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SOME FRACTIONAL CALCULUS PROPERTIES OF BIVARIATE MITTAG-LEFFLER FUNCTION

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Abstract. We study bivariate Mittag-Leffler function which is an extension of several known Mittag-Leffler functions. We derive the Riemann-Liouville fractional derivatives and integrals of these function and solve a singular integral equation with the bivariate Mittag-Leffler function in the kernel. We also introduce and investigate a fractional integral operator involving the bivariate Mittag-Leffler function.

1. Introduction

The Mittag-Leffler function is an important function that finds widespread use in the subject of fractional calculus. Indeed solutions of various differential and integral equations involving fractional derivatives can be derived in form of Mittag-Leffler functions. Also, the Mittag-Leffler function can be seen in the solution of the same boundary value problems. For details, we refer to works by Gorenflo and Mainard[3], Kilbas and Saigo[7], Kilbas et al. [8] and Rahman et al. [13]. For the purposes of this paper, we will review some Mittag-Leffler function definitions of one and more parameters (see[5],[10],[12],[20]):

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\alpha n+1)}, \tag{1}$$

$$\left(\Re(\alpha)>0, z\in\mathbb{C}\right),$$

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\alpha n+\beta)}, \tag{2}$$

$$\left(\alpha,\beta\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0\right),$$

where as usual notation $\Gamma(\lambda)$ for the Gamma function and $(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$, $n\geq 0,\ \lambda\neq 0,-1,-2,...$, the Pochhammer symbol (see e.g. [4],[14] and [15]). The

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generalization of (1) and (2) was introduced by Prabhakar [12] in terms of the series representation:

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!}, \tag{3}$$

$$(\alpha,\beta,\gamma\in\mathbb{C},\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0).$$

Prabhakar [12] studied some properties of the integral operator

$$\mathrm{E}^{(\gamma)}_{\rho,\mu,w:a^{+}\varphi}(x)=\int_a^x(x-t)^{\mu-1}\ E_{\rho.\mu}[w(x-t)^{\rho}]\varphi(t)dt,(x>a), \tag{4}$$

$$(w\in\mathbb{C},\Re(\gamma)>0,\Re(\rho)>0,\Re(\mu)>0),$$

containing the function (3) in the kernel and applied the results obtained to prove the existence and uniqueness of the solution for the corresponding integral equation of the first kind

$$\int_a^x(x-t)^{\mu-1}\ E_{\rho,\mu}[w(x-t)^{\rho}]\varphi(t)dt=f(x),(x>a). \tag{5}$$

In [1], Bin-Saad et al. established some properties for the bivariate Mittag-Leffler function

$$E^{(1)}_{\alpha,\beta,\gamma}(z_1,z_2)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_{m+n}z_1^m z_2^n}{\Gamma(\alpha m+\gamma n+\beta)m!n!}, \tag{6}$$

or in an equivalent form

$$E_{\alpha,\beta,\gamma}(z_1,z_2)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{m!\ z_1^{m-n}\ z_2^n}{\Gamma(\alpha m+(\gamma-\alpha)n+\beta)(m-n)!n!}, \tag{7}$$

$$(\alpha,\beta,\gamma,z_1,z_2\in\mathbb{C},\min\{\Re(\beta),\Re(\gamma),\Re(\alpha)\}>0),$$

which has been introduced by Luchko and Gorenflo [9]. In our present study, we also propose to use the Riemann-Liouville fractional derivative operator (see e.g. [4],[11] and [15]):

$$\left(D^{\alpha}_{a^+}y\right)(x)=\left(\frac{d}{dx}\right)^n\left(I^{n-\alpha}_{a^+}y\right)(x), \tag{8}$$

$$(\alpha\in\mathbb{C},\Re(\alpha)>0,n=[\Re(\alpha)]+1),$$

defined for $a<x\leq b$, by ([9], Section 2.3 and 2.4), in terms of Riemann-Liouville fractional calculus

$$\left(I^{\alpha}_{a^+}\ \varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{\varphi(t)}{(x-t)^{1-\alpha}}dt, \qquad (9),$$

$$(\alpha\in\mathbb{C},\Re(\alpha)>0,a<x\leq b).$$

Here the the operator $I^{\alpha}_{a^+}$ is defined on the space $\mathbb{L}(a,b)$ (see for details [4],[7]; see also [13]):

$$\mathbb{L}(a,b)=\left\{f(x):\parallel f\parallel_l=\int_a^b|f(t)|dt<\infty\right\}. \tag{10}$$

For the case $\varphi(x)=x^{\alpha},\alpha>-1$, we get the result

$$D^{\nu}_x x^{\alpha}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\nu+1)}x^{\alpha-\nu}, \tag{11}$$

where $D_x^\nu = \left(\frac{d}{dx}\right)^\nu, x > 0, \alpha > -1, \nu \geq 0$ is not restricted to integer values. Also, we recall the relation

$$D_{a^+}^\nu \ I_{a^+}^\nu = \varphi, \tag{12}$$
$$(\alpha \in \mathbb{C}, \Re(\alpha) > 0, \varphi \in \mathbb{L}[a,b]),$$

and the Dirichlet formula [15]

$$\int_a^b dx \ \int_a^x f(x,y)dy = \int_a^b dy \ \int_y^b f(x,y)dx. \tag{13}$$

The present paper sequel to the work in [1] and is tended for the derivation of certain fractional calculus properties for a bivariate Mittag- Leffler function with three parameters $E_{\alpha,\beta,\gamma}^{(1)}$. The paper is organized as follows. Section 2 deals with the computation of the Riemann- Liouville fractional derivatives and integrals of $E_{\alpha,\beta,\gamma}^{(1)}$. Section 3 is devoted to introducing and solving the singular integral equation with $E_{\alpha,\beta,\gamma}^{(1)}$ in the kernel. In Section 4, we introduce an integral operator $\mathbf{E}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ which contains $E_{\alpha,\beta,\gamma}^{(1)}$ in the kernel and investigate its transformation properties in the space of Lebesque summable continuous functions. Composition of the operator $\mathbf{E}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ with Riemann- Liouville fractional integration and differentiation are established in Section 5.

2. Relations via fractional integrals and derivatives

In this section, we derive five theorems relating to the Riemann-Liouville fractional integral and derivative of the bivariate Mittag-Leffler function $E_{\alpha,\beta,\gamma}^{(1)}$.

Theorem 2.1. Let $\alpha, \beta, \gamma \in \mathbb{C}, \{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(w_1), \Re(w_2)\} > 0$. Then the following fractional integral formula holds.

$$
\begin{aligned}
&{}_xI_{a^+}^\lambda \left[(x-a)^{\beta-1} \ E_{\alpha,\beta,\gamma}^{(1)}(w_1(x-a)^\alpha, w_2(x-a)^\gamma)\right] \\
&= (x-a)^{\beta+\lambda-1} E_{\alpha,\beta+\lambda,\gamma}^{(1)}(w_1(x-a)^\alpha, w_2(x-a)^\gamma).
\end{aligned} \tag{14}
$$

Proof. Upon interchanging the order of summation and fractional integration, which is permissible under the assumption started in the theorem and using definitions (6) and (9), we find

$$
\begin{aligned}
&{}_xI_{a^+}^\lambda \left[(x-a)^{\beta-1} \ E_{\alpha,\beta,\gamma}^{(1)}(w_1(x-a)^\alpha, w_2(x-a)^\gamma)\right] \\
&= \int_a^x \frac{(x-t)^{\lambda-1}}{\Gamma(\lambda)} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(1)_{m+n} \ w_1^m w_2^n}{\Gamma(\alpha m + \gamma n + \beta) m! n!} (t-a)^{\alpha m + \gamma n + \beta - 1} dt, \ (x > a), \\
&= \frac{1}{\Gamma(\lambda)} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(1)_{m+n} \ w_1^m w_2^n}{\Gamma(\alpha m + \gamma n + \beta) m! n!} \int_a^x (x-t)^{\lambda-1} (t-a)^{\alpha m + \gamma n + \beta - 1} dt, \\
&= (x-a)^{\beta+\lambda-1} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(1)_{m+n} \ w_1^m w_2^n (x-a)^{\alpha m + \gamma n}}{\Gamma(\alpha m + \gamma n + \beta + \lambda) m! n!},
\end{aligned} \tag{15}
$$

which gives us the desired result (14).

Theorem 2.2. Let $\alpha, \beta, \gamma \in \mathbb{C}, \{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(1-\lambda)), \Re(w_1), \Re(w_2)\} > 0$. Then the following fractional integral formula holds.

$$
\begin{aligned}
&{}_xI_{a^+}^{\lambda}\left[(x-a)^{(1-\lambda)-1}\ E^{(1)}_{\alpha,\beta,\gamma}(w_1(x-a), w_2(x-a))\right] \\
&= \Gamma(1-\lambda)(x-a)^{(1-\lambda)-1}E^{(1-\lambda)}_{\alpha,\beta,\gamma}(w_1(x-a), w_2(x-a)). \qquad (16)
\end{aligned}
$$

Proof. We refer to the proof of Theorem (2.1).

Corollary 2.1. As a consequence of (3) and Theorem 2.2, we have

$$
\begin{aligned}
&{}_xI_{a^+}^{\lambda}\left[(x-a)^{(1-\lambda)-1}\ E^{(1)}_{\alpha,\beta}(w_1(x-a))\right] \\
&= \Gamma(1-\lambda)(x-a)^{(1-\lambda)-1}E^{(1-\lambda)}_{\alpha,\beta}(w_1(x-a)). \qquad (17)
\end{aligned}
$$

where $\alpha, \beta, \mathbb{C}, \{\Re(\alpha), \Re(\beta), \Re(1-\lambda), \Re(w_1)\} > 0$.
We now proceed to find the fractional derivative of the bivariate Mittag-Leffler function $E^{(1)}_{\alpha,\beta,\gamma}$

Theorem 2.3. Let $\alpha, \beta, \gamma \in \mathbb{C}, \{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(w_1), \Re(w_2)\} > 0$. Then the following fractional derivative formula holds.

$$
\begin{aligned}
&{}_xD_{a^+}^{\lambda}\left[(x-a)^{\beta-1}\ E^{(1)}_{\alpha,\beta,\gamma}(w_1(x-a)^{\alpha}, w_2(x-a)^{\gamma})\right] \\
&= (x-a)^{\beta+\lambda-1}E^{(1)}_{\alpha,\beta-\lambda,\gamma}(w_1(x-a)^{\alpha}, w_2(x-a)^{\gamma}). \qquad (18)
\end{aligned}
$$

Proof. Upon using (8), interchanging the order of summation and fractional integration, which is permissible under the assumption started in the theorem and using definitions (6) and (9), we find

$$
\begin{aligned}
&{}_xD_{a^+}^{\lambda}\left[(x-a)^{\beta-1}\ E^{(1)}_{\alpha,\beta,\gamma}(w_1(x-a)^{\alpha}, w_2(x-a)^{\gamma})\right] \\
&= {}_xD_{a^+}^{k}\,{}_xI_{a^+}^{k-\lambda}\left[(x-a)^{\beta-1}\ E^{(1)}_{\alpha,\beta,\gamma}(w_1(x-a)^{\alpha}, w_2(x-a)^{\gamma})\right] \\
&= \frac{1}{\Gamma(k-\lambda)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_{m+n}\ w_1^m w_2^n}{\Gamma(\alpha m+\gamma n+\beta)m!n!}\,{}_xD_{a^+}^{k}\int_a^x (x-t)^{k-\lambda-1}(t-a)^{\alpha m+\gamma n+\beta-1}dt, \\
&= (x-a)^{\beta-\lambda-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_{m+n}\ w_1^m w_2^n (x-a)^{\alpha m+\gamma n}}{\Gamma(\alpha m+\gamma n+\beta-\lambda)m!n!},
\end{aligned}
$$

which gives us the desired result (18).

Theorem 2.4. Let $\alpha, \beta, \gamma \in \mathbb{C}, \{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\lambda), \Re(w_1), \Re(w_2)\} > 0$,. Then the following fractional derivative formula holds.

$$
\begin{aligned}
&{}_xD_{a^+}^{\lambda}\left[(x-a)^{(1+\lambda)-1}\ E^{(1)}_{\alpha,\beta,\gamma}(w_1(x-a), w_2(x-a))\right] \\
&= \Gamma(1+\lambda)(x-a)^{(1+\lambda)-1}E^{(1+\lambda)}_{\alpha,\beta,\gamma}(w_1(x-a), w_2(x-a)). \qquad (19)
\end{aligned}
$$

Proof. We refer to the proof of Theorem 2.3.

Corollary 2.2. As a consequence of (3) and Theorem (2.4), we have

$$
\begin{aligned}
&{}_xD_{a^+}^{\lambda}\left[(x-a)^{(1+\lambda)-1}\ E^{(1)}_{\alpha,\beta}(w_1(x-a))\right] \\
&= \Gamma(1+\lambda)E^{(1+\lambda)}_{\alpha,\beta+\lambda,}(w_1(x-a)), \qquad (20)
\end{aligned}
$$

where $\alpha,\beta\in\mathbb{C},\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\lambda),\Re(w_1),\Re(w_2)\}>0,.$

Theorem 2.5. Let $\alpha,\beta,\gamma,\lambda,w_1,w_2\in\mathbb{C},\{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\lambda)\}>0$. Then the following formulas hold.

$$D_x^{\lambda}\left[x^{\beta-1}\ E^{(1)}_{\alpha,\beta,\gamma}(w_1x^{\alpha},w_2x^{\gamma})\right]=x^{\beta-\lambda-1}\ E^{(1)}_{\alpha,\beta-\lambda,\gamma}(w_1x^{\alpha},w_2x^{\gamma}). \tag{21}$$

In particular

$$D_x^{k}\left[x^{\beta-1}\ E^{(1)}_{\alpha,\beta,\gamma}(w_1x^{\alpha},w_2x^{\gamma})\right]=x^{\beta-\lambda-1}\ E^{(1)}_{\alpha,\beta-k,\gamma}(w_1x^{\alpha},w_2x^{\gamma}). \tag{22}$$

Proof. We have

$$\begin{aligned}
&D_x^{\lambda}\left[x^{\beta-1}\ E^{(1)}_{\alpha,\beta,\gamma}(w_1x^{\alpha},w_2x^{\gamma})\right]\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_{m+n}\ w_1^m w_2^n}{\Gamma(\alpha m+\gamma n+\beta)m!n!}\left(D_x^{\lambda}\ x^{\beta+\alpha m+\gamma n-1}\right),\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_{m+n}\ w_1^m w_2^n}{\Gamma(\alpha m+\gamma n+\beta-\lambda)m!n!}x^{\beta+\alpha m+\gamma n-\lambda-1},\\
&=x^{\beta-\lambda-1}\ E^{(1)}_{\alpha,\beta-\lambda,\gamma}(w_1x^{\alpha},w_2x^{\gamma}).
\end{aligned}$$

which proves (21). The relations (22) follow from (21) when $\lambda=k,(k\in\mathbb{N})$.

3. Singular integral equation with $E^{(1)}_{\alpha,\beta,\gamma}$ in kernel

In this section, we solve a singular integral equation with the bivariate Mittag-Leffler function $E^{(1)}_{\alpha,\beta,\gamma}$ in the kernel. We denote the Laplace transform of a function f (see [15] and [17]) by

$$\mathbb{L}\left[f(t)\right](p)=\breve{f}(p)=\int_0^{\infty}e^{-pt}f(t)dt,(\Re(p)>0). \tag{23}$$

Theorem 3.1. Let $\alpha,\beta,\gamma,w_1,w_2\in\mathbb{C},\ \{\Re(\alpha),\Re(\beta),\Re(\gamma)\}>0$. Then

$$\mathbb{L}\left[x^{\beta-1}\ E^{(1)}_{\alpha,\beta,\gamma}\left((wx)^{\alpha},(wx)^{\gamma}\right)\right](p)=p^{-\beta}\ \left[1-\left(\frac{w}{p}\right)^{\alpha}-\left(\frac{w}{p}\right)^{\gamma}\right]^{-1}. \tag{24}$$

Proof. We have

$$\begin{aligned}
&\mathbb{L}\left[x^{\beta-1}\ E^{(1)}_{\alpha,\beta,\gamma}\left((wx)^{\alpha},(wx)^{\gamma}\right)\right](p)\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_{m+n}(wx)^{\alpha m+\gamma n}}{m!n!\Gamma(\alpha m+\gamma n+\beta)}\int_0^{\infty}e^{-px}x^{\alpha m+\gamma n+\beta-1}dx\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_{m+n}}{m!n!}p^{\alpha m+\gamma n+\beta}w^{\alpha m+\gamma n}\\
&=p^{-\beta}\left[1-\left(\frac{w}{p}\right)^{\alpha}-\left(\frac{w}{p}\right)^{\gamma}\right]^{-1},
\end{aligned} \tag{25}$$

which is the desired result.

Theorem 3.2. Let $\alpha, \beta, \gamma, \lambda, w \in \mathbb{C}$ such that $\{ \Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\lambda)\} > 0$. Then

$$\int_0^x (x-t)^{\beta-1} E^{(1)}_{\alpha,\beta,\gamma}\left[(w(x-t))^{\alpha}, (w(x-t))^{\gamma}\right] \times t^{\lambda-1} E^{(1)}_{\alpha,\lambda,\gamma}\left[(wt))^{\alpha}, (wt)^{\gamma}\right] dt$$
$$= t^{\beta+\lambda-1}\ E^{(2)}_{\alpha,\beta+\lambda,\gamma}\left[(wt)^{\alpha}, (wt)^{\gamma}\right]. \tag{26}$$

Proof. With the help of the convolution theorem for the Laplace transform (see [19]):

$$\mathbb{L}\left[\int_0^x f(x-t)g(t)dt\right](p) = \mathbb{L}\left[f(x)\right](p)\ \mathbb{L}\left[g(x)\right](p),$$

we have

$$\mathbb{L}\left[\int_0^x (x-t)^{\beta-1}\ E^{(1)}_{\alpha,\beta,\gamma}\left((w(x-t))^{\alpha}, (w(x-t))^{\gamma}\right) \times t^{\lambda-1}\ E^{(1)}_{\alpha,\lambda,\gamma}\left((wt)^{\alpha}, (wt)^{\gamma}\right)\right](p).$$
$$= \mathbb{L}\left[(t^{\beta-1}\ E^{(1)}_{\alpha,\beta,\gamma}\left((wt)^{\alpha}, (wt)^{\gamma}\right)\right](p) \times\ \cdot \mathbb{L}\left[t^{\lambda-1} E^{(1)}_{\alpha,\lambda,\gamma}\left((wt)^{\alpha}, (wt)^{\gamma}\right)\right](p).$$

Now, from Theorem 3.1

$$\mathbb{L}\left[\int_0^x (x-t)^{\beta-1}\ E^{(1)}_{\alpha,\beta,\gamma}\left((w(x-t))^{\alpha}, (w(x-t))^{\gamma}\right) \times t^{\lambda-1}\ E^{(1)}_{\alpha,\lambda,\gamma}\left((wt)^{\alpha}, (wt)^{\gamma}\right)\right](p)$$
$$= p^{-(\beta+\lambda)}\ \left[1 - \left(\frac{w}{p}\right)^{\alpha} - \left(\frac{w}{p}\right)^{\gamma}\right]^{-(2)}.$$

Hence

$$\mathbb{L}\left[\int_0^x (x-t)^{\beta-1}\ E^{(1)}_{\alpha,\beta,\gamma}\left((w(x-t))^{\alpha}, (w(x-t))^{\gamma}\right) \times t^{\lambda-1}\ E^{(1)}_{\alpha,\lambda,\gamma}\left((wt)^{\alpha}, (wt)^{\gamma}\right)\right](p)$$
$$= \mathbb{L}\left[t^{\beta+\lambda-1}\ E^{(2)}_{\alpha,\beta+\lambda,\gamma}\left((wt))^{\alpha}, (wt)^{\gamma}\right)\right](p). \tag{27}$$

Finally, taking the inverse Laplace transform of (27), the result follows.

Now, let us consider the following convolution equation involving the bivariate Mittag-Leffler function $E^{(1)}_{\alpha,\beta,\gamma}$ in the kernel.

$$\int_0^x (x-t)^{\beta-1} E^{(1)}_{\alpha,\beta,\gamma}\left[(w(x-t))^{\alpha}, (w(x-t))^{\gamma}\right] \cdot \phi(t)dt = \psi(x), (\Re(\beta) > -1). \tag{28}$$

Theorem 3.3. The singular integral equation (28) admits a locally integrable solution

$$\phi(x) = \int_0^x (x-t)^{\nu-\beta-1} E^{(1)}_{\alpha,\nu-\beta,\gamma}\left[(w(x-t))^{\alpha}, (w(x-t))^{\gamma}\right] \cdot {}_tI^{-\nu}_{0^+}\psi(t)dt, \tag{29}$$

provided that ${}_tI^{-\nu}_{0^+}\varphi(t)$ exists for $\Re(\nu) > \Re(\beta+1)$ and locally integrable for $0 < t < \mu < \infty$.

Proof. Applying the Laplace transform on both sides of (28), using the convolution theorem as well as Theorem 3.1, we get

$$p^{-\beta}\ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1)_{m+n}}{m!n!} \left(\frac{w}{p}\right)^{\alpha m+\gamma n}\ \mathbb{L}\left[\phi(t)\right](p) = \mathbb{L}\left[\psi(t)\right](p), \tag{30}$$

which under the assumptions that $\left|\left(\frac{w}{p}\right)^{\alpha}+\left(\frac{w}{p}\right)^{\gamma}\right|<1$, can be written in the form

$$p^{-\beta}\left[1-\left(\frac{w}{p}\right)^{\alpha}-\left(\frac{w}{p}\right)^{\gamma}\right]^{-1}\mathbb{L}\left[\phi(t)\right](p)=\mathbb{L}\left[\psi(t)\right](p), \tag{31}$$

Therefore, we have

$$\mathbb{L}\left[\phi(t)\right](p)=\left\{\left[1-\left(\frac{w}{p}\right)^{\alpha}-\left(\frac{w}{p}\right)^{\gamma}\right]^{1}p^{\beta-\nu}\right\}\left\{p^{\nu}\mathbb{L}\left[\psi(t)\right](p)\right\}. \tag{32}$$

Taking the inverse Laplace transform of both sides of (32) and with the aid of the following property ([19],p.217, Eq. (3.8))

$$p^{\mu}\mathbb{L}\left[f(t)\right](p)=\mathbb{L}\left[{}_{t}I_{0^+}^{-\mu}\ f(t)\right](p),$$
$$(\mu,p\in\mathbb{C};\Re(p)>0),$$

which holds true for suitable f, we thus find

$$\phi(x)=\int_0^x(x-t)^{\nu-\beta-1}E^{(1)}_{\alpha,\nu-\beta,\gamma}\left[(w(x-t))^{\alpha},(w(x-t))^{\gamma}\right]\cdot{}_{t}I_{0^+}^{-\nu}\psi(t)dt.$$

4. An integral operator with $E^{(1)}_{\alpha,\beta,\gamma}$ in kernel

In this section, we consider the integral operator $\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ defined by

$$\left(\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\right)(x)$$
$$=\int_a^x(x-t)^{\beta-1}E^{(1)}_{\alpha,\beta,\gamma}\left(w_1(x-t)^{\alpha},w_2(x-t)^{\gamma}\right)\varphi(t)dt,(x>a), \tag{33}$$

with $\alpha,\beta,\gamma,w_1,w_2\in\mathbb{C},\ \{\Re(\beta),\Re(\alpha),\Re(\gamma)\}>0.$

In particular, for $w_2=0$, (33) gives the integral operator by Prabhakar [12]

$$\left(\mathbf{E}^{(1)}_{\alpha,\beta,w:a^+}\varphi\right)(x)=\int_a^x(x-t)^{\beta-1}E^{(1)}_{\alpha,\beta}\left(w(x-t)^{\alpha}\right)\varphi(t)dt,(x>a), \tag{34}$$

with the generalized Mittag-Leffler function defined by (3) in the kernel.
We now show that the integral operator $\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ is bounded on the space $L(a,b)$.

Theorem 4.1. Let $\alpha,\beta,\gamma,w_1,w_2\in\mathbb{C}$, $\{\Re(\alpha),\Re(\beta),\Re(\gamma)\}>0$, then the operator $\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ is bounded on $L(a,b)$ and

$$\parallel\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\parallel_l\le A\parallel\varphi\parallel_l, \tag{35}$$

where

$$A=(b-a)^{\Re(\beta)}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{|(1)_{m+n}|}{|\Gamma(\alpha m+\gamma n+\beta)|[\Re(\alpha)m+\Re(\gamma)n+\Re(\beta)]}$$
$$\frac{|w_1(b-a)^{\Re(\alpha)}|^m}{m!}\ \frac{|w_2(b-a)^{\Re(\gamma)}|^n}{n!}. \tag{36}$$

Proof. Using (9) and (6) and interchanging the order of integration and applying the Dirichlet formula (13), we find that

$$\begin{aligned}
&\| \mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi \|_l \\
&= \int_a^b \left| \int_a^x (x-t)^{\beta-1} E^{(1)}_{\alpha,\beta,\gamma}\left(w_1(x-t)^\alpha, w_2(x-t)^\gamma\right)\varphi(t)dt \right| dx, \\
&\le \int_a^b \left\{ \int_t^b (x-t)^{\Re(\beta)-1} \left| E^{(1)}_{\alpha,\beta,\gamma}\left(w_1(x-t)^\alpha, w_2(x-t)^\gamma\right) \right| dx \right\} |\varphi(t)|\, dt \\
&= \int_a^b \left\{ \int_0^{b-t} u^{\Re(\beta)-1} \left| E^{(1)}_{\alpha,\beta,\gamma}\left(w_1 u^\alpha, w_2 u^\gamma\right) \right| du \right\} |\varphi(t)|\, dt \\
&\le \int_a^b \left\{ \int_0^{b-a} u^{\Re(\beta)-1} \left| E^{(1)}_{\alpha,\beta,\gamma}\left(w_1 u^\alpha, w_2 u^\gamma\right) \right| du \right\} |\varphi(t)|\, dt. \qquad (37)
\end{aligned}$$

Using (6), carrying out term-by-term integration, and taking into account (36), we obtain

$$\int_0^{b-a} u^{\Re(\beta)-1} \left| E^{(1)}_{\alpha,\beta,\gamma}\left(w_1 u^\alpha, w_2 u^\gamma\right) \right| du \le \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{|(1)_{m+n}|\ |w_1|^m\ |w_2|^n}{|\Gamma(\alpha m+\gamma n+\beta)|\ m!n!}$$

$$\int_0^{b-a} u^{\Re(\beta)+\Re(\alpha)m+\Re(\gamma)n-1} du = A, \qquad (38)$$

and (37) yields (36), which completes the proof of the theorem.

The following corollary easily follows from Theorem 4.1.

Corollary 4.1. Let $\alpha,\beta,\gamma,w_1,w_2 \in \mathbb{C}$, $\{\Re(\alpha),\Re(\beta),\Re(\gamma)\} > 0$ and $b > a$, then the integral operator $\mathbf{E}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ is bounded on $L(a,b)$ and

$$\| \mathbf{E}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi \|_l \le B \| \varphi \|_l, \qquad (39)$$

where

$$B = (b-a)^{\Re(\beta)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{|w_1(b-a)^{\Re(\alpha)}|^m |w_2(b-a)^{\Re(\gamma)}|^n}{|\Gamma(\alpha m+\gamma n+\beta)|[\Re(\alpha)m+\Re(\gamma)n+\Re(\beta)]m!n!}. \qquad (40)$$

Note that for $w_2 = 0$, (36) gives the known result equation (4.10) in [16]

$$\| \mathbf{E}^{(\gamma)}_{\alpha,\beta,w:a^+}\varphi \|_l \le D \| \varphi \|_l, \qquad (41)$$

where

$$D = (b-a)^{\Re(\beta)} \sum_{m=0}^{\infty} \frac{|(\delta)_m|\ |w_1(b-a)^{\Re(\alpha)}|^m}{|\Gamma(\alpha m+\beta)|[\Re(\alpha)m+\Re(\beta)]m!}. \qquad (42)$$

Next, we show that the integral operator $\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ is bounded in the space $C[a,b]$ of continuous function h on $[a,b]$ with a finite norm

$$\| h \|_C = \max_{a\le x\le b} |h(x)|. \qquad (43)$$

Theorem 4.2. Let $\alpha,\beta,\gamma,w_1,w_2\in\mathbb{C}$, $\{\Re(\alpha),\Re(\beta),\Re(\gamma)\}>0$ and $b>a$, then the operator $\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ is bounded on $C[a,b]$ and

$$\parallel \mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi \parallel_C \le A \parallel \varphi \parallel_C, \tag{44}$$

where A is given by (36).

Proof. Using (33) and (43), we have for any $x\in[a,b]$ and $\varphi\in C[a,b]$

$$\begin{aligned}
&\left|\left(\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\right)(x)\right| \\
&=\left|\int_a^x (x-t)^{\beta-1}E^{(1)}_{\alpha,\beta,\gamma}\left(w_1(x-t)^{\alpha},w_2(x-t)^{\gamma}\right)\varphi(t)dt\right| \\
&\le \int_a^x \left|(x-t)^{\beta-1}E^{(1)}_{\alpha,\beta,\gamma}\left(w_1(x-t)^{\alpha},w_2(x-t)^{\gamma}\right)\right| |\varphi(t)|\, dt \\
&\le \parallel \varphi \parallel_C \int_a^x (x-t)^{\Re(\beta)-1}\left|E^{(1)}_{\alpha,\beta,\gamma}\left(w_1(x-t)^{\alpha},w_2(x-t)^{\gamma}\right)\right| dt \\
&= \parallel \varphi \parallel_C \int_0^{x-a} u^{\Re(\beta)-1}\left|E^{(1)}_{\alpha,\beta,\gamma}\left(w_1u^{\alpha},w_2u^{\gamma}\right)\right| du \\
&\le \parallel \varphi \parallel_C \int_0^{b-a} u^{\Re(\beta)-1}\left|E^{(1)}_{\alpha,\beta,\gamma}\left(w_1u^{\alpha},w_2u^{\gamma}\right)\right| du \\
&= A \parallel \varphi \parallel_C,
\end{aligned}$$

where A is the same as in (36).

As a consequence of Theorem 4.2, we can easily establish the following result.

Corollary 4.2. Let $\alpha,\beta,\gamma,w_1,w_2\in\mathbb{C}$, $\{\Re(\alpha),\Re(\beta),\Re(\gamma)\}>0$ *and* $b>a$, then the operator $\mathbf{E}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ in (34) is bounded on $C[a,b]$ and

$$\parallel \mathbf{E}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi \parallel_C \le B \parallel \varphi \parallel_C, \tag{45}$$

where B is given by (40).

The following Lemma is an application of the previous results.

Lemma 4.1. Let $\alpha,\beta,\gamma,w_1,w_2\in\mathbb{C}$, $\{\Re(\alpha),\Re(\beta),\Re(\gamma)\}>0$ and $b>a$, then

$$\begin{aligned}
&\left(\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}(t-a)^{\lambda-1}\right)(x) \\
&=\Gamma(\lambda)(x-a)^{\beta+\lambda-1}E^{(1)}_{\alpha,\beta+\lambda,\gamma}\left[w_1(x-a)^{\alpha},w_2(x-a)^{\gamma}\right].
\end{aligned}\tag{46}$$

Proof. Making use of (6) and the operator (33), term-by-term integrating and applying the formula [2] for the Beta and Gamma functions, it gives

$$\begin{aligned}
&\left(\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}(t-a)^{\lambda-1}\right)(x) \\
&=\int_a^x (x-t)^{\beta-1}E^{(1)}_{\alpha,\beta,\gamma}\left(w_1(x-t)^{\alpha},w_2(x-t)^{\gamma}\right)(t-a)^{\lambda-1}dt, \\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_{m+n}\ w_1^m w_2^n}{\Gamma(\alpha m+\gamma n+\beta)m!n!}\ \int_a^x (x-a)^{\beta+\alpha m+\gamma n-1}(t-a)^{\lambda-1}, \\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(1)_{m+n}\ w_1^m w_2^n}{\Gamma(\alpha m+\gamma n+\beta)m!n!}B(\beta+\alpha m+\gamma n,\lambda)(x-a)^{\beta+\lambda+\alpha m+\gamma n-1},
\end{aligned}$$

which yields (46) according to the definition (6).

5. Composition properties for the integral operator $\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$

First, in this section, we establish compositions of the integral operator $\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ with different indices.

Theorem 5.1. Let $\alpha,\beta,\gamma,\lambda\in\mathbb{C}, \{\Re(\alpha),\Re(\beta),\Re(\gamma),\Re(\lambda)\}>0$. Then the relation

$$\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\ \mathbf{E}^{(1)}_{\alpha,\lambda,\gamma,w_1,w_2:a^+}\varphi=\mathbf{E}^{(2)}_{\alpha,\beta+\lambda,\gamma,w_1,w_2:a^+}\varphi, \tag{47}$$

holds for any summable function for $\varphi\in L[a,b]$.
In particular

$$\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\ \mathbf{E}^{(-1)}_{\alpha,\lambda,\gamma,w_1,w_2:a^+}\varphi=I^{(\beta+\lambda)}_{a^+}\varphi. \tag{48}$$

*Proof.*Using (6) and (33) and applying the Dirichlet integral formula (13), we find that

$$\begin{aligned}
&\left(\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\ \mathbf{E}^{(1)}_{\alpha,\lambda,\gamma,w_1,w_2:a^+}\ \varphi\right)(x)\\
&=\left(\int_a^x (x-u)^{\beta-1}E^{(1)}_{\alpha,\beta,\gamma}\left(w_1(x-u)^{\alpha},w_2(x-u)^{\gamma}\right)du\right.\\
&\left.\times\int_a^u (u-t)^{\lambda-1}E^{(1)}_{\alpha,\lambda,\gamma}\left(w_1(u-t)^{\alpha},w_2(u-t)^{\gamma}\right)\right)\varphi(t)dt\\
&=\int_a^x\left[\int_t^x (x-u)^{\beta-1}E^{(1)}_{\alpha,\beta,\gamma}\left(w_1(x-u)^{\alpha},w_2(x-u)^{\gamma}\right)\right.\\
&\left.\times(u-t)^{\lambda-1}E^{(1)}_{\alpha,\lambda,\gamma}\left(w_1(u-t)^{\alpha},w_2(u-t)^{\gamma}\right)\right]\varphi(t)dt\\
&=\int_a^x\left[\int_0^{x-t}(x-t-\tau)^{\beta-1}E^{(1)}_{\alpha,\beta,\gamma}\left(w_1(x-t-\tau)^{\alpha},w_2(x-t-\tau)^{\gamma}\right)\right.\\
&\left.\times\tau^{\lambda-1}E^{(1)}_{\alpha,\lambda,\gamma}\left(w_1\tau^{\alpha},w_2\tau^{\gamma}\right)\right]\varphi(t)dt.
\end{aligned}$$

Now, by using Theorem 4.2, we obtain

$$\begin{aligned}
&\left(\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\ \mathbf{E}^{(1)}_{\alpha,\lambda,\gamma,w_1,w_2:a^+}\ \varphi\right)(x)\\
&=\int_a^x (x-t)^{\beta-1}E^{(2)}_{\alpha,\beta+\lambda,\gamma}\left(w_1(x-t)^{\alpha},w_2(x-t)^{\gamma}\right)\ \varphi(t)dt.\\
&=\left(\mathbf{E}^{(2)}_{\alpha,\beta+\lambda,\gamma,w_1,w_2:a^+}\varphi\right)(x),
\end{aligned} \tag{49}$$

which is the desired result.

The following corollary easily follows from Theorem 5.1.

Corollary 5.1. Let $\alpha,\beta,\lambda\in\mathbb{C}, \{\Re(\alpha),\Re(\beta),\Re(\lambda)\}>0$, then we obtain

$$\mathbf{E}^{(1)}_{\alpha,\beta,w:a^+}\ \mathbf{E}^{(1)}_{\alpha,\lambda w:a^+}\varphi=\mathbf{E}^{(2)}_{\alpha,\beta+\lambda w:a^+}\varphi, \tag{50}$$

holds for any summable function for $\varphi\in L[a,b]$.
In particular

$$\mathbf{E}^{(1)}_{\alpha,\beta,w:a^+}\ \mathbf{E}^{(-1)}_{\alpha,\lambda w:a^+}\varphi=I^{(\beta+\lambda)}_{a^+}\varphi. \tag{51}$$

where (5.4) and (5.5) are special cases of known results due to Kiblas et al. [6].

Corollary 5.2. Let $\alpha, \beta, \gamma, \lambda \in \mathbb{C}, \{\Re(\alpha), \Re(\beta), \Re(\lambda)\} > 0$, then

$$\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\ \mathbf{E}^{(1)}_{\alpha,\lambda,\gamma,w_1,w_2:a^+}\varphi = \mathbf{E}^{(2)}_{\alpha,\beta+\lambda,\gamma,w_1,w_2:a^+}\varphi, \tag{52}$$

holds for any summable function $\varphi \in L[a,b]$.
Now, we consider the composition of the fractional integral operator $I^{\alpha}_{a^+}$ with the operator $\mathbf{E}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$

Theorem 5.2. Let $\alpha, \beta, \gamma, \lambda \in \mathbb{C}, \{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\lambda)\} > 0$, then

$$\left(I^{\lambda}_{a^+}\left[\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\right]\right)(x) = \left(\mathbf{E}^{(1)}_{\alpha,\beta+\lambda,\gamma,w_1,w_2:a^+}\varphi\right)(x) = \left(\ \mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\left[I^{\lambda}_{a^+}\varphi\right]\right)(x), \tag{53}$$

holds for any summable function $\varphi \in L[a,b]$.
Proof. Using (47) and (9), we have

$$\left(I^{\lambda}_{a^+}\left[\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\right]\right)(x) = \frac{1}{\Gamma(\lambda)}\int_a^x \frac{\left(\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\right)(\tau)}{(x-\tau)^{1-\lambda}}d\tau$$

$$= \frac{1}{\Gamma(\lambda)}\int_a^x (x-\tau)^{\lambda-1}\left[\int_a^{\tau}(\tau-u)^{\beta-1}E^{(1)}_{\alpha,\beta,\gamma}(w_1(\tau-u)^{\alpha}, w_2(\tau-u)^{\gamma})\varphi(u)du\right]d\tau.$$

Next, applying the Dirichlet formula (13), we obtain

$$\left(I^{\alpha}_{a^+}\left[\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\right]\right)(x)$$

$$= \int_a^x\left[\frac{1}{\Gamma(\lambda)}\int_u^x (x-\tau)^{\lambda-1}(\tau-u)^{\beta-1}E^{(1)}_{\alpha,\beta,\gamma}(w_1(\tau-u)^{\alpha}, w_2(\tau-u)^{\gamma})d\tau\right]\varphi(u)du.$$

Let $\tau - u = \sigma$ in the above equation, we have

$$\left(I^{\alpha}_{a^+}\left[\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\right]\right)(x)$$

$$= \int_a^x\left[\frac{1}{\Gamma(\lambda)}\int_0^{x-u}(x-u-\sigma)^{\lambda-1}\sigma^{\beta-1}E^{(1)}_{\alpha,\beta,\gamma}(w_1\sigma^{\alpha}, w_2\sigma^{\gamma})d\sigma\right]\varphi(u)du.$$

$$= \int_a^x\left(I^{\lambda}_{a^+}\left[\sigma^{\beta-1}E^{(1)}_{\alpha,\beta,\gamma}(w_1\sigma^{\alpha}, w_2\sigma^{\gamma})\right]\right)(x-u)\varphi(u)du.$$

Now, on using (23), we get

$$\left(I^{\lambda}_{a^+}\left[\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\right]\right)(x)$$

$$= \int_a^x (x-u)^{\beta+\lambda-1}E^{(1)}_{\alpha,\beta+\lambda,\gamma}(w_1(x-u)^{\alpha}, w_2(x-u)^{\gamma})\varphi(u)du = \left(\mathbf{E}^{(1)}_{\alpha,\beta+\lambda,\gamma,w_1,w_2:a^+}\varphi\right)(x),$$

which establishes the first equality in (53). Similarly, we can prove the second formula in (53).

Remark 5.1. The above theorem generalized the results obtained by Kilbas et al. [6] and Srivastava and Tomovski [18].
Finally, we discuss the case of the fractional calculus operator $D^{\lambda}_{a^+}$

Theorem 5.3. Let Let $\alpha, \beta, \gamma, w_1, w_2 \in \mathbb{C}$, $\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0$ and $b > a$, then the relation for $\varphi \in L[a,b]$

$$D^{\lambda}_{a^+} \left(\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\right) = \left(\mathbf{E}^{(1)}_{\alpha,\beta-\lambda,\gamma,w_1,w_2:a^+}\varphi\right), \tag{54}$$

holds for any function $\varphi \in C[a,b]$.
In particular for $k \in \mathbb{N}$ and $\Re(\beta) > k$,

$$\left(\frac{d}{dx}\right)^k \left(\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\right)(x) = \mathbf{E}^{(1)}_{\alpha,\beta-k,\gamma,w_1,w_2:a^+}\varphi. \tag{55}$$

Proof. Let $n = [\Re(\lambda)] + 1$. Using (8) and (33) and applying (53) with λ being replaced by $n - \lambda$, we have for $x > a \geq 0$:

$$D^{\lambda}_{a^+} \left(\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\right)$$

$$= \left(\frac{d}{dx}\right)^n \left(I^{n-\lambda}_{a^+}\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\right)(x) = \left(\frac{d}{dx}\right)^n \left(\mathbf{E}^{(1)}_{\alpha,\beta+n-\lambda,\gamma,w_1,w_2:a^+}\varphi\right)(x),$$

$$= \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{\beta+n-\lambda-1} E^{(1)}_{\alpha,\beta+n-\lambda,\gamma}\left(w_1(x-t)^{\alpha}, w_2(x-t)^{\gamma}\right)\varphi(t)dt, \tag{56}$$

Since the integrand in (56) is a continuous function in $C[a,b]$, we can apply the formula

$$\frac{d}{dx}\int_a^x f(x,t)dt = \int_a^x \frac{\partial}{\partial x} f(x,t) + f(x,x), \tag{57}$$

to obtain

$$D^{\lambda}_{a^+} \left(\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\right)$$

$$= \left(\frac{d}{dx}\right)^{n-1} \int_a^x \frac{\partial}{\partial x}(x-t)^{\beta+n-\lambda-1} E^{(1)}_{\alpha,\beta+n-\lambda,\gamma}\left(w_1(x-t)^{\alpha}, w_2(x-t)^{\gamma}\right)\varphi(t)dt$$

$$+ \lim_{t\to x-} (x-t)^{\beta+n-\lambda-1} E^{(1)}_{\alpha,\beta+n-\lambda,\gamma}\left(w_1(x-t)^{\alpha}, w_2(x-t)^{\gamma}\right)\varphi(t)$$

$$= \left(\frac{d}{dx}\right)^{n-1} \int_a^x \frac{\partial}{\partial x}(x-t)^{\beta+n-\lambda-2} E^{(1)}_{\alpha,\beta+n-\lambda-1,\gamma}\left(w_1(x-t)^{\alpha}, w_2(x-t)^{\gamma}\right)\varphi(t)dt$$

in accordance with the result(31). Now applying (57) further $n-1$ times, we get

$$D^{\lambda}_{a^+} \left(\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}\varphi\right) = \int_a^x \frac{\partial}{\partial x}(x-t)^{\beta-\lambda-1} E^{(1)}_{\alpha,\beta-\lambda,\gamma}\left(w_1(x-t)^{\alpha}, w_2(x-t)^{\gamma}\right)\varphi(t)dt,$$

which gives (54). Equation (55) follows from (54), if we take into account the formula

$$\left(D^k_{a^+}\ \mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}y\right)(x) = y^{(k)}(x), (k \in \mathbb{N}), \tag{58}$$

and this completes the proof of the theorem.

6. Conclusions

We used the Riemann-Liouville fractional integral and derivative operators to investigate several properties of the Mittag-Leffler function $E^{(1)}_{\alpha,\beta,\gamma}$. We established the fractional integration and derivation of the function $E^{(1)}_{\alpha,\beta,\gamma}$. Based on careful analysis of function $E^{(1)}_{\alpha,\beta,\gamma}$, we succeed in dominating the solution of the singular integral equation involving the function $E^{(1)}_{\alpha,\beta,\gamma}$ in the kernel. Further, we introduced an integral operator $\mathbf{E}^{(1)}_{\alpha,\beta,\gamma,w_1,w_2:a^+}$ with $E^{(1)}_{\alpha,\beta,\gamma}$ in the kernel and derived its transformation properties in the space of Lebesgue summable continuous function spaces. Future work using these Mittag-Leffler function $E^{(1)}_{\alpha,\beta,\gamma}$ is expected to include, for example, numerical approximation of the functions by solving fractional differential equations; numerical approximation of the operators by Bernstein-polynomial techniques; asymptotic analysis of the functions; and many more properties waiting to be proved mathematically and then applied in practice.

Data Availability Statement

The results data used to support the findings of this study are included within the article.

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