\eta(1-\alpha)}}{1-\alpha} \right] \|u-v\|_\alpha.
\end{aligned}
$$

From (12), it follows that P is a contraction on $\mathcal{W}$. By the Banach contraction mapping principle, the map P has fixed point in $\mathcal{W}$.

□

4. Application

In this section, we consider the following fractional differential equation with a deviating argument to illustrate the theory. For $0 < T < \infty$ and $(x,t) \in (0,1) \times (0,T)$, [9, 11]

$$
\left.
\begin{aligned}
\frac{\partial^\beta u}{\partial t^\beta} &= \frac{\partial^2 u}{\partial x^2} + \tilde{H}(x,u(x,t)) + G(t,x,u(x,t)) \\
&\qquad + \int_0^t a(t,\tau) g(\tau,u(\tau)) d\tau, \\
\Delta u|_{t=\frac{1}{2}} &= \frac{2u(\frac{1}{2}^-)}{2+u(\frac{1}{2}^-)} \\
u(0,t) &= u(1,t) = 0, \\
u(x,0) &= u_0(x),\ x \in (0,1),
\end{aligned}
\right\} \tag{15}
$$

where $\beta \in (0,1)$,

$$
\tilde{H}(x,u(x,t)) = \int_0^x K(x,y) u(y, g_0(t)|u(y,t)|) dy,
$$

and the function $G : \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is measurable in x, locally Hölder continuous in t, locally Lipschitz continuous in u, uniformly in x. Assume that $g_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Hölder continuous in t with $g_0(0) = 0$ and $K \in C^1([0,1] \times [0,1]; \mathbb{R})$. We take $X = L^2((0,1); \mathbb{R})$, $Au = \dfrac{d^2u}{dx^2}$, $D(A) = H^2(0,1) \cap H_0^1(0,1)$ and $X_{1/2} = D((-A)^{1/2}) = H_0^1(0,1)$ and $X_{-1/2} = (H_0^1(0,1))^* = H^{-1}(0,1) \equiv H^1(0,1)$. For $x \in (0,1)$, we define $F : \mathbb{R}_+ \times H_0^1(0,1) \times H^1(0,1) \to L^2(0,1)$ by

$$
F(t,\phi,\psi) = H(x,\psi) + G(t,x,\phi),
$$

where

$$
H(x,\psi(x,t)) = \int_0^x K(x,y) \psi(y,t) dy.
$$

Then the semigroup is given by

$$S(t)u = \sum_{n\in\mathbb{N}} \exp(-n^2\pi^2 t)\langle u, u_m\rangle u_m$$

for $u \in D(A)$, where $u_n(x) = \sin(n\pi x)$. Also the assumptions **(B3)** and **(B4)** are satisfied [11]. If $u, v \in D((-A)^{1/2})$, then

$$\|I_k(u) - I_k(v)\|_{\frac{1}{2}} \leq \frac{2\|u-v\|_{\frac{1}{2}}}{\|(2+u)(2+v)\|_{\frac{1}{2}}} \leq \frac{1}{2}\|u-v\|_{\frac{1}{2}}.$$

That is the assumption **(B7)** is satisfied. Thus Problem (15) has a unique solution if we choose some appropriate function g that satisfies **(B5)** and a continues function a.

Acknowledgements

The authors would like to thank the referees for their valuable remarks that have helped us to improve the original manuscript.

References

[1] R. P. Agarwal, M. Benchohra and S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, Acta Appl. Math. 109 (2010) 973–1033.

[2] M. Benchohra and B. A. Slimani, Existence and uniqueness of solutions to impulsive fractional differential equations, Electron. J. Differential Equations 10 (2009) 1–11.

[3] K. Balachandran, S. Kiruthika and J. J. Trujillo, Remark on the existence results for fractional impulsive integrodifferential equations in Banach spaces, Commun. Nonlinear Sci. Numer. Simul. 17 (2012), no. 6, 2244–2247.

[4] M. M. El-Borai, Some probability densities and fundamental solutions of fractional evolution equations, Chaos Solitons Fractals 149 (2004) 823–831.

[5] Mahmoud M. El-Borai and A. Debbouche, On Some Fractional Integro-Differential Equations with Analytic Semigroups, Int. J. Contemp. Math. Sciences, Vol. 4, 2009, no. 28, 1361 - 1371.

[6] M. G. Crandall, S. O. Londen, and J. A. Nohel, An abstract nonlinear Volterra integro-differential equation, J. Math. Anal. Appl., 64 (1978), pp. 701–735.

[7] L. E. El'sgol'ts and S. B. Norkin, : *Introduction to the Theory of Differential Equations with Deviating Arguments*, Academic Press (1973).

[8] M. Feckan, Y. Zhou and J. R. Wang, On the concept and existence of solution for impulsive fractional differential equations, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 3050–3060.

[9] C. G. Gal, Nonlinear abstract differential equations with deviated argument, J. Math. Anal. Appl., 333(2)(2007), pp. 971–983.

[10] R. Haloi, D. Bahuguna and D. N. Pandey, Existence and uniqueness of solutions for quasi-linear differential equations with deviating arguments, Electron. J. Differential Equations, 2012 (13) (2012), , pp. 1–10.

[11] R. Haloi, D. N. Pandey, and D. Bahuguna, Existence and Uniqueness of a Solution for a Non-Autonomous Semilinear Integro-Differential Equation with Deviated Argument, Differ. Equ. Dyn. Syst, 20(1), 2012, 1–16.

[12] M. L. Heard, S. M. Rankin, A semilinear parabolic Volterra intgro-differential equation, Journal of Differential Equations, 71(1988), pp. 201–233.

[13] H. Jiang, Existence results for fractional order functional differential equations with impulse. Comput. Math. Appl. 64 (2012), no. 10, 3477–3483.

[14] H. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publ. Co., Singapore, 2000.

[15] A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, in: North-Holland Mathematics Studies, vol. 204, Amsterdam, 2006.

[16] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, Worlds Scientific, Singapore, 1989.

[17] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, *Theory of Fractional Dynamic Systems, Cambridge Academic*, Cambridge, UK, (2009).

[18] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations* Wiley, New York, 1993.

[19] G. M. Mophou, Existence and uniqueness of mild solutions to impulsive fractional differential equations. Nonlinear Anal. 72 (2010), no. 3-4, 1604–1615.

[20] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, 1974.

[21] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, 1983.

[22] I. Podlubny, *Fractional Differential Equations*, Math Science and Eng., 198. Academic Press, San Diego 1999.

[23] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[24] X. B. Shu, Y. Lai and Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, Nonlinear Anal. 74, 2003–2011 (2011).

[25] G. T. Wang, B. Ahmad and L. H. Zhang, Some existence results for impulsive nonlinear fractional differential equations with mixed boundary conditions, Comput. Math. Appl. 62 (2011) 1389–1397.

[26] J. Wang, M. Fečkan and Y. Zhou, On the new concept of solutions and existence results for impulsive fractional evolution equations. Dyn. Partial Differ. Equ. 8 (2011), no. 4, 345–361.

[27] Y. Zho and F. Jiao, Existence of mild solutions for fractional neutral evoultion equations, Comp. Math. Appl., 59 (2010), 1063–1077.

[28] X. Zhang, C. Zhu, Z. Wu, The Cauchy problem for a class of fractional impulsive differential equations with delay. Electron. J. Qual. Theory Differ. Equ. 2012, No. 37, 1–13.

Rajib Haloi
Department of Mathematical Sciences, Tezpur University, INDIA, Pin- 784028.
E-mail address: rajib.haloi@gmail.com

Pradeep Kumar, Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, INDIA, Pin- 208016.
E-mail address: prdipk@gmail.com

Dwijendra N. Pandey, Department of Mathematics, Indian Institute of Technology Roorkee, INDIA, Pin- 247667.
E-mail address: dwij.iitk@gmail.com