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A FAMILY OF EXTENSIONS AND GENERALIZATIONS OF KÜMMER'S SECOND SUMMATION THEOREM

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Abstract. In recent years, various extensions of the popular and useful Kümmer's second summation theorem have been given by Rathie-Pogany and Rakha-Rathie. The purpose of this paper is to acquire the extensions and generalizations of Kümmer's second summation theorem in the form of

$$_{r+2}F_{r+1}[a,\ b,\ \{n_r+\zeta_r\}\,;\ \frac{1+a+b-m}{2},\ \{\zeta_r\}\,;\ \frac{1}{2}],$$

with suitable convergence conditions. Their derivations are presented by using the summation formula recorded by Prudnikov *et. al.* All the obtained results are believed to be new, interesting and may be useful in the applicable sciences.

1. Introduction

A natural generalization of the Gaussian hypergeometric series $_2F_1[\alpha,\beta;\gamma;z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$_pF_q\left[\begin{array}{ccc}(\alpha_p); & & \\ & & z \\ (\beta_q); & & \end{array}\right] = {}_pF_q\left[\begin{array}{ccc}\alpha_1,\alpha_2,\ldots,\alpha_p; & & \\ & & z \\ \beta_1,\beta_2,\ldots,\beta_q; & & \end{array}\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}, \qquad (1)$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that

$$\beta_j \neq 0,-1,-2,\ldots\ ;\ j=1,2,\ldots,q. \qquad (2)$$

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the $_pF_q$ series defined by equation (1):

(i) converges for $|z|<\infty$, if $p\leq q$,
(ii) converges for $|z|<1$, if $p=q+1$,

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(iii) diverges for all z, $z \neq 0$, if $p > q+1$,
(iv) converges absolutely for $|z| = 1$, if $p = q+1$ and $\Re(\omega) > 0$,
(v) converges conditionally for $|z| = 1$ $(z \neq 1)$, if $p = q+1$ and $-1 < \Re(\omega) \leq 0$,
(vi) diverges for $|z| = 1$, if $p = q+1$ and $\Re(\omega) \leq -1$,

where, by convention, a product over an empty set is interpreted as 1 and

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \tag{3}$$

= Sum of denominator parameters − Sum of numerator parameters.

Pochhammer symbol $(\alpha)_p$ $(\alpha, p \in \mathbb{C})$ is defined, in terms of Gamma function Γ (see, e.g., [21, p.2 and p.5]), see also [14, p.22, Eq.(1), p.32, Q.N.(8) and Q.N.(9)] by

$$(\alpha)_p = \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)} \quad \left(\alpha + p \in \mathbb{C}\backslash\mathbb{Z}_0^-,\, p \in \mathbb{C}\setminus\{0\};\, \alpha \in \mathbb{C}\backslash\mathbb{Z}_0^-,\, p = 0\right)$$
$$= \begin{cases} 1 & \left(p = 0\ ;\ \alpha \in \mathbb{C}\backslash\mathbb{Z}_0^-\right), \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & (p = n \in \mathbb{N}\ ;\ \alpha \in \mathbb{C}), \\ \frac{(-1)^k\, n!}{(n-k)!} & (p = k,\ \alpha = -n\ ;\ n,\, k \in \mathbb{N}_0,\ 0 \leq k \leq n), \\ 0 & (p = k,\ \alpha = -n\ ;\ n,\, k \in \mathbb{N}_0,\ k > n), \\ \frac{(-1)^n}{(1-\alpha)_n} & (p = -n\ ;\ \alpha \in \mathbb{C}\setminus\mathbb{Z},\ n \in \mathbb{N}), \end{cases} \tag{4}$$

it being understood that $(0)_0 = 1$ (see, e.g., [14, 22]) and assumed tacitly that the Gamma quotient exists.

Kümmer's second summation theorem [10, p. 134] :

$${}_2F_1\left[\begin{array}{cc} a,\ b; & \\ & \frac{1}{2} \\ \frac{1+a+b}{2}; & \end{array}\right] = \frac{\sqrt{(\pi)}\ \Gamma(\frac{1+a+b}{2})}{\Gamma(\frac{1+a}{2})\Gamma(\frac{1+b}{2})}, \tag{5}$$

where $\frac{1+a+b}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
In the literature above summation theorem is also known as Gauss second summation theorem.

An interesting extension of Kümmer's second summation theorem was given by Rathie and Pogany [19] in the form

$$e^{-\frac{x}{2}}\ {}_2F_2\left[\begin{array}{cc} a,\ 1+d; & \\ & x \\ 2a+1,\ d; & \end{array}\right] = {}_0F_1\left[\begin{array}{cc} -\ ; & \\ & \frac{x^2}{16} \\ a+\frac{1}{2}; & \end{array}\right] - \frac{x(1-\frac{2a}{d})}{2(2a+1)}\ {}_0F_1\left[\begin{array}{cc} -\ ; & \\ & \frac{x^2}{16} \\ a+\frac{3}{2}; & \end{array}\right]. \tag{6}$$

An attractive extension of Kümmer's second summation theorem was given by Kim et al. [9] in the form

$$e^{-\frac{x}{2}}\ {}_1F_1\left[\begin{array}{cc} a\ ; & \\ & x \\ 2a+j; & \end{array}\right] \text{ for } j = 0, \pm1, \pm2, ..., \pm5. \tag{7}$$

Extension of Kümmer's second summation theorem was given by Rakha et al. [16] in the form

$$e^{-\frac{x}{2}}\ {}_2F_2\left[\begin{array}{cc} a,\ 2+d; & \\ & x \\ 2a+2,\ d; & \end{array}\right] = {}_0F_1\left[\begin{array}{cc} -\ ; & \\ & \frac{x^2}{16} \\ a+\frac{3}{2}; & \end{array}\right] + \frac{x(\frac{2a}{d}-\frac{1}{2})}{(a+1)}\ {}_0F_1\left[\begin{array}{cc} -\ ; & \\ & \frac{x^2}{16} \\ a+\frac{3}{2}; & \end{array}\right] +$$

$$+\frac{cx^2}{2(2a+3)}\ {}_0F_1\left[\begin{array}{cc} -\ ; & \\ & \frac{x^2}{16} \\ a+\frac{5}{2}; & \end{array}\right], \tag{8}$$

where $d \neq 0, -1, -2...$ and c is given by

$$c = \left(\frac{1}{a+1}\right)\left(\frac{1}{2}-\frac{a}{d}\right) + \left(\frac{a}{d(a+1)}\right).$$

Summation formula for Gauss' series [11, p. 491, Entry(7.3.7.2)]:

$${}_2F_1\left[\begin{array}{cc} \alpha,\ \beta; & \\ & \frac{1}{2} \\ \frac{1+\alpha+\beta-j}{2}; & \end{array}\right] = \frac{2^{\beta-1}\Gamma(\frac{1+\alpha+\beta-j}{2})}{\Gamma(\beta)} \sum_{r=0}^{j} \left\{ \binom{j}{r} \frac{\Gamma\left(\frac{\beta+r}{2}\right)}{\Gamma(\frac{1+\alpha+r-j}{2})} \right\}, \tag{9}$$

where $\beta, \frac{1+\alpha+\beta-j}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-,\ j \in \mathbb{N}_0$.

Motivated by the work given beautifully by Adunkudi [1], Andrews [2], Bailey[3], Carlson[4] ,Choi[5, 6], Erdélyi et al. [7], Exton [8], Kim et al.[9], Prudnikov et al.[11], Qureshi et al.[12, 13], Rakha-Rathie [15, 17], Ramesh [18], Slatter [20], Srivastava[21, 22] and Vidunas [23], we mention some summation theorems for ordinary generalized hypergeometric series having the argument $(\frac{1}{2})$. Their derivations are given by using series rearrangement technique and summation formula recorded by Prudnikov *et. al*.

Any values of numerator and denominator parameters in Sections 2 leading to the results which do not make sense are tacitly excluded. It is very interesting to mention here that we have verified the summation theorems using *MATHEMATICA* software, a general system of doing mathematics by computer.

2. MAIN SUMMATION THEOREMS

Theorem 1. *The following summation theorem holds true:*

$${}_6F_5\left[\begin{array}{cc} a,\ b,\ c+1,\ d+1,\ g+1,\ h+1; & \\ & \frac{1}{2} \\ \frac{1+a+b-m}{2},\ c,\ d,\ g,\ h; & \end{array}\right] = \frac{2^{b-1}\Gamma(\frac{1+a+b-m}{2})}{\Gamma(b)} \sum_{r=0}^{m} \binom{m}{r} \times$$

$$\times\left\{\frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{1+a+r-m}{2}\right)} + \frac{a(1+cdg+cdh+cgh+dgh+cd+cg+ch+dg+dh+gh+}{cdgh}\right.$$

$$\frac{+c+g+h+d)\Gamma\left(\frac{b+1+r}{2}\right)}{\Gamma\left(\frac{2+a+r-m}{2}\right)} + \frac{(7+cd+cg+ch+dg+dh+gh+3c+3d+3g+3h)}{cdgh}\times$$

$$\times\frac{a(a+1)\Gamma\left(\frac{b+2+r}{2}\right)}{\Gamma\left(\frac{3+a+r-m}{2}\right)} + \frac{(6+c+d+g+h)a(a+1)(a+2)}{cdgh}\frac{\Gamma\left(\frac{b+3+r}{2}\right)}{\Gamma\left(\frac{4+a+r-m}{2}\right)} +$$

$$+\frac{a(a+1)(a+2)(a+3)}{cdgh}\frac{\Gamma\left(\frac{b+4+r}{2}\right)}{\Gamma\left(\frac{5+a+r-m}{2}\right)}\Bigg\}, \tag{10}$$

where $a, b, c, d, g, h, \frac{1+a+b-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$, and $m \in \mathbb{N}_0$.

Proof: Using the definition of Pochhammer symbol

$$(\alpha)_p = \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)},$$

we can see that

$$\frac{(c+1)_r(d+1)_r(g+1)_r(h+1)_r}{(c)_r(d)_r(g)_r(h)_r} =$$

$$\times\Bigg[1+\frac{(1+cdg+cdh+cgh+dgh+cd+cg+ch+dg+dh+gh+c+d+g+h)r}{cdgh}+$$
$$+\frac{(7+cd+cg+ch+dg+dh+gh+3c+3d+3g+3h)r(r-1)}{cdgh}+$$
$$+\frac{(6+c+d+g+h)r(r-1)(r-2)}{cdgh}+\frac{r(r-1)(r-2)(r-3)}{cdgh}\Bigg], \tag{11}$$

Now, in order to establish our Theorem 1, we expand the left hand side in the series form

$${}_6F_5\left[\begin{array}{c} a,\ b,\ c+1,\ d+1,\ g+1,\ h+1; \\ \frac{1+a+b-m}{2},\ c,\ d,\ g,\ h; \end{array} \frac{1}{2}\right] =$$

$$= {}_2F_1\left[\begin{array}{c} a,\ b\ ; \\ \frac{1+a+b-m}{2}\ ; \end{array} \frac{1}{2}\right]+\frac{(1+cdg+cdh+cgh+dgh+cd+cg+ch+dg+dh+}{cd}$$

$$\frac{+gh+c+d+g+h)}{gh}\sum_{r=1}^{\infty}\frac{(a)_r(b)_r(\frac{1}{2})^r}{(\frac{1+a+b-m}{2})_r(r-1)!}+\frac{(7+cd+cg+ch+dg+dh+gh+}{cd}$$

$$\frac{+3c+3d+3g+3h)}{gh}\sum_{r=2}^{\infty}\frac{(a)_r(b)_r(\frac{1}{2})^r}{(\frac{1+a+b-m}{2})_r(r-2)!}+\frac{(6+c+d+g+h)}{cdgh}\times$$

$$\times\sum_{r=3}^{\infty}\frac{(a)_r(b)_r(\frac{1}{2})^r}{(\frac{1+a+b-m}{2})_r(r-3)!}+\frac{1}{cdgh}\sum_{r=4}^{\infty}\frac{(a)_r(b)_r(\frac{1}{2})^r}{(\frac{1+a+b-m}{2})_r(r-4)!}. \tag{12}$$

Replacing r by $r+1$ in second term, r by $r+2$ in third term, r by $r+3$ in fourth term and r by $r+4$ in fifth term on the right hand side of the equation (12), we come to

$${}_6F_5\left[\begin{array}{c} a,\ b,\ c+1,\ d+1,\ g+1,\ h+1; \\ \frac{1+a+b-m}{2},\ c,\ d,\ g,\ h; \end{array} \frac{1}{2}\right]$$

$$= {}_2F_1\left[\begin{array}{c} a,\ b\ ; \\ \frac{1+a+b-m}{2}\ ; \end{array} \frac{1}{2}\right]+\frac{ab\,(1+cdg+cdh+cgh+dgh+cd+cg+ch+dg+dh+}{cdgh}$$

$$\frac{+gh+c+d+g+h)}{(1+a+b-m)}\,{}_2F_1\left[\begin{array}{c} a+1,\ b+1\ ; \\ \frac{3+a+b-m}{2}\ ; \end{array} \frac{1}{2}\right]+\frac{(7+cd+cg+ch+dg+dh+gh+}{cdgh}$$

$$\frac{+3c+3d+3g+3h)a(a+1)b(b+1)}{(1+a+b-m)(3+a+b-m)}\,{}_2F_1\left[\begin{array}{cc} a+2,\ b+2\ ; & \\ & \frac{1}{2} \\ \frac{5+a+b-m}{2}\ ; & \end{array}\right]+\frac{(6+c+d+g+h)}{cdgh(1+a+b-m)}\times$$

$$\times\frac{a(a+1)(a+2)b(b+1)(b+2)}{(3+a+b-m)(5+a+b-m)}\,{}_2F_1\left[\begin{array}{cc} a+3,\ b+3\ ; & \\ & \frac{1}{2} \\ \frac{7+a+b-m}{2}\ ; & \end{array}\right]+\frac{a(a+1)(a+2)(a+3)}{cdgh(1+a+b-m)}\times$$

$$\times\ \frac{b(b+1)(b+2)(b+3)}{(3+a+b-m)(5+a+b-m)(7+a+b-m)}\ 2F_1\left[\begin{array}{cc} a+4,\ b+4\ ; & \\ & \frac{1}{2} \\ \frac{9+a+b-m}{2}\ ; & \end{array}\right]. \tag{13}$$

Finally using summation formula recorded by Prudnikov *et al.* (9), on the right hand side of the equation (13), we arrive at the right hand side of Theorem 1.

Theorem 2. *The following summation theorem holds true:*

$$ {}_5F_4\left[\begin{array}{cc} a,\ b,\ c+1,\ d+1,\ g+1; & \\ & \frac{1}{2} \\ \frac{1+a+b-m}{2},\ c,\ d,\ g; & \end{array}\right]=\frac{2^{b-1}\Gamma(\frac{1+a+b-m}{2})}{\Gamma(b)}\sum_{r=0}^{m}\binom{m}{r}\left\{\frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{1+a+r-m}{2}\right)}+\right.$$

$$+\frac{a(1+c+d+g+cd+cg+dg)}{cdg}\frac{\Gamma\left(\frac{b+1+r}{2}\right)}{\Gamma\left(\frac{2+a+r-m}{2}\right)}+\frac{a(a+1)(c+d+g+3)}{cdg}\times$$

$$\left.\times\frac{\Gamma\left(\frac{b+2+r}{2}\right)}{\Gamma\left(\frac{3+a+r-m}{2}\right)}+\frac{a(a+1)(a+2)}{cdg}\frac{\Gamma\left(\frac{b+3+r}{2}\right)}{\Gamma\left(\frac{4+a+r-m}{2}\right)}\right\}, \tag{14}$$

where $a,b,c,d,g,\frac{1+a+b-m}{2}\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $m\in\mathbb{N}_0$.

Proof: In order to derive the Theorem 2, we shall use the following:

$$\frac{(c+1)_r(d+1)_r(g+1)_r}{(c)_r(d)_r(g)_r}=$$

$$=\left[1+\frac{(1+c+d+g+cd+cg+dg)r}{cdg}+\frac{(c+d+g+3)r(r-1)}{cdg}+\frac{r(r-1)(r-2)}{cdg}\right]. \tag{15}$$

After some simplification and using the summation formula recorded by Prudnikov *et al.* (9), we will get the right hand side of Theorem 2.

Theorem 3. *The following summation theorem holds true:*

$$ {}_5F_4\left[\begin{array}{cc} a,\ b,\ c+1,\ d+1,\ g+2; & \\ & \frac{1}{2} \\ \frac{1+a+b-m}{2},\ c,\ d,\ g; & \end{array}\right]=\frac{2^{b-1}\Gamma(\frac{1+a+b-m}{2})}{\Gamma(b)}\sum_{r=0}^{m}\binom{m}{r}\left\{\frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{1+a+r-m}{2}\right)}+\right.$$

$$+\frac{a(dg+cg+2cd+2c+2d+g+2)}{cdg}\frac{\Gamma\left(\frac{b+1+r}{2}\right)}{\Gamma\left(\frac{2+a+r-m}{2}\right)}+\frac{(10+cd+2dg+2cg+}{cdg}$$

$$\frac{+g^2+4c+4d+7g)a(a+1)}{(g+1)}\frac{\Gamma\left(\frac{b+2+r}{2}\right)}{\Gamma\left(\frac{3+a+r-m}{2}\right)}+\frac{(7+c+d+2g)a(a+1)(a+2)}{cdg(g+1)}\times$$

$$\left.\times\frac{\Gamma\left(\frac{b+3+r}{2}\right)}{\Gamma\left(\frac{4+a+r-m}{2}\right)}+\frac{a(a+1)(a+2)(a+3)}{cdg(g+1)}\frac{\Gamma\left(\frac{b+4+r}{2}\right)}{\Gamma\left(\frac{5+a+r-m}{2}\right)}\right\}, \tag{16}$$

where $a, b, c, d, g, \frac{1+a+b-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $m \in \mathbb{N}_0$.

Proof : The proof would flow along the lines of that of Theorem 2, with the aid of result (9). So we prefer to omit the details.

Theorem 4. *The following summation theorem holds true:*

$$
{}_4F_3\left[\begin{array}{c} a,\ b,\ c+1,\ d+1; \\ \frac{1+a+b-m}{2},\ c,\ d; \end{array} \frac{1}{2}\right] = \frac{2^{b-1}\Gamma(\frac{1+a+b-m}{2})}{\Gamma(b)} \sum_{r=0}^{m} \binom{m}{r} \left\{ \frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{1+a+r-m}{2}\right)} + \right.
$$

$$
\left. + \frac{a(c+d+1)}{cd} \frac{\Gamma\left(\frac{b+r+1}{2}\right)}{\Gamma\left(\frac{2+a+r-m}{2}\right)} + \frac{a(a+1)}{cd} \frac{\Gamma\left(\frac{b+r+2}{2}\right)}{\Gamma\left(\frac{3+a+r-m}{2}\right)} \right\}, \tag{17}
$$

where $a, b, c, d, \frac{1+a+b-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $m \in \mathbb{N}_0$.

Proof: The proof would run in parallel with that of Theorem 1, with the aid of result (9). The details are omitted.

Theorem 5. *The following summation theorem holds true:*

$$
{}_4F_3\left[\begin{array}{c} a,\ b,\ c+1,\ d+2; \\ \frac{1+a+b-m}{2},\ c,\ d; \end{array} \frac{1}{2}\right] = \frac{2^{b-1}\Gamma(\frac{1+a+b-m}{2})}{\Gamma(b)} \sum_{r=0}^{m} \binom{m}{r} \left\{ \frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{1+a+r-m}{2}\right)} + \frac{a(2c+d+2)}{cd} \times \right.
$$

$$
\left. \times \frac{\Gamma\left(\frac{b+1+r}{2}\right)}{\Gamma\left(\frac{2+a+r-m}{2}\right)} + \frac{a(a+1)(c+2d+4)}{cd(d+1)} \frac{\Gamma\left(\frac{b+2+r}{2}\right)}{\Gamma\left(\frac{3+a+r-m}{2}\right)} + \frac{a(a+1)(a+2)}{cd(d+1)} \frac{\Gamma\left(\frac{b+3+r}{2}\right)}{\Gamma\left(\frac{4+a+r-m}{2}\right)} \right\}, \tag{18}
$$

where $a, b, c, d, \frac{1+a+b-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $m \in \mathbb{N}_0$.

Proof: In order to acquire the Theorem 5, we shall use the following:

$$
\frac{(c+1)_r (d+2)_r}{(c)_r (d)_r} = \left(1 + \frac{(2c+d+2)r}{cd} + \frac{(c+2d+4)r(r-1)}{cd(d+1)} + \frac{r(r-1)(r-2)}{cd(d+1)}\right) \tag{19}
$$

After further simplification and using the summation formula recorded by Prudnikov *et al.* (9), we will arrive at the right hand side of Theorem 5. We omit specifics.

Theorem 6. *The following summation theorem holds true:*

$$
{}_4F_3\left[\begin{array}{c} a,\ b,\ c+2,\ d+2\ ; \\ \frac{1+a+b-m}{2},\ c,\ d; \end{array} \frac{1}{2}\right] = \frac{2^{b-1}\Gamma(\frac{1+a+b-m}{2})}{\Gamma(b)} \sum_{r=0}^{m} \binom{m}{r} \left\{ \frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{1+a+r-m}{2}\right)} + \right.
$$

$$
+ \frac{a(4+2c+2d)}{cd} \frac{\Gamma\left(\frac{b+1+r}{2}\right)}{\Gamma\left(\frac{2+a+r-m}{2}\right)} + \frac{(14+c^2+d^2+4cd+9c+9d)a(a+1)}{cd(c+1)(d+1)} \frac{\Gamma\left(\frac{b+2+r}{2}\right)}{\Gamma\left(\frac{3+a+r-m}{2}\right)} +
$$

$$
\left. + \frac{(8+2c+2d)a(a+1)(a+2)}{cd(c+1)(d+1)} \frac{\Gamma\left(\frac{b+3+r}{2}\right)}{\Gamma\left(\frac{4+a+r-m}{2}\right)} + \frac{a(a+1)(a+2)(a+3)}{cd(c+1)(d+1)} \frac{\Gamma\left(\frac{b+4+r}{2}\right)}{\Gamma\left(\frac{5+a+r-m}{2}\right)} \right\}, \tag{20}
$$

where $a, b, c, d, \frac{1+a+b-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$, and $m \in \mathbb{N}_0$.

Proof: The proof would be accomplished by following the lines of that of Theorem 5, with the aid of result (9). The involved details are omitted.

Theorem 7. *The following summation theorem holds true:*

$$
{}_4F_3\left[\begin{array}{c} a,\ b,\ c+1,\ d+3\ ; \\ \frac{1+a+b-m}{2},\ c,\ d; \end{array} \frac{1}{2}\right] = \frac{2^{b-1}\Gamma(\frac{1+a+b-m}{2})}{\Gamma(b)} \sum_{r=0}^{m} \binom{m}{r} \left\{ \frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{1+a+r-m}{2}\right)} + \right.
$$

$$
+\frac{a(3+3c+d)}{cd}\frac{\Gamma\left(\frac{b+1+r}{2}\right)}{\Gamma\left(\frac{2+a+r-m}{2}\right)} + \frac{(9+3c+3d)a(a+1)}{cd(d+1)}\frac{\Gamma\left(\frac{b+2+r}{2}\right)}{\Gamma\left(\frac{3+a+r-m}{2}\right)} +
$$

$$
\left. +\frac{(9+c+3d)a(a+1)(a+2)}{cd(d+1)(d+2)}\frac{\Gamma\left(\frac{b+3+r}{2}\right)}{\Gamma\left(\frac{4+a+r-m}{2}\right)} + \frac{a(a+1)(a+2)(a+3)}{cd(d+1)(d+2)}\frac{\Gamma\left(\frac{b+4+r}{2}\right)}{\Gamma\left(\frac{5+a+r-m}{2}\right)} \right\}, \tag{21}
$$

where $a, b, c, d, \frac{1+a+b-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$, and $m \in \mathbb{N}_0$.

Proof: In order to establish the Theorem 7, we shall use the following:

$$
\frac{(c+1)_r(d+3)_r}{(c)_r(d)_r} = \left[1 + \frac{(3+3c+d)r}{cd} + \frac{(9+3c+3d)r(r-1)}{cd(d+1)} + \right.
$$

$$
\left. +\frac{(9+c+3d)r(r-1)(r-2)}{cd(d+1)(d+2)} + \frac{r(r-1)(r-2)(r-3)}{cd(d+1)(d+2)}\right]. \tag{22}
$$

After straight forword calculation and using the summation formula recorded by Prudnikov *et al.* (9), we will arrive at the right hand side of Theorem 7. The involved details are omitted.

Theorem 8. *The following summation theorem holds true:*

$$
{}_3F_2\left[\begin{array}{c} a,\ b,\ c+4; \\ \frac{1+a+b-m}{2},\ c; \end{array} \frac{1}{2}\right] = \frac{2^{b-1}\Gamma(\frac{1+a+b-m}{2})}{\Gamma(b)} \sum_{r=0}^{m} \binom{m}{r} \left\{ \frac{\Gamma\left(\frac{b+r}{2}\right)}{\Gamma\left(\frac{1+a+r-m}{2}\right)} + \right.
$$

$$
+\frac{4a}{c}\frac{\Gamma\left(\frac{b+1+r}{2}\right)}{\Gamma\left(\frac{2+a+r-m}{2}\right)} + \frac{6a(a+1)}{c(c+1)}\frac{\Gamma\left(\frac{b+2+r}{2}\right)}{\Gamma\left(\frac{3+a+r-m}{2}\right)} + \frac{4a(a+1)(a+2)}{c(c+1)(c+2)}\frac{\Gamma\left(\frac{b+3+r}{2}\right)}{\Gamma\left(\frac{4+a+r-m}{2}\right)} +
$$

$$
\left. +\frac{a(a+1)(a+2)(a+3)}{c(c+1)(c+2)(c+3)}\frac{\Gamma\left(\frac{b+4+r}{2}\right)}{\Gamma\left(\frac{5+a+r-m}{2}\right)} \right\}, \tag{23}
$$

where $a, b, c, \frac{1+a+b-m}{2} \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $m \in \mathbb{N}_0$.

Proof: In order to derive the Theorem 8, we will use the following:

$$
\frac{(c+4)_r}{(c)_r} = 1 + \frac{4r}{c} + \frac{6r(r-1)}{c(c+1)} + \frac{4r(r-1)(r-2)}{c(c+1)(c+2)} + \frac{r(r-1)(r-2)(r-3)}{c(c+1)(c+2)(c+3)}. \tag{24}
$$

After further simplification and using the summation formula recorded by Prudnikov *et al.* (9), we will arrive at the right hand side of Theorem 8. We omit specifics.

3. CONCLUSION

In this paper, we have derived some extensions and generalizations of Kümmer's second summation theorem (5) for ${}_6F_5[\frac{1}{2}]$, ${}_5F_4[\frac{1}{2}]$, ${}_4F_3[\frac{1}{2}]$ and ${}_3F_2[\frac{1}{2}]$, where certain numerator and denominator parameters differ by a positive integer, as claimed in theorem 1 to 8. We conclude this paper with the note that some other summation theorems can be obtained in a same way. Besides the derived result are quite significant, we believe that these would be able to find wide range of applications and advantages in obtaning the exact mathematical expressions in place of already approximate mathematical expressions scattered in the fields of Applied Mathematics and Engineering Sciences to improve the accuracy.

Conflicts of interests: The authors declare that there are no conflicts of interests.

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$$1+\frac{\alpha.\beta}{1.\gamma}\ x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}\ x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}\ x^3+\cdots,$$

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