Journal of Fractional Calculus and Applications
Vol. 14(1) Jan. 2023, pp. 228-235.
ISSN: 2090-5858.
http://math-frac.org/Journals/JFCA/

BERNSTEIN COLLOCATION TECHNIQUE FOR VOLTERRA-FREDHOLM FRACTIONAL ORDER INTEGRO-DIFFERENTIAL EQUATIONS

T. OYEDEPO , G. AJILEYE, A. M. AYINDE, I. J. OTAIDE

Abstract. In this study, we solve Fractional Volterra-Fredholm Integro-Differential Equations (FVFIDEs) using the Bernstein Collocation Technique (BCT). The approach breaks the problem down into a set of linear algebraic equations, which are then resolved by matrix inversion to get the unknown constants. The accuracy and effectiveness of the procedure are demonstrated using numerical examples in tables and figures. The outcomes demonstrate that the strategy worked better in terms of increasing accuracy and necessitating less strenuous labour.

1. Introduction

This work focuses on fractional calculus, which is calculus with fractional derivatives. The ideal is that we have the first derivative, which is velocity, and the second derivative, which is acceleration, and to be able to have any derivative between the first and second derivatives. [1], [2], [3], [4], and among others claim that Leibniz made the discovery in 1695, just a few years after making the discovery of ordinary calculus, but due to the complicated formula for these fractional derivatives, it was later forgotten, making it difficult to work with ordinary pencil and paper, but now that we have computers and machines running, complexity is no longer a problem. The best way to model anomalous phenomena, such as heat spreading in a furnace, plasma, or the flow of water beneath the ground, is with fractional calculus. It is also used to model the spread of virus, satellite disposition in space, and system memory behavior. Mathematicians and other scientists have developed a keen interest in fractional calculus, which has led to a great deal of recent attention being paid to fractional differential and FVFIDE solutions. Finding accurate approximations utilizing numerical techniques would be very helpful because many FVFIDEs cannot be solved analytically. Many authors have presented numerical methods for solving the FVFIDEs, including the following: Adomian decomposition technique (ADM) was utilized by [5] to solve Fractional Integro-Differential Equations

2010 *Mathematics Subject Classification.* 49M27, 45J05, 26A33.
Key words and phrases. Approximate solutions, Bernstein collocation technique, Caputo derivative, Fractional order integro-differential equations, Volterra-Fredholm.
Submitted Sep. 28, 2022. Revised Dec. 12, 2022.

(FIDEs), Bernstein polynomials were employed as basis functions by [6] to approximate the solution of FIDEs. [7] and [8] presented the Least Squares Method (LSM) for solving FIDEs. [9] and [10] used the collocation method for solving FIDEs.[11] used Laguerre polynomials as a basis, and [12] presented fractional order approximations to FVFIDEs. [13] used the Chebyshev wavelet method to solve nonlinear FVFIDEs with mixed boundary conditions. [14] introduced numerical solution of FVFIDEs with mixed boundary conditions using the Chebyshev wavelet method; [15] used a combination of Lucas wavelets and Legendre-Gauss quadrature; [16]used Lagrange polynomials; and so on. Motivated and inspired by the preceding work, we propose Bernstein Collocation Techniques with improving accuracy and less rigorous work for FVFIDEs. In this work, the fractional derivative for the problem under consideration is taken for α=0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1, yielding various approximate solutions. The class of problem studied in this work is:

$$\mu_2\varphi^{''}(x)+\mu_1\varphi^{'}(x)+\mu_\alpha D^\alpha\varphi(x)+\mu_0\varphi(x)=f(x)+\lambda_1\int_0^x K_1(x,t)\varphi(x)dt + \lambda_2\int_0^1 K_2(x,t)\varphi(x)dt, \tag{1}$$

Subject to this boundary conditions

$$\varphi(a)=0,\varphi(b)=0,a<x<b \tag{2}$$

$K_1(x,t)$ and $K_2(x,t)$ are the Fredholm and Volterra intergral kernel functions, μ_1, μ_2, μ_α, λ_1 and λ_2 are known constants, $f(x)$ is a known function and $\varphi(x)$ is the unknown function to be determined. Where $D^\alpha\varphi(x)$ indicates the α^{th} Caputo fractional derivative of $\varphi(x)$.

Definition 1. The Caputo Fractional Derivative is defined as [17]:

$$D^\alpha f(x)=\frac{1}{\Gamma(j-\alpha)}\int_0^x (x-s)^{j-\alpha-1}f^j(s)ds \tag{3}$$

where j is a positive integer with the property that $j-1<\alpha<j$. For example if $0<\alpha<1$ the Caputo fractional derivative is

$$D^\alpha f(x)=\frac{1}{\Gamma(1-\alpha)}\int_0^x (x-s)^{-\alpha}f^{'}(s)ds \tag{4}$$

Definition 2. Bernstein basis polynomials: A Bernstein polynomial [18] of degree j is defined by:

$$\varphi(x)=\xi_{i,j}(x)=\binom{j}{i}x^i(1-x)^{j-i}c_i \; i=0,1...j \tag{5}$$

where

$$\binom{j}{i}=\frac{j!}{i!(j-i)!}$$

and c_i, $i=0,1,2,\cdots$

The following are the few Bernstein basis polynomials:

when $j=0$, $\varphi(x)=1$

when $j=1$, $\varphi(x)=c_0(x-1)+c_1x$

when $j = 2$, $\varphi(x) = c_0(1 - 2x + x^2) + c_1(2x - 2x^2) + c_2x^2$

Definition 3. Here, we defined Absolute Error (AE) as follows:

$$Absolute\ Error = |\Phi(x) - \varphi(x)|; 0 \le x \le 1 \tag{6}$$

where the Exact Solution (ES) is $\Phi(x)$ and the Approximate Solution (AS) is $\varphi(x)$.

2. Demonstration of the suggested method

Bernstein Collocation Technique (BCT). The approach relies on approximating the unknown function $\varphi(x)$ by assuming an approximation solution of the kind specified in equation (5), using equation (2) on equation (1), and getting the following result for equation (1) :

$$\mu_2\varphi^{''}(x) + \mu_1\varphi^{'}(x) + \mu_\alpha(\frac{1}{\Gamma(1-\alpha)}\int_0^x (x-t)^{j-\alpha-1}\frac{d^j}{dt^j}\varphi(t)dt) + \mu_0\varphi(x) = f(x) +$$

$$\lambda_1\int_0^x K_1(x,t)\varphi(x)(t)dt + \lambda_2\int_0^1 K_2(x,t)\varphi(x)(t)dt, \tag{7}$$

$$Let\ \zeta(x) = \mu_\alpha(\frac{1}{\Gamma(1-\alpha)}\int_0^x (x-s)^{j-\alpha-1}\frac{d^j}{dt^j}\varphi(t)dt),\ \eta(x) = \lambda_1\int_0^x K_1(x,t)\varphi(x)(t)dt,$$

$$\tau(x) = \lambda_2\int_0^1 K_2(x,t)\varphi(x)(t)dt$$

Substituting $\zeta(x)$, $\eta(x)$ and $\tau(x)$ in equation (7)

$$\mu_2\varphi^{''}(x) + \mu_1\varphi^{'}(x) + \zeta(x) + \mu_0\varphi(x) - \eta(x) - \tau(x) = f(x) \tag{8}$$

Collocating equation (8) at $x_i = a + \frac{(b-a)i}{j+1}$, $(i = 1(1)(j+1))$ gives linear system algebraic of equations in $(j+1)$ unknown constants $c_i's$. Additional two equations are obtained using the boundary conditions, which are represented in matrix form:

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & \cdots & \cdots & \cdots & M_{1n} \\ M_{21} & M_{22} & M_{23} & \cdots & \cdots & \cdots & M_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ M_{m1} & M_{m2} & M_{m3} & \cdots & \cdots & \cdots & M_{mn} \\ M^*_{11} & M^*_{12} & M^*_{13} & \cdots & \cdots & \cdots & M^*_{1n} \\ M^*_{21} & M^*_{22} & M^*_{23} & \cdots & \cdots & \cdots & M^*_{2n} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} N_{11} \\ N_{22} \\ \vdots \\ \vdots \\ Nmn \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

where M_{is} and M^*_{is} are the coefficients of c_{is} given as:
$M_{11}, M_{12}, M_{13}, ...\ M_{1n} = \mu_2\varphi^{''}(x_1) + \mu_1\varphi^{'}(x_1) + \zeta(x_1) + \mu_0\varphi(x_1) - \eta(x_1) - \tau(x_1)$,
$M_{21}, A_{22}, M_{23}, ...\ M_{2n} = \mu_2\varphi^{''}(x_2) + \mu_1\varphi^{'}(x_2) + \zeta(x_2) + \mu_0\varphi(x_2) - \eta(x_2) - \tau(x_2)$,
$M_{31}, M_{32}, A_{33}, ...\ M_{3n} = \mu_2\varphi^{''}(x_3) + \mu_1\varphi^{'}(x_3) + \zeta(x_3) + \mu_0\varphi(x_3) - \eta(x_3) - \tau(x_3)$
$M^*_{11}, M^*_{12}, M^*_{13}, ...\ M^*_{1n} = \varphi(a), M^*_{21}, M^*_{22}, M^*_{23}, ...\ M^*_{2n} = \varphi(b)$, and N_{is} are values of

$f(x_i)$. The matrix inversion approach is then used to solve the system of equations in order to get the unknown constants.

$$\begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & \cdots & \cdots & \cdots & M_{1n} \\ M_{21} & M_{22} & M_{23} & \cdots & \cdots & \cdots & M_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \\ \vdots & \vdots & \vdots & & \vdots & & \\ M_{m1} & M_{m2} & M_{m3} & \cdots & \cdots & \cdots & M_{mn} \\ M_{11}^* & M_{12}^* & M_{13}^* & \cdots & \cdots & \cdots & M_{1n}^* \\ M_{21}^* & M_{22}^* & M_{23}^* & \cdots & \cdots & \cdots & A_{2n}^* \end{pmatrix}^{-1} \begin{pmatrix} N_{11} \\ N_{22} \\ \vdots \\ \vdots \\ Nmn \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

Solving equation (10) to get the values of unknown constant which are substituted back into the assumed approximate solution to get the required approximate solution.

3. Numerical Applications

Example 1: Consider the following FIDE [12]

$$\varphi^{''}(x) + \frac{1}{x}D^{\alpha}\varphi(x) + \frac{1}{x^2}\varphi(x) - \int_0^x \sin(x-t)\varphi(t)dt - \int_0^1 \cos(x-t)\varphi(t)dt$$

$$= 1.50451x^{\frac{1}{2}} - 13x - \frac{180541}{100000}x^{\frac{3}{2}} - x^2 + x^3 - \frac{2067}{1000}x\cos(x) + \frac{595385}{100000}\sin(x) \tag{11}$$

$\varphi(0) = 0$, $\varphi(1) = 0$, for $\alpha = 0.5$, the exact solution is $\varphi(x) = x^2 - x^3$. Using the suggested method for various values of $\alpha = 0.1, 0.2, o.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$, we have the following approximate solutions.

For $\alpha = 0.1$, $\varphi(x) = -0.01098588449x^5 + 0.07304092856x^4 - 0.995230193x^3 + 0.879438378x^2 + 0.01106390911x + 3.552713679 \times 10^{-14}$

For $\alpha = 0.2, -0.008943751595x^5 + 0.06025371458x^4 - 0.9986701444x^3 + 0.9029646681x^2 + 0.009100041105x - 5.329070518 \times 10^{-15}$

For $\alpha = 0.3, -0.0064430822x^5 + 0.044109271x^4 - 1.001215500x^3 + 0.9309930461x^2 + 0.006638926830$

For $\alpha = 0.4, -0.003461973034x^5 + 0.02415682014x^4 - 1.002005896x^3 + 0.9634405890x^2 + 0.003621558466x + 7.105427358 \times 10^{-15}$

For $\alpha = 0.5, 1.170384552 \times 10^{-7}x^5 - 1.595538298 \times 10^{-7}x^4 - 0.9999997042x^3 + 0.9999989556x^2 -$
$3.536126058 \times 10^{-8}x - 5.684341886 \times 10^{-14}$

For $\alpha = 0.6, -0.003906450454x^5 - 0.02859369682x^4 - 0.9940850253x^3 + 1.040072119x^2 - 0.004251101546x - 7.105427358 \times 10^{-14}$

For $\alpha = 0.7, 0.008171856510x^5 - 0.06156205804x^4 - 0.9832340789x^3 + 1.082713192x^2 - 0.009115484491x + 1.136868377 \times 10^{-13}$

For $\alpha = 0.8, 0.01265688334x^5 - 0.09840341234x^4 - 0.9667374650x^3 + 1.126590898x^2 - 0.01451302266x + 1.705302566 \times 10^{-13}$

For $\alpha = 0.9, 0.01716593076x^5 - 0.1380207200x^4 + 0.9445330664x^3 + 1.170035228x^2 - 0.02028349224x + 1.136868377 \times 10^{-13}$

For $\alpha = 1, 0.01260222492x^5 + 0.08291190306x^4 - 0.9916111581x^3 + 0.8603389039x^2 + 0.01259110448x + 2.131628207 \times 10^{-13}$

TABLE 1. Comparison of the AE is shown in Table 1 for example 1

x	[12] AE j=4	[12] AE j=32	Our Method AE J=5
0.0	−	−	$5.684E-14$
0.2	$2.330E-2$	$2.048E-5$	$6.670E-8$
0.4	$2.690E-2$	$2.503E-5$	$1.652E-7$
0.6	$1.011E-2$	$1.789E-5$	$3.449E-7$
0.8	$3.697E-3$	$7.682E-5$	$5.723E-7$
1.1	−	-	$8.265E-7$

Table 1 demonstrates that our approach performed more accurately than [12], and Figure 1 displays the approximate solution for various values of $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$.

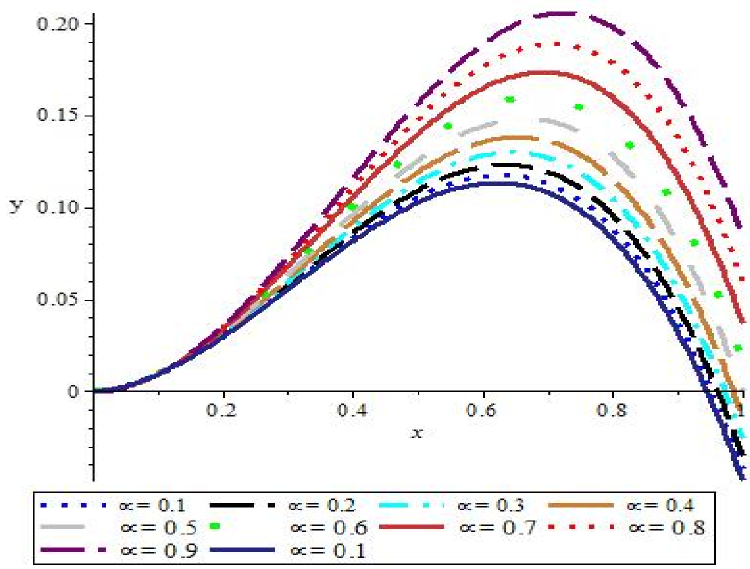

FIGURE 1. Displaying the graphical behavior of example 1's ES and AS

Example 2: Reference considers [12] FIDE

$$\varphi^{''}(x) + D^{\alpha}\varphi(x) - 2\int_0^x (x-t)\varphi(t)dt - \int_0^1 (x^2-t)\varphi(t)dt$$
$$= \frac{1}{30} - 6x - \frac{181x^2}{20} + 4x^3 - \frac{x^5}{10} + \frac{x^6}{15} \quad (12)$$

$\varphi(0) = 0$, $\varphi(1) = 0$, for $\alpha = 1$, the exact solution is $\varphi(x) = x^4 - x^3$. Using the suggested method for various values $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$, we have the following approximate solutions.

For $\alpha = 0.1$, $\varphi(x) = 0.009175645816x^5 + 0.9921654864x^4 - 1.005058512x^3 + 0.001416362828x^2 + 0.002301126448x - 4.955635769 \times 10^{-8}$

For $\alpha = 0.2$, $\varphi(x) = 0.02073681840x^5 + 0.9815451666x^4 - 1.010391949x^3 + 0.00291542454x^2 + 0.005194750461x - 9.1923244818 \times 10^{-8}$

For $\alpha = 0.3$, $\varphi(x) = 0.03487738114x^5+0.9675347720x^4-1.015494815x^3+0.00435094048x^2+0.008732014642x - 1.194294611 \times 10^{-8}$
For $\alpha = 0.4$, $\varphi(x) = 0.05174521116x^5+0.9495191575x^4-1.019764534x^3+0.0055416762x^2+0.01295882468x - 1.263124507 \times 10^{-7}$
For $\alpha = 0.5$, $\varphi(x) = 0.07140013477x^5+0.9269557121x^4-1.022544839x^3+0.00628135746x^2+0.01790796527x - 1.100546865 \times 10^{-7}$
For $\alpha = 0.6$, $\varphi(x) = 0.09376065321x^5+0.8994913703x^4-1.023197684x^3+0.00635853356x^2+0.02358740594x - 7.420652044 \times 10^{-8}$
For $\alpha = 0.7$, $\varphi(x) = 0.1185439063x^5+0.8671071458x^4-1.021202676x^3+0.0055875413x^2+0.02996427503x - 2.874659927 \times 10^{-8}$
For $\alpha = 0.8$, $\varphi(x) = 0.1452072917x^5+0.8302743464x^4-1.016175743x^3+0.038494765x^2+0.03694472287x - 9.990275198 \times 10^{-9}$
For $\alpha = 0.9$, $\varphi(x) = 0.1729049655x^5+0.7900931958x^4-1.008487683x^3+0.0011378685x^2+0.04435167467x + 2.268940275 \times 10^{-9}$
For $\alpha = 1$, $\varphi(x) = 6.354667771 \times 10^{-9}x^5 + x^4 - x^3 - 7.003654019 \times 10^{-10}x^2 + 7.477633656 \times 10^{-10}x + 1.139230281 \times 10^{-12}$

Table 2. Comparison of the AE is shown in Table 2 for example 2

x	ES	AS	AE
0.0	0.0000	0.0000	0.0000
0.2	−0.0064	−0.0064	0.0000
0.4	−0.0384	−0.0384	0.0000
0.6	−0.0684	−0.0684	0.0000
0.8	−0.1024	−0.1024	0.0000
1.1	0.0000	0.0000	0.0000

The results of our method were exact, as shown in Table 2.2. Figure 2 displays the approximate solution for various values of $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$, indicating that the calculation was more accurately done as the table of error obtained is smaller than [12].

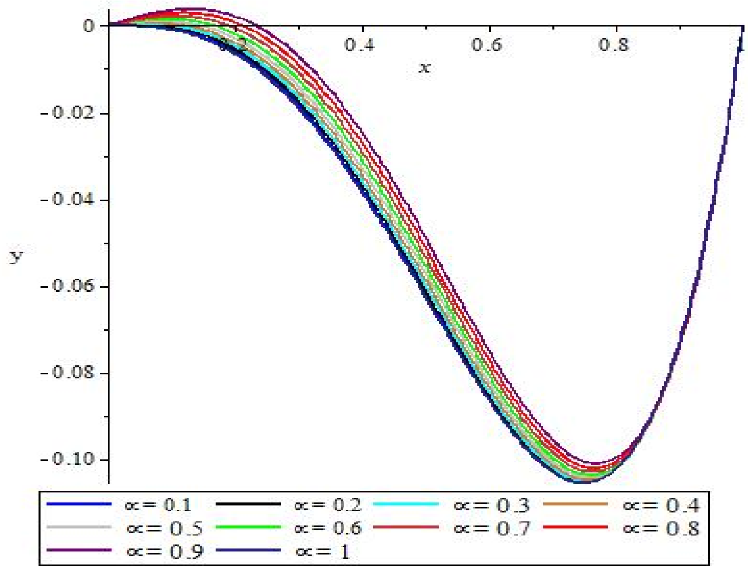

FIGURE 2. Displaying the graphical behavior of example 2's exact and approximate solutions

4. Numerical Applications

This research reported on the numerical solution of FVFIDEs using BCT. Using numerical computations, we verified that the proposed strategy is in superb agreement with the precise result. In comparison to [12] result, the BCT solution is more accurate. The researchers can use this technique on other FVFIDEs based on their findings.

Acknowledgments: Thank you to the reviewers for your insightful comments and improvements.

Conflicts of Interest: The authors say they have no competing interests.

References

[1] L. Adam, Fractional calculus: History, definition and application for the engineer, Department of Aerospace and Mechanical Engineering University of Notre Dame IN 46556, U.S.A. 2004.

[2] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, Geophysical J. Int., Vol. 13, 5, 529539, 1967, 529539. doi.org/10.1111/j.1365- 246X.1967.tb02303.x.

[3] S. Momani and A. Qaralleh, An efficient method for solving systems of fractional integro-differential equations, Comput. and Math. with Appl., Vol. 52, 3, 459- 570, 2006. doi.org/10.1016.

[4] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional integrals and derivatives, theory and applications, Gordon and Breach, Yverdon 1993.

[5] R.C. Mittal and R. Nigam, Solution of fractional integro-differential equations by Adomian decomposition method, Int. J. of Appl. Math. and Mech., Vol. 4, 2, 87- 94, 2008.

[6] H.M. Osama and A.A. Sarmad, Approximate solution of fractional integro- differential equations by using Bernstein polynomials, Eng. and Tech. J., Vol. 30, 8,1362-1373, 2012.

[7] D. Sh. Mohammed, Numerical solution of fractional integro-differential equations by least squares method and shifted Chebyshev polynomial. Math. Problems in Eng., 2014, Article ID 431965, 1, 5.
[8] A.M.S. Mahdy, and E.M.H. Mohamed, Numerical studies for solving system of linear fractional integro-differential equations by using least squares method and Shifted Chebyshev polynomials. J. of Abstract and Comput. Math., Vol. 1, 24, 24- 32, 2016.
[9] V.D. Dilek and D. Aysegl, Applied collocation method using Laguerre polynomials as the basis functions, Advances in Differ. Equ. Springer Open J., 1- 11, 2018. doi.org/10.1186/s13662-018-1924-0.
[10] T. Oyedepo, C.Y. Ishola, T.F. Aminu, A.A. Victor, Bernstein collocation method for the solution of fractional integro-differential equations. J. of Sci. Tech. and Edu., Vol. 8, 1, 65 72, 2020.
[11] D. Aysegul and V.B. Dilek, solving fractional Fredholm integro-differential equations by Laguerre polynomials. Sains Malaysiana, Vol. 48, 1, 251-257, 2019. dio .org/10.17576/jsm-2019-4801-29.
[12] S. Alkan, V.F. Hatipoglu, Approximate solutions of Volterra-Fredholm integro- differential equations of fractional order. Tbilisi Math. J., Vol. 10, 2, 1-13, 2017. doi: 10.1515/tmj-2017-0021.
[13] S.T . Mohyud-Din, H. Khan, M. Arif and M. Rafiq, Chebyshev wavelet method to nonlinear fractional VolterraFredholm integro-differential equations with mixed boundary conditions, Advances in Mech. Eng. Vol. 9, 3, 1-8, 2017. doi.org/10.1177/1687814017694802.
[14] F. Zhou, F. and X. Xu, Numerical solution of fractional Volterra-Fredholm integro- differential equations with mixed boundary conditions via Chebyshev wavelet Method, Int. J. of Comput. Math., Vol. 96, 118, 2018. doi:10.1080/00207160.2018.1521517.
[15] H. Dehestani, Y. Ordokhani, and M. Razzaghi, Combination of Lucas wavelets with LegendreGauss quadrature for fractional Fredholm-Volterra integro- differential equations. J. of Comput. and Appl. Math., Vol. 382, 135, 2021. doi: 10.1016/j.cam.2020.113070.
[16] N.K. Salman and M.M. Mustafa, Numerical solution of fractional Volterra- Fredholm integro-differential equation using Lagrange polynomials, Baghdad Sci. J. Vol. 17, 4, 1234-1240, 2020. doi.org/10.21123/bsj.2020.17.4.1234.
[17] C. Edwards, Math 312 Fractional calculus final presentation. [video] Retrieved fromhttps://www.youtube.com/watch?v=CsJa3XiOmfs [Accessed 20 Sept. 2018].
[18] Grant, T. Math 336 Approximating continuous functions and curves using Bernstein polynomials 2014.

T. Oyedepo
Department of Applied Sciences, Federal College of Dental Technology and Therapy ,Enugu, Nigeria
E-mail address: `oyedepotaiye@yahoo.com`

G. Ajileye
Department of Mathematics and Statistics, Federal University Wukari, Taraba, Nigeria
E-mail address: `ajileye@fuwukari.edu.ng`

Abdullahi. M. Ayinde
Department of Mathematics, University Abuja, Abuja, Nigeria
E-mail address: `ayinde.abdullahi@uniabuja.edu.ng`

Ikechukwu. J. Otaide
Department of Mathematics, Edwin Clark University, Kiagbodo, Nigeria
E-mail address: `otaideikechukwu@gmail.com`